where $\hat{y}(t;\theta)$ is the output of the neural control scheme, $J_{\rm track}(\theta_c)$ is a cost function for the tracking error, defined on a given specific reference input $d(t)$ and t_f is a finite time horizon. The matrix inequality constraint can be related to Lemma 1 as well as to Lemma 2, respectively, for imposing global asymptotic stability or I/O stability with a fixed disturbance attenuation level ζ^*. Such methods have been successfully applied for the discrete-time recurrent neural networks using NL_q theory in [19].

VII. Conclusion

In this paper absolute stability and dissipativity of continuous-time recurrent neural networks with two hidden layers have been studied. These types of models occur when one considers nonlinear models and controllers that are parameterized by multilayer perceptrons with one hidden layer. For the autonomous case a classical Lur'e system representation and Lur'e system with multilayer perceptron nonlinearity is given. Sufficient conditions for absolute stability and dissipativity have been derived from a Lur'e–Postnikov Lyapunov function and a storage function of the same form. The criteria are expressed as matrix inequalities. They can be employed in order to impose closed-loop stability in Narendra's dynamic backpropagation procedure and for nonlinear H_∞ control.

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Noninteracting Control via Static Measurement Feedback for Nonlinear Systems with Relative Degree

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***Abstract*— In this paper the authors give a necessary and sufficient geometric condition for achieving noninteraction via static measurement feedback for nonlinear systems with vector relative degree. Their analysis relies on the theory of connections and as a result gives systematic procedures for constructing a decoupling feedback law.**

***Index Terms*—Measurement feedback, noninteracting control.**

I. The Class of Systems and Control Laws

Let us consider the affine nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x)u_j \\ y_i &= h_i(x), \qquad i = 1, \cdots, m, \\ z &= k(x) \end{aligned} \tag{1}$$

where $x \in \mathcal{M}$, a smooth (Hausdorff) manifold, the u_i's are input functions from a suitable function space (e.g., measurable $\mathbb{R}$-valued functions defined on closed intervals of the form $[0, T]$), the y_i's are the $\mathbb{R}$-valued output functions and $z \in \mathbb{R}^s$ is the vector of

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variables which are available for feedback, and $f, g_1 \cdots, g_m$ are smooth vector fields on some open subset $\mathcal{U} \subset \mathcal{M}$ with $g_1, \cdots, g_m$ linearly independent on $\mathcal{U}$ and $f(0) = 0$. Moreover, $h_1, \cdots, h_m$ and k are smooth functions defined on $\mathcal{U}$, with $h_i(0) = 0$ and $k(0) = 0$. Denote by h the vector $(h_1 \cdots h_m)^T$ and by g the matrix $(g_1 \cdots g_m)$.

We will consider *static measurement-feedback laws* of the form

$$u = \alpha(k(x)) + \beta(k(x))v \tag{2}$$

with smooth functions α and β defined on some open subset $\mathcal{Z} \subset \mathbb{R}^s$, $\beta(k(x)) \in \mathcal{GL}(\mathbb{R}, m)$, where $\mathcal{GL}(\mathbb{R}, m)$ is the general linear group of $m \times m$ invertible matrices over $\mathbb{R}$, and v the new input vector.

II. Motivations and Problem Statement

Assume that $k(x) = x$, i.e., the state vector is available for feedback. Moreover, if not otherwise stated, assume that $\mathcal{U} = \mathcal{M}$. Let $\mathcal{G}$ be the distribution spanned by $g_1, \cdots, g_m$. A distribution Δ on $\mathcal{M}$ is said to be *weakly* (f, g)*-invariant at* $x \in \mathcal{M}$, if there exists a neighborhood $\mathcal{U}_0$ of x such that on $\mathcal{U}_0$

$$[f, \Delta] \subset \mathcal{G} + \Delta \tag{4}$$

$$[g, \Delta] \subset \mathcal{G} + \Delta. \tag{5}$$

Δ is said to be *globally weakly* (f, g)*-invariant* if it is weakly (f, g)-invariant at each $x \in \mathcal{M}$.

If (4) and (5) are not satisfied, one can try to modify the behavior of (1) through (2) in such a way that these properties are achieved for the closed-loop system. A distribution Δ on $\mathcal{M}$ is said to be (f, g)*-invariant at* $x \in \mathcal{M}$, if there exists a *static state* feedback law (2), defined in a neighborhood $\mathcal{U}_0$ of x, such that

$$[f + g\alpha, \Delta] \subset \Delta \tag{6}$$

$$[g\beta, \Delta] \subset \Delta. \tag{7}$$

If the feedback law is defined on $\mathcal{M}$, then Δ is said to be *globally* (f, g)*-invariant*.

Weak (f, g)-invariance was first introduced for linear systems in [3] and, independently, in [4], for nonlinear systems affine in the input in [5] and for general nonlinear systems in [6]. If (1) is a *linear* system and Δ is a subspace of $\mathbb{R}^n$, (5) and (7) are trivially satisfied, *local* and *global* definitions coincide, and weak (f, g)-invariance is equivalent to (f, g) invariance. For nonlinear system (1), weak (f, g)-invariance is implied by (f, g)-invariance but the converse is not true in general. A powerful lemma for studying (f, g)-invariance is given by the *Quaker lemma* [1]–[6]. The main difficulty in extending the Quaker lemma to a *global* setting hides behind using arguments heavily based on partial differential equations (PDE's). The key observation, which gives a deep insight into global obstructions, has been first noted in [7], followed by [8], where it is shown that weak (f, g)-invariance can be interpreted geometrically saying that a subbundle of the normal bundle of Δ is "invariant under parallel transport" along leaves of Δ.

Now, assume that only $k(x)$ is available for feedback. A distribution Δ on $\mathcal{M}$ is said to be *weakly* (f, g, k)*-invariant at* $x \in \mathcal{M}$, if it is *weakly* (f, g)*-invariant at* $x \in \mathcal{M}$ and there exists a neighborhood $\mathcal{U}_0$ of x such that

$$[f, \Delta \cap \ker dk] \subset \Delta \tag{8}$$

$$[g, \Delta \cap \ker dk] \subset \Delta. \tag{9}$$

Δ is said to be *globally weakly* (f, g, k)*-invariant* if it is weakly (f, g, k)-invariant at each $x \in \mathcal{M}$.

A distribution Δ on $\mathcal{M}$ is said to be (f, g, k)*–invariant at* $x \in \mathcal{M}$, if there exists a *static measurement* feedback law (2), defined in neighborhood $\mathcal{U}_0$ of x, such that

$$[f + g(\alpha \circ k), \Delta] \subset \Delta \tag{10}$$

$$[g(\beta \circ k), \Delta] \subset \Delta. \tag{11}$$

If the feedback law is defined on $\mathcal{M}$, then Δ is said to be *globally* (f, g, k)*-invariant*.

Weak (f, g, k) invariance was first introduced in [3] and, independently, in [4] for *linear* systems and in [5] for nonlinear systems, affine in the input. If (1) is a *linear* system and Δ is spanned by a set of constant vectors of $\mathbb{R}^n$, *local* and *global* definitions coincide and weak (f, g, k)-invariance is equivalent to (f, g, k) invariance [3], [4]. For nonlinear systems (1), weak (f, g, k)-invariance is implied by (f, g, k)-invariance but the converse is not true in general. Necessary and sufficient (existence) conditions for *local* (f, g, h) invariance are given in [9].

As is well known, weak (f, g)-invariance is a natural tool for solving many control problems, such as *disturbance decoupling* and *noninteracting* control, as long as the state x is available for feedback and *local* solutions are sought (see [2] and [10] for an exhaustive discussion). For global (f, g) invariance, additional assumptions to weak (f, g)-invariance must be imposed (see above). If only z is available for feedback, weak (f, g, k) invariance still does the job for *linear* systems [3], [4]. However, in a nonlinear setting, one needs additional assumptions to fully characterize (global) (f, g, k) invariance in terms of weak (f, g, k) invariance. In general, these extra assumptions are tailored to guarantee that a *state*-feedback law (2), which renders a given distribution Δ invariant, can be "expressed" as a *measurement* feedback law (2).

In this paper, exploring the route of (f, g, k) invariance, we will focus our attention on the problem of rendering (1) *noninteractive* via *static measurement* feedback (2). We say that the system (1), (2) is *noninteractive* if each output is influenced (or "controlled") only by one input. Moreover, we consider the class of nonlinear systems (1) which have *uniform vector relative degree* on $\mathcal{M}$ [10], i.e., the *same* vector relative degree at each $x \in \mathcal{M}$ (see [12] for motivations).

Noninteracting Control via Static Measurement Feedback Laws (NSM): Find, if possible, a static feedback law (2), defined on a neighborhood $\mathcal{U}$ of $x = 0$, such that the system (1), (2) is noninteractive and has uniform vector relative degree on $\mathcal{U}$.

When $\mathcal{U} = \mathcal{M}$, we will refer to the noninteracting control problem by global noninteracting control via static measurement (GNSM). When the functions f, g, h, and k are analytic, a stronger problem can be formulated (*strong input–output decoupling*; see [10]).

The *local* noninteracting control problem via *static* output feedback (i.e., $k = h$) was first solved in [12] under an additional assumption. This assumption was removed in [13], where a necessary and sufficient condition for solving NSM is given. Unfortunately, although the *global* validity of this condition is equivalent to GNSM being solvable, no global constructive method is pointed out in [13] for checking this condition. In this paper, we give a necessary and sufficient (checkable) condition for solving GNSM, using a complete different approach, advocated in [8] for global (f, g) invariance and leading naturally to a global solution of the noninteracting control problem. We point out the obstruction lying between *linear* and *nonlinear* (f, g, k)-invariance. Our proof gives a constructive procedure for obtaining the decoupling feedback law and has, in local coordinates, a very simple interpretation. Moreover, our condition recovers the ones given in [13] and in [9] for local (f, g, k) invariance. One of the contributions of this paper is, in our opinion, to give (global) constructive tools which can also be used to solve measurement feedback problems different from noninteracting control such as either *disturbance decoupling* or *model matching*.

III. Theoretical Background and Definitions

- We assume that concepts, as bundles, sections, etc., are familiar to the reader. We refer to standard textbooks such as [14] and [15]. By $T(\mathcal{M})$ we will denote the *tangent bundle* on $\mathcal{M}$

and by the same symbol we will denote both distributions and their associated subbundles. Let $C^\infty(\mathcal{M})$ be the set of smooth functions on $\mathcal{M}$ and $\mathcal{X}^\infty(\mathcal{M})$ be the set of smooth vector fields on $\mathcal{M}$.

- A smooth *section* X of a vector bundle Δ over $\mathcal{M}$ is a smooth mapping $X: \mathcal{M} \to \Delta$ such that $\pi \circ X = id_\mathcal{M}$, where $id_\mathcal{M}$ is the identity on $\mathcal{M}$. A vector field on $\mathcal{M}$ is a section of $T(\mathcal{M})$. If q is the dimension of Δ and if there exist q everywhere independent sections (*frame* of Δ), then Δ is said to be *trivial*.
- A smooth *connection* ∇ on $T(\mathcal{M})$ is a mapping $(X,Y) \mapsto \nabla_X Y$, with ∇: $\mathcal{X}^\infty(\mathcal{M}) \times \mathcal{X}^\infty(\mathcal{M}) \to \mathcal{X}^\infty(\mathcal{M})$, which is $C^\infty(\mathcal{M})$-linear w.r.t. $X, \mathbb{R}$-linear w.r.t. Y and such that $\nabla_X(fY) = (L_X f)Y + f\nabla_X Y$, $f \in C^\infty(\mathcal{M})$. A smooth *connection* ∇ on a vector bundle π: $\mathcal{D} \to \mathcal{M}$ is a mapping $(X,Y) \mapsto \nabla_X Y$, with ∇: $\mathcal{X}^\infty(\mathcal{M}) \times \mathcal{D} \to \mathcal{D}$, which is $C^\infty(\mathcal{M})$-linear w.r.t. $X, \mathbb{R}$-linear w.r.t. Y and such that $\nabla_X(fY) = (L_X f)Y + f\nabla_X Y$, $f \in C^\infty(\mathcal{M})$.
- The subbundle Δ determines the *quotient bundle* $\mathcal{Q} = T(\mathcal{M})/\Delta$, having $\mathcal{Q}_x = (T_x\mathcal{M})/\Delta_x$ as fiber over x. If a Riemaniann metric is introduced on $\mathcal{M}$, $\mathcal{Q}$ can be identified with a subbundle $\mathcal{Q}'$ of $T(\mathcal{M})$ (*normal bundle* of Δ, denoted by $\Delta^\perp$), having as fiber over x the subspace $\mathcal{Q}'_x = \Delta^\perp_x \subset T_x\mathcal{M}$. As a consequence, any section Y of $T(\mathcal{M})$ decomposes uniquely as $Y_\Delta + Y_{\mathcal{Q}'}$. According to this decomposition, one can define $\nabla_X \overline{Y} = \nabla_X Y_{\mathcal{Q}'} = [X, Y_{\mathcal{Q}'}]_{\mathcal{Q}'}$, with $X \in \Delta$, $\overline{Y}$ is a section of $\mathcal{Q}$ and $Y_{\mathcal{Q}'}$ is the unique vector field in $\mathcal{Q}'$ which projects onto $\overline{Y}$ at each x by the natural projection $T_x(\mathcal{M}) \to T_x(\mathcal{M})/\Delta$. ∇ is shown to define a connection on $\mathcal{Q}$ and to be independent of the choice of $\mathcal{Q}'$. If $\nabla_X Y_{\mathcal{Q}'} = 0$, we say that $\overline{Y}$ (or $Y_{\mathcal{Q}'}$, once $\overline{Y}$ and $Y_{\mathcal{Q}'}$ are identified as above) is *parallel* along leaves of Δ. Such a connection is commonly referred to as the *Bott connection*.

IV. Necessary and Sufficient Condition for Noninteracting Control via Static Measurement Feedback

Let $\mathcal{R}^*_i$ be the *maximal controllability* distribution for (1) contained in $\bigcap_{j\neq i} \ker dh_j$, supposed to exist on all $\mathcal{M}$. This distribution can be obtained through a constructive algorithm (see [2]). *We will assume that at each step of this algorithm we obtain a nonsingular distribution on* $\mathcal{M}$. Under this assumption, such an algorithm ends in a finite number of steps. More simply, we will say that $\mathcal{R}^*_i$ is *regularly computable* on $\mathcal{M}$.

Let $\mathcal{R}^* = \bigcap_{i=1}^m; (\Sigma_{j\neq i}\ \mathcal{R}^*_j)$ and let $\mathcal{R}$ be the *strong accessibility distribution* or, equivalently, the smallest distribution which is *invariant under* f and g and contains $\mathcal{G}$ (see [2] for constructive algorithms). Moreover, let $\mathcal{D}_i = \mathcal{R}^*_i \cap \mathcal{G}$ and $\mathcal{E}_i = \Sigma_{j\neq i}\ \mathcal{R}^*_j$.

A1) $\mathcal{R}$ and $\mathcal{R}^*_i$ are regularly computable on $\mathcal{M}$ for $i = 1,\cdots,m, \mathcal{E}_i$, $\mathcal{R}^*$, $\ker dk$ and $\ker dk + \mathcal{E}_i$ have constant dimension on $\mathcal{M}$. Moreover, $\ker dk + \mathcal{E}_i$ is involutive. •

A2) System (1) has uniform vector relative degree on $\mathcal{M}$. •

A3) $\mathcal{R}$ has constant dimension n on $\mathcal{M}$. •

The distribution $\ker dk + \mathcal{E}_i$ is *not* involutive, in general. However, if, in particular, $k(x) = h(x)$ and under Assumptions A1) and A2), it is easy to show that $\ker dk + \mathcal{E}_i$ is indeed involutive (see [12, Sec. 6.1]).

In the framework of *local* noninteracting control with internal *stability* (see the end of this section), A3) is a standard assumption [4], [16] and in the *linear* case amounts to require that the systems be *controllable* [4]. If internal stability is not required, Assumption A3) is not needed but for simplifying the formulas involved.

Before stating the main result of this section, we will discuss some basic facts.

For each $i \in \{1,\cdots,m\}$, let decompose $T(\mathcal{M})$ as follows. If a Riemaniann metric is introduced on $\mathcal{M}$, the *quotient bundle* $\mathcal{Q}_i = T(\mathcal{M})/\mathcal{E}_i$ can be identified with a subbundle $\mathcal{Q}'_i$ of $T(\mathcal{M})$. Thus, any vector field X on $\mathcal{M}$ can be uniquely decomposed as $X_{\mathcal{Q}'_i} + X_{\mathcal{E}_i}$. Moreover, under Assumptions A1) and A2), $(\mathcal{E}_i \cap \mathcal{G}) \cap \mathcal{D}_i = 0$ ([16] and [12]). This implies that $\mathcal{Q}'_i$ can be chosen in such a way to include $\mathcal{D}_i$. In particular, $\mathcal{Q}'_i = \mathcal{D}_i + (\mathcal{D}_i + \mathcal{E}_i)^\perp$. In what follows, by ∇^i we will denote the Bott connection such that $\nabla_X Y' = [X, Y']_{\mathcal{Q}'_i}$.

Given any distribution Δ or vector field X on $\mathcal{M}$, we will denote its projection into $\mathcal{Q}'_i$ (or $\mathcal{Q}_i$, once $\mathcal{Q}_i$ is identified with $\mathcal{Q}'_i$) by $\overline{\Delta}^i$ or $\overline{X}^i$, respectively. The projection of X onto $\mathcal{D}_i$ will be also denoted by $X_{\mathcal{D}_i}$.

By definition, for each $i \in \{1,\cdots,m\}$ the distribution $\mathcal{E}_i$ is globally weakly (f,g)-invariant and, under Assumption A1), it is also involutive (see [2] or [12] for a detailed proof). Thus, through each $p \in \mathcal{M}$ there is a maximal connected integral manifold $\mathcal{L}^{\mathcal{E}_i}_p$, the leaf of $\mathcal{E}_i$ passing through p. As pointed out in [8], global weak (f,g)-invariance of $\mathcal{E}_i$ can be interpreted from a geometric point of view by saying that *the projection of* $\mathcal{G}$ *into* $\mathcal{Q}_i$ *is invariant under parallel transport along leaves* (of $\mathcal{E}_i$). This can be stated equivalently as follows: if $(\mathcal{W}_\alpha, \phi_\alpha)$ is a Frobenius chart and $\tilde{\phi}_\alpha$ is ϕ_α followed by projection into $\mathbb{R}^q$ (q is the codimension of $\mathcal{E}_i$), then, at each point x of any leaf of $\mathcal{E}_i$ intersected with $\mathcal{W}_\alpha, (\mathcal{G} + \mathcal{E}_i)_x$ projects to the same subspace of $T_{\tilde{\phi}_\alpha(x)}\mathbb{R}^q$.

Let $\mathcal{F}_i$ denote the folation of $\mathcal{E}_i$. Assume that $\mathcal{M}/\mathcal{F}_i$ is a smooth manifold and let $\pi_{\mathcal{F}_i}$: $\mathcal{M} \to \mathcal{M}/\mathcal{F}_i$ be the projection map. At each $p \in \mathcal{M}$ one can assign a subspace of $T_{\pi_{\mathcal{F}_i}(p)}(\mathcal{M}/\mathcal{F}_i)$ by $\widehat{\mathcal{D}_i}^i(\pi_{\mathcal{F}_i}(p)) = \{(\pi_{\mathcal{F}_i})_{*p}X | X \in \mathcal{D}_i(p)\}$. Since $\mathcal{G} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_m$ and $(\Sigma_{j\in I}\ \mathcal{R}^*_j) \cap \mathcal{G} = \Sigma_{j\in I}\ \mathcal{D}_j$, with $I \subset \{1,\cdots,m\}$ ([16]), global weak (f,g) invariance of $\mathcal{E}_i$ implies that the projection of $\mathcal{D}_i$ onto $\mathcal{Q}_i$ is invariant under parallel transport along leaves. Thus, $\widehat{\mathcal{D}_i}^i(\pi_{\mathcal{F}_i}(p))$ does not depend on the choice of p on the leaf of $\mathcal{E}_i$ and $\widehat{\mathcal{D}_i}^i$ *is a smooth distribution on* $\mathcal{M}/\mathcal{F}_i$ (see also [8]).

Given any distribution Δ or vector field X on $\mathcal{M}$, we will denote (when this does make sense) its projection onto $T(\mathcal{M}/\mathcal{F}_i)$ by $\hat{\Delta}^i$ or $\hat{X}^i$, respectively.

If there exists a complementary subbundle $\mathcal{H}_i$ to $\widehat{\mathcal{D}_i}^i$ in $T(\mathcal{M}/\mathcal{F}_i)$, one can pull back $\mathcal{H}_i$ to obtain a complementary subbundle $\mathcal{N}_i$ to $\mathcal{D}_i$ in $\mathcal{Q}_i$. With $\mathcal{Q}'_i$ chosen as above, the subbundle $\mathcal{N}_i$ is isomorphic to the subbundle $\mathcal{N}'_i = (\mathcal{D}_i + \mathcal{E}_i)^\perp$ under the projection $T(\mathcal{M}) \to \mathcal{Q}_i$ and, thus, by construction $\mathcal{N}'_i$ *is invariant under parallel transport along leaves*.

Denoting by $\mathcal{G}_i$ the distribution spanned by g_i, it can be easily seen that Assumption A2) implies that for each $i \in \{1,\cdots,m\}$ and in a *neighborhood* $\mathcal{U}_0$ of each point $p \in \mathcal{M}$ there exists $k_i \in \{1,\cdots,m\}$ such that $\overline{\mathcal{G}_{k_i}}^i = \mathcal{D}_i$. For achieving a *global* result and according to our previous discussion, we assume the following.

A4) After possibly renumbering the inputs and for each $i \in \{1,\cdots,m\}$, $\overline{\mathcal{G}_i}^i = \mathcal{D}_i$. Moreover, $\mathcal{M}/\mathcal{F}_i, i = 1,\cdots,m$, are smooth (not necessarily Hausdorff) manifolds, $\widehat{\mathcal{D}_i}^i$ is trivial and there exists a complementary subbundle $\mathcal{H}_i$ to $\widehat{\mathcal{D}_i}^i$ in $T(\mathcal{M}/\mathcal{F}_i)$. •

When $k(x) = x$, Assumption A4) is exactly the one invoked in [8] (Proposition 5.1). If $\mathcal{M}/\mathcal{F}_i$ is also Hausdorff, a Riemaniann metric can be introduced and the existence of a complementary subbundle $\mathcal{H}_i$ to $\widehat{\mathcal{D}_i}^i$ in $T(\mathcal{M}/\mathcal{F}_i)$ is automatically guaranteed.

Let $\mathcal{K}$ be the foliation of $\ker dk$ and $\mathcal{K}_i$ be the foliation of $(\ker dk) \cap \mathcal{E}_i$. Let $\mathcal{M}/\mathcal{K}_i$ be a smooth manifold, $\imath_{\mathcal{K}_i}$: $\mathcal{M}/\mathcal{K}_i \to \mathcal{M}$ be the inclusion map, and $\pi_{\mathcal{K}_i}$: $\mathcal{M} \to \mathcal{M}/\mathcal{K}_i$ be the projection map. Note that, since $\ker dk$ is involutive, $[X, \ker dk] \subset \ker dk$ for all $X \in (\ker dk) \cap \mathcal{E}_i$. Thus, the distribution $\widehat{\mathcal{S}}_i = \{(\pi_{\mathcal{K}_i})_{*p}X | X \in (\ker dk)(p)\}$ is a smooth distribution on $\mathcal{M}/\mathcal{K}_i$.

Moreover, by $\mathcal{L}_p^{\Delta}$ we will denote the leaf, passing through p, of an involutive and nonsingular distribution Δ. Our last assumption is the following and its role and interpretation will be clear in the proof of the main theorem.

A5) $\mathcal{M}/\mathcal{K}$, $\mathcal{M}/\mathcal{K}_i$, and $\imath_{\mathcal{K}_i}^{-1}\mathcal{L}_p^{\mathcal{E}_i}$ are smooth manifolds and there exists a smooth manifold $\mathcal{O}_i \subset \mathcal{M}/\mathcal{K}_i$, transversal to each $\imath_{\mathcal{K}_i}^{-1}\mathcal{L}_p^{\mathcal{E}_i}$ in $\mathcal{M}/\mathcal{K}_i$, intersecting each $\imath_{\mathcal{K}_i}^{-1}\mathcal{L}_p^{\mathcal{E}_i}$ only once and such that $\widehat{\mathcal{S}}_i(p) \subset T_p\mathcal{O}_i$. •

Our main result is the following.

Theorem 1: Assume A1)–A5). GNSM is solvable if and only if for each $i \in \{1, \cdots, m\}$ and for each $p \in \mathcal{M}$ there exists a neighborhood $\mathcal{U}_p$ of p such that for any frame $Z_1, \cdots, Z_d, \cdots, Z_l$ of $\ker dk$, defined on $\mathcal{U}_p$, with $\overline{Z_j}^i$ parallel for all $j \le d$ and $Z_{d+1}, \cdots, Z_l$ a frame of $(\ker dk) \cap \mathcal{E}_i$

- for $j \le d$ and for all $X \in \mathcal{E}_i$

$$[Z_j, \overline{g_s}^i]_{\mathcal{D}_i} = \psi_{ji}\overline{g_s}^i \qquad s = 1, \cdots, m \tag{12}$$
$$\nabla_X^i [Z_j, f_{\mathcal{D}_i}]_{\mathcal{D}_i} = \psi_{ji}\nabla_X^i f_{\mathcal{D}_i} \tag{13}$$

  with ψ_{ji} constant along leaves of $\mathcal{E}_i$ in $\mathcal{U}_0$
- for $j \ge d+1$

$$\nabla_{Z_j}^i \overline{g_s}^i = 0 \qquad s = 1, \cdots, m \tag{14}$$
$$\nabla_{Z_j}^i \overline{f}^i = 0. \tag{15}$$

•

Remark: Conditions (12) and (13) are trivially satisfied for *linear* systems and (14) and (15) are nothing but (8) and (9) (i.e., weak (f, g, k) invariance). Thus, system (12), (13) is the *nonlinear* obstruction to (f, g, k) invariance of each distribution $\mathcal{R}_i^*$. Moreover, as far as a *local* solution of noninteracting control via *static* measurement feedback is sought, the condition $\overline{\mathcal{G}}_i^{\,i} = \mathcal{D}_i$ (after possibly renumbering the inputs) is guaranteed by Assumption A2) while the existence of a complementary subbundle $\mathcal{H}_i$ to $\widehat{\mathcal{D}}_i^{\,i}$ in $T(\mathcal{M}/\mathcal{F}_i)$ and A5) are automatically satisfied. It is easy to show that (12)–(15) recover the local condition given in [13] and [9]. •

Proof (Only If): Fix $i \in \{1, \cdots, m\}$. Since GNSM is solvable, following the proof of [12, Proposition 3.1], one can prove that there exist everywhere nonzero vector fields $W_1, \cdots, W_m$ and a static feedback law (2), both defined on $\mathcal{M}$ and with $\beta \circ k$: $\mathcal{M} \to \mathcal{GL}(\mathbb{R}, m)$, such that for each $i = 1, \cdots, m$

$$\mathcal{D}_i = \operatorname{span}\{W_i\} \tag{16a}$$
$$(W_1 \ \cdots \ W_m) = g(\beta \circ k) \tag{16b}$$
$$[f + g(\alpha \circ k), \mathcal{E}_i] \subset \mathcal{E}_i \tag{16c}$$
$$[g(\beta \circ k), \mathcal{E}_i] \subset \mathcal{E}_i \tag{16d}$$

on $\mathcal{M}$. This, together with A4), implies (14) and (15).

As a consequence of (17a), (17b) and A4), there exists $\gamma_{ii} \in C^\infty(\mathcal{M})$ nonzero everywhere on $\mathcal{M}$ and constant along leaves of $\ker dk$ such that

$$\overline{g_i}^i = \gamma_{ii} W_i. \tag{17}$$

By global weak (f, g)-invariance of $\mathcal{E}_i, \nabla_X^i \overline{g_i}^i = c_{X,i}\ \overline{g_i}^i$ for $X \in \mathcal{E}_i$ and for some $c_{X,i} \in C^\infty(\mathcal{M})$, and by (16d) and (17) $[\gamma_{ii}^{-1}c_{X,i} + L_X(\gamma_{ii}^{-1})]\overline{g_i}^i = \nabla_X^i W_i = 0$ for $X \in \mathcal{E}_i$. Thus, since γ_{ii} is constant along leaves of $\ker dk$

$$\begin{aligned} L_X(\gamma_{ii}^{-1}) &= -\gamma_{ii}^{-1}c_{X,i} \\ L_{Z_j}(\gamma_{ii}^{-1}) &= 0 \end{aligned} \tag{18}$$

for $X \in \mathcal{E}_i$ and $Z_j \in \ker dk$. A necessary condition for (18) is that

$$L_{Z_j} c_{X,i} = \sum_s b_{js} c_{X_s,i} \tag{19}$$

where $[Z_j, X] = \Sigma_s\ b_{js}X_s$ and $X_1, \cdots, X_k$ is a frame of $\mathcal{E}_i$ (see [2, Th. 6.2.3]).

Let $Z_1, \cdots, Z_d, \cdots, Z_l$ be any local frame of $\ker dk$, defined in $\mathcal{U}_p$, with $\overline{Z_j}^i$ parallel for $j \le d$ and $Z_{d+1}, \cdots, Z_l$ a frame of $(\ker dk) \cap \mathcal{E}_i$ (this frame always exists, since $(\ker dk) \cap \mathcal{E}_i$ has constant dimension on $\mathcal{M}$ and $\ker dk + \mathcal{E}_i$ is involutive). Moreover, let us decompose $T(\mathcal{M})$ as $\mathcal{E}_i + \mathcal{Q}_i'$, with $\mathcal{Q}_i' = \mathcal{D}_i + \mathcal{N}_i'$, and let

$$[Z_j, \overline{g_i}^i]_{\mathcal{D}_i} = \psi_{ji}\overline{g_i}^i \tag{20}$$

for $j \le d$ and some $\psi_{ji} \in C^\infty(\mathcal{M})$. Since $\overline{Z_j}^i$ is parallel and $\mathcal{N}_i'$ is invariant under parallel transport along leaves, i.e., $[\mathcal{N}_i', \mathcal{E}_i] \subset \mathcal{N}_i' + \mathcal{E}_i$, from the Jacobi identity

$$0 = [Z_j, [X, \overline{g_i}^i]] - [X, [Z_j, \overline{g_i}^i]] - [[Z_j, X], \overline{g_i}^i]$$

it follows that

$$0 = \overline{[Z_j, \nabla_X^i \overline{g_i}^i]}^i - \nabla_X^i \overline{[Z_j, \overline{g_i}^i]}^i - \nabla_{[Z_j, X]}^i \overline{g_i}^i \tag{21}$$

for all $X \in \mathcal{E}_i$ and $j \le d$. As a consequence of (17)–(21), $L_X\psi_{ji} = 0$ for all $X \in \mathcal{E}_i$. This, together with (20), implies (12) with $s = i$. Moreover, from (17a) and (17b)

$$\overline{g_j}^i = \gamma_{ji} W_i$$

for $j \ne i$ and for some $\gamma_{ji} \in C^\infty(\mathcal{M})$, constant along leaves of $\ker dk$. Thus, (12) holds for all $s = 1, \cdots, m$.

Finally, let us prove (13). For the vector field f decomposes uniquely as

$$f = f_{\mathcal{E}_i} + f_{\mathcal{D}_i} + f_{\mathcal{N}_i'}. \tag{22}$$

Since $\mathcal{E}_i$ is globally weakly (f, g) invariant and $\mathcal{N}_i'$ is invariant under parallel transport along leaves, it must be

$$[f_{\mathcal{N}_i'}, \mathcal{E}_i] \subset \mathcal{E}_i \tag{23}$$

(see [8]). From (22) and (23)

$$\nabla_X^i \overline{f}^i = \nabla_X^i f_{\mathcal{D}_i} = c_{Xi0} W_i \tag{24}$$

for some $c_{Xi0} \in C^\infty(\mathcal{M})$ and for all $X \in \mathcal{E}_i$. By (17), (24) and since $\nabla_X^i W_i = 0$ for all $X \in \mathcal{E}_i, (L_X\tilde{\alpha}_i + c_{Xi0})W_i = \nabla_X^i(f_{\mathcal{D}_i} + \tilde{\alpha}_i W_i) = 0$ for some $\tilde{\alpha}_i \in C^\infty(\mathcal{M})$, constant along leaves of $\ker dk$, and for all $X \in \mathcal{E}_i$ or equivalently

$$\begin{aligned} L_X\tilde{\alpha}_i &= -c_{Xi0} \\ L_{Z_j}\tilde{\alpha}_i &= 0 \end{aligned} \tag{25}$$

A necessary condition for (24) is

$$L_{Z_j} c_{Xi0} = \sum_s b_{js} c_{X_s i0} \tag{26}$$

where $[Z_j, X] = \Sigma_s\ b_{js}X_s$ (see [2, Th. 6.2.3]). Moreover,

$$[Z_j, W_i]_{\mathcal{D}_i} = \psi_{ji} W_i. \tag{27}$$

Using (20), with $\overline{g_i}^i$ replaced with $f_{\mathcal{D}_i}$, and (26) and (27), one obtains (13).

(If): First, we prove (12). Let decompose $T(\mathcal{M})$ as $\mathcal{E}_i + \mathcal{Q}_i'$, with $\mathcal{Q}_i' = \mathcal{D}_i + \mathcal{N}_i'$. From A4) one obtains

$$\overline{g_i}^i \gamma_{ii} = \tilde{W}_i \tag{28}$$

for some $\gamma_{ii} \in C^\infty(\mathcal{M})$, nonzero everywhere on $\mathcal{M}$, and for some parallel $\tilde{W}_i$ which spans $\mathcal{D}_i$. Indeed, $\overline{\mathcal{G}}_i^{\,i}$ is invariant under parallel transport along leaves and $\widehat{\mathcal{D}}_i^{\,i}$ is a smooth distribution on $\mathcal{M}/\mathcal{F}_i$. Since $(\pi_{\mathcal{F}_i})_{*p}$: $\mathcal{D}_i \to T_{\pi_{\mathcal{F}_i}(p)}(\mathcal{M}/\mathcal{F}_i)$ is one-to-one with image equal to $\widehat{\mathcal{D}}_i^{\,i}(\pi_{\mathcal{F}_i}(p))$ and by triviality of $\widehat{\mathcal{D}}_i^{\,i}$, one can define $\tilde{W}_i(p) = ((\pi_{\mathcal{F}_i})_{*p})^{-1}(Y_i(\pi_{\mathcal{F}_i}(p)))$, where Y_i is a smooth, never vanishing vector field of $\widehat{\mathcal{D}}_i{}^i$. This automatically defines a function $\gamma_{ii} \in C^\infty(\mathcal{M})$, which is nonzero on $\mathcal{M}$ and satisfies (28) (see [8]). Moreover, $\tilde{W}_i$ is parallel and spans $\mathcal{D}_i$.

Let $Z_1, \cdots, Z_d, \cdots, Z_l$ be any frame, defined in a neighborhood $\mathcal{U}_p$ of $p \in \mathcal{M}$, with $\overline{Z_j}^i$ parallel for $j \le d$ and $Z_{d+1}, \cdots, Z_l$ a frame of $(\ker dk) \cap \mathcal{E}_i$. Along any integral curve $s(t)$ of $Z_{d+1}, \cdots, Z_l$, entirely contained in $\mathcal{U}_p$

$$\nabla^i_{Z_j} \tilde{W}_i = \frac{d\gamma_{ii}}{dt} \overline{g_i}^i + \gamma_{ii} \nabla^i_{Z_j} \overline{g_i}^i \tag{29}$$

for $j \ge d+1$ and with $Z_j(s(t)) = \dot{s}(t)$. Since $\tilde{W}_i$ is parallel and $Z_j \in \mathcal{E}_i$ for $j \ge d+1$, it follows from (14) and (29) that

$$L_{Z_j} \gamma_{ii} = 0 \tag{30}$$

in $\mathcal{U}_p$ and for all $j \ge d+1$.

On the other hand, since $\overline{Z_j}^i$ is parallel for $j \le d$ and ψ_{ji} is constant along leaves of $\mathcal{E}_i$, one can find a parallel W_i, which spans $\mathcal{D}_i$ and is defined on $\mathcal{M}$, such that for $j \le d$

$$[Z_j, W_i]_{\mathcal{D}_i} = \psi_{ji} W_i. \tag{31}$$

Indeed, let $\tilde{W}_i$ as above. Let $\pi_{\mathcal{M}/\mathcal{K}_i}$ be the projection $\mathcal{M}/\mathcal{K}_i \to \mathcal{O}_i$ and let $\pi_{\mathcal{E}_i} = \imath_{\mathcal{O}_i} \circ \pi_{\mathcal{M}/\mathcal{K}_i} \circ \pi_{\mathcal{K}_i}$, where $\imath_{\mathcal{O}_i}$: $\mathcal{O}_i \to \mathcal{M}$ is the inclusion map. We will show that the vector field $W_i = \tilde{W}_i(\gamma_{ii} \circ \pi_{\mathcal{E}_i})^{-1}$ satisfies (30). From Jacobi identity

$$0 = [Z_j, [X, \tilde{W}_i]] - [X, [Z_j, \tilde{W}_i]] - [[Z_j, X], \tilde{W}_i]$$

for $X \in \mathcal{E}_i$, since $\overline{Z_j}^i$ is parallel for all $j \le r$ and $\tilde{W}_i$ is also parallel, it follows that

$$[Z_j, \tilde{W}_i]_{\mathcal{D}_i} = \tilde{\psi}_{ji} \tilde{W}_i \tag{32}$$

for $j \le d$, with $\tilde{\psi}_{ji}$ constant along leaves of $\mathcal{E}_i$. As a consequence of (12), (28), and (32)

$$L_{Z_j} \gamma_{ii} = (-\psi_{ji} + \tilde{\psi}_{ji}) \gamma_{ii}. \tag{33}$$

From (33), since $\overline{Z_j}^i$ is parallel for all $j \le d$ and $\psi_{ji} - \tilde{\psi}_{ji}$ is constant along leaves, it is easy to realize (in local coordinates) that $\gamma_{ii} \circ \pi_{\mathcal{E}_i} \in C^\infty(\mathcal{M})$ and satisfies

$$L_{Z_j}(\gamma_{ii} \circ \pi_{\mathcal{E}_i}) = (-\psi_{ji} + \tilde{\psi}_{ji})(\gamma_{ii} \circ \pi_{\mathcal{E}_i}). \tag{34}$$

Indeed, choose local coordinates $T(x) = (\zeta_0^T(x), \zeta_1^T(x), \zeta_2^T(x), \zeta_3^T(x))$ such that

$$\begin{aligned} \mathcal{E}_i &= \operatorname{span}\left\{ \frac{\partial}{\partial \zeta_2}, \frac{\partial}{\partial \zeta_3} \right\} \\ \ker dk &= \operatorname{span}\left\{ \frac{\partial}{\partial \zeta_1}, \frac{\partial}{\partial \zeta_3} \right\}. \end{aligned}$$

Locally around p, $\pi_{\mathcal{E}_i}$ is the map $T^{-1}(\zeta_0, \zeta_1, 0, 0)$. Note that, since $\overline{Z_j}^i$ is parallel for $j \le d$, the projection of Z_j onto $\operatorname{span}(\partial/\partial\zeta_0), (\partial/\partial\zeta_1)\}$ depends only on ζ_0 and ζ_1. For similar reasons, the functions ψ_{ji} and $\tilde{\psi}_{ji}$ depend only on ζ_0 and ζ_1. Moreover, by (30), γ_{ii} is independent of ζ_3. Since $\mathcal{E}_i \cap (\ker dk) = \operatorname{span}\{(\partial/\partial\zeta_3)\}$, (34) follows from (33).

From (32), it follows that (31) holds with $W_i = \tilde{W}_i(\gamma_{ii} \circ \pi_{\mathcal{E}_i})^{-1}$. Indeed, by (32) and (34)

$$\begin{aligned} &W_i \tilde{\psi}_{ji}(\gamma_{ii} \circ \pi_{\mathcal{E}_i}) \\ &\quad = [Z_j, W_i](\gamma_{ii} \circ \pi_{\mathcal{E}_i}) + W_i L_{Z_j}(\gamma_{ii} \circ \pi_{\mathcal{E}_i}) \\ &\quad = [Z_j, W_i](\gamma_{ii} \circ \pi_{\mathcal{E}_i}) + W_i(\tilde{\psi}_{ji} - \psi_{ji})(\gamma_{ii} \circ \pi_{\mathcal{E}_i}) \end{aligned}$$

which, since $\gamma_{ii} \circ \pi_{\mathcal{E}_i}$ is nonzero on $\mathcal{M}$, implies (31).

Let $\tilde{\gamma}_{ii} = \gamma_{ii}(\gamma_{ii} \circ \pi_{\mathcal{E}_i})^{-1}$. Thus

$$W_i = \tilde{\gamma}_{ii} \overline{g_i}^i \tag{35}$$

and

$$L_{Z_j} \tilde{\gamma}_{ii} = 0, \qquad j \ge d+1 \tag{36}$$

in $\mathcal{U}_p$ [by (30)]. From (31) and (35), for $j \le d$

$$\begin{aligned} \psi_{ji} W_i &= [Z_j, W_i]_{\mathcal{D}_i} = (L_{Z_j} \tilde{\gamma}_{ii}) \overline{g_i}^i + \tilde{\gamma}_{ii} [Z_j, \overline{g_i}^i]_{\mathcal{D}_i} \\ &= (L_{Z_j} \tilde{\gamma}_{ii}) \overline{g_i}^i + \psi_{ji} W_i \end{aligned}$$

which implies

$$L_{Z_j} \tilde{\gamma}_{ii} = 0 \qquad j \le d \tag{37}$$

on $\mathcal{U}_p$. Repeating the above arguments for a neighborhood $\mathcal{U}_p$ of each point $p \in \mathcal{M}$, from (36) and (37) one obtains $L_X \tilde{\gamma}_{ii} = 0$ on $\mathcal{M}$ and for all $X \in \ker dk$.

Since $\overline{\mathcal{G}}_i{}^i = \mathcal{D}_i$, then $\overline{g_s}^i = \gamma_{si} W_i$ for some $\gamma_{si} \in C^\infty(\mathcal{M})$ and for all $s \ne i$. As a consequence of (12), (14), and (31)

$$\psi_{ji} \gamma_{si} W_i = [Z_j, \gamma_{si} W_i]_{\mathcal{D}_i} = \psi_{ji} \gamma_{si} W_i + (L_{Z_j} \gamma_{si}) W_i$$

which proves that

$$L_{Z_j} \gamma_{si} = 0 \tag{38}$$

for all $s \ne i$ and for all j.

Since $\mathcal{M}/\mathcal{K}$ is a smooth manifold, as a consequence of (36)–(38), there exist functions $\hat{\gamma}_{ii} \in C^\infty(\mathcal{M}/\mathcal{K})$ and $\hat{\gamma}_{ji} \in C^\infty(\mathcal{M}/\mathcal{K})$ such that $\tilde{\gamma}_{ii} = \hat{\gamma}_{ii} \circ \pi_{\mathcal{K}}$ and $\gamma_{ji} = \hat{\gamma}_{ji} \circ \pi_{\mathcal{K}}$, where $\pi_{\mathcal{K}}$: $\mathcal{M} \to \mathcal{M}/\mathcal{K}$ is the projection map. Moreover, since $\mathcal{E}_i \cap \mathcal{G} = \Sigma_{j \ne i} \mathcal{D}_j$, $\mathcal{D}_i \cap (\Sigma_{j \ne i} \mathcal{D}_j) = 0$ and since $\mathcal{G} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_m$ and $\tilde{\gamma}_{ii}$ is nonzero on $\mathcal{M}$, there exists a smooth function $\beta \circ k$: $\mathcal{M} \to \mathcal{GL}(\mathbb{R}, m)$ such that

$$\begin{aligned} &g(\beta \circ k) = (W_1 \ \ldots \ W_m) \\ &\mathcal{D}_i = \operatorname{sp}\{W_i\} \\ &\nabla^i_X \overline{g(\beta \circ k)}^i = 0 \end{aligned} \tag{39}$$

for all $X \in \mathcal{E}_i$.

Finally, using similar arguments to those above (a detailed proof is omitted for lack of space), one proves the existence of $\alpha \circ k$: $\mathcal{M} \to \mathbb{R}^m$ such that

$$\nabla^i_X \overline{f + g(\alpha \circ k)}^i = 0 \tag{40}$$

for all $X \in \mathcal{E}_i$. In particular, if we decompose f as in (22) and write $f_{\mathcal{D}_i} = \omega_i W_i$, defining $\tilde{\omega}_i = \omega_i - \omega_i \circ \pi_{\mathcal{E}_i}$ and $\tilde{\alpha} = -(\beta \circ k)(\tilde{\omega}_1 \cdots \tilde{\omega}_m)^T$, one has

$$\begin{aligned} [X, f + g\tilde{\alpha}]_{\mathcal{D}_i} &= [X, f_{\mathcal{D}_i} - W_i \tilde{\omega}_i]_{\mathcal{D}_i} \\ &= [X, W_i(\omega_i \circ \pi_{\mathcal{E}_i})]_{\mathcal{D}_i} = 0 \end{aligned}$$

for all $X \in \mathcal{E}_i$. This implies (40) with $\tilde{\alpha} = \alpha \circ k$.

From (39) and (40) and by direct inspection in coordinates $x_1, \cdots, x_{m+1}$ of the closed-loop system $\dot{x} = f + g(\alpha \circ k) + g(\beta \circ k)v$, $y = h(x)$, with v being the new input vector, we conclude that the closed-loop system is noninteractive (see also [11, Proposition 3.3]). Moreover, since $\beta \circ k$: $\mathcal{M} \to \mathcal{GL}(m, \mathbb{R})$, the closed-loop system has also uniform vector relative degree on $\mathcal{M}$. •

We want to remark that also the *stability* issue can be discussed and results similar to those contained in [12, Ch. 4] can be derived.

We conclude with a simple example. Let us study the noninteracting control problem for the following system:

$$\begin{aligned} \dot{x}_1 &= -e^{x_1}x_3^2 + e^{x_1}u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 + x_2 + u_1 + u_2 \\ y_i &= x_i \qquad i = 1, 2, \\ z &= x_3 \end{aligned}$$

with $x = (x_1, x_2, x_3)^T$. We have $\mathcal{E}_1 = \mathrm{span}(\partial/\partial x_2), (\partial/\partial x_3)\}$, $\mathcal{E}_2 = \mathrm{span}\{(\partial/\partial x_1), (\partial/\partial x_3)\}$, $\ker dz \cap \mathcal{E}_1 = \mathrm{span}\{(\partial/\partial x_2)\}$ and $\ker dz \cap \mathcal{E}_2 = \mathrm{span}\{(\partial/\partial x_1)\}$. Assumptions A1)–A3) are trivially satisfied. Moreover, $\mathcal{D}_1 = \mathrm{span}\{e^{x_1}(\partial/\partial x_1) + (\partial/\partial x_3)\}$ and $\mathcal{D}_2 = \mathrm{span}\{(\partial/\partial x_2) + (\partial/\partial x_3)\}$. Moreover, $\widehat{\mathcal{D}}_i{}^i$ is trivial and since $\mathcal{M}/\mathcal{F}_i = \mathbb{R}$ we can choose $\mathcal{H}_i = 0$. Thus, Assumption A4) is satisfied. Finally, since $\mathcal{M}/\mathcal{K}_i = \mathbb{R}^2$ choose $\mathcal{O}_i = \{x_i \in \mathbb{R}\}$ so that also A5) is satisfied. If $\ker dz = \mathrm{span}\{(\partial/\partial x_1), (\partial/\partial x_2)\}$, it is easy to check that (12)–(15) hold true. Thus, Theorem 1 applies. Following the proof of Theorem 1, we can easily derive the decoupling controller. Indeed, we have $\pi_{\mathcal{E}_1}(x) = (x_1, 0, 0)^T$ and $\pi_{\mathcal{E}_2}(x) = (0, x_2, 0)^T$. Moreover, $\omega_1(x) = -x_3^2$, $\omega_1 \circ \pi_{\mathcal{E}_1}(x) = 0$, $\omega_2(x) = 0$ and $\omega_2 \circ \pi_{\mathcal{E}_2}(x) = 0$, so that $u_1 = -\omega_1(x) + \omega_1 \circ \pi_{\mathcal{E}_1}(x) + v_1 = x_3^2 + v_1$ and $u_2 = v_2$ is the desired decoupling control law.

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Optimal Average Cost Manufacturing Flow Controllers: Convexity and Differentiability

Michael H. Veatch and Michael C. Caramanis

***Abstract*— The authors consider the control of a production facility consisting of a single workstation with multiple failure modes and part types using a continuous flow control model. Technical issues concerning the convexity and differentiability of the differential cost function are investigated. It is proven that under an optimal control policy the differential cost is C^1 on attractive control switching boundaries.**

***Index Terms*— Average cost minimization, differentiability, manufacturing flow control, value function.**

I. INTRODUCTION

Manufacturing systems subject to discrete disturbances (failures, setup changes, and the like) have been studied extensively using a *fluid model* approximation, where surplus or backlog of production is represented by a continuous variable (see [5] for justification). The goal is to control production with a state feedback policy that minimizes the average cost of production surplus and backlog under a constant demand rate and stochastic production capacity. Little is known about the structure of the *optimal* policy for systems involving more than one part type; see Srivatsan and Dallery [12] Perkins and Srikant [8] and Veatch and Caramanis [14] for some recent exceptions. Instead, algorithms have been developed to compute a reasonable control policy using infinitesimal perturbation analysis or direct computation of average cost [2], [6], [7]. However, some of these algorithms rely on properties of the differential cost functions that have not been rigorously proven. Sethi *et al.* [10] prove the existence of the potential cost function that is closely related to the differential cost.

This paper investigates the continuity of the differential cost function's derivative on control switching surfaces, which are hypersurfaces in the state space that form the boundaries between state space regions characterized by a constant optimal control. We show that the differential cost is, at least in some cases, continuously differentiable, justifying the assumption made in some previous papers and supporting the quadratic approximation used in [2]. Convexity of the differential cost is also established.

II. THE FLOW CONTROL MODEL

We consider the flow control model of Liberopoulos and Caramanis [6], which generalizes the multiple unreliable machine model of [2]. The system state is $(x(t), \alpha(t))$, where $x = (x_1, \cdots, x_n)$, x_i is the continuous production surplus of part type i, and α is the discrete machine state. When $x_i(t) > 0$ there is a surplus and when $x_i(t) < 0$ there is a shortage and demand is backlogged. The machine state is governed by a continuous-time irreducible Markov chain on a finite state space $\mathcal{E}$. Let $Q = [q_{\alpha\beta}]$, $\alpha, \beta \in \mathcal{E}$ be the generator, i.e., $q_{\alpha\beta}$ is the transition rate from state α to state β and $q_{\alpha\alpha} = -\sum_{\beta \neq \alpha} q_{\alpha\beta}$.

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