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STABILIZATION OF NONLINEAR SYSTEMS WITH FILTERED LYAPUNOV FUNCTIONS AND FEEDBACK PASSIVATION

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ABSTRACT

In this paper we introduce a generalized class of filtered Lyapunov functions, which are Lyapunov functions with time-varying parameters satisfying certain differential equations. Filtered Lyapunov functions have the same stability properties as Lyapunov functions. We give tools for designing composite filtered Lyapunov functions for cascaded systems. These functions are used to design globally stabilizing dynamic feedback laws for block-feedforward systems with stabilizable linear approximation.

Key Words: Filtered Lyapunov functions, dynamic feedback, stabilization.

I. INTRODUCTION

Dissipativity ([6]) and Lyapunov-based small-gain theorems ([13], [10]) establish elegant methods for the stability analysis of interconnected systems but do not point out any constructive procedure for a Lyapunov function. The design of composite Lyapunov functions has been investigated in [11] and [9] in the case of triangular systems $\Sigma_1 : \dot{\mathbf{x}} = f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z}),\ \Sigma_2 : \dot{\mathbf{z}} = a(\mathbf{z})$. In [11] the composite Lyapunov function is the sum of suitable nonlinear scalings of the Lyapunov functions W_1 and W_2 of $\dot{\mathbf{x}} = f(\mathbf{x})$ and, respectively, $\dot{\mathbf{z}} = a(\mathbf{z})$, viz. $\alpha_1(W_1) + \alpha_2(W_2)$. A term $Mz + x^T Hz$ can be tolerated in $h(x, z)$ by using a Lyapunov function $\alpha_1(W_1 + \Psi) + \alpha_2(W_2)$ with "relaxed" cross term Ψ provided $h(x, z)$ has the form $h_0(z)x + h_1(z)$ ([11]). In [9] no assumption is made on $h(x, z)$ except for a linear growth with respect to x and the composite Lyapunov function is the sum of the Lyapunov functions of each system Σ_1 and Σ_2 plus a suitable cross term, viz. $W_1 + \Psi + W_2$. The definition of Ψ involves a line integral along the system trajectories, which can be calculated only in some specific cases. We remark that no iISS property is required for Σ_1. Recently in [8] the constructive aspects of a composite Lyapunov function have been investigated for non-triangular systems $\Sigma_1 : \dot{\mathbf{x}} = f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z}),\ \Sigma_2 : \dot{\mathbf{z}} = a(\mathbf{z}) + b(\mathbf{x}, \mathbf{z})$, where each single system is assumed to be iISS.

Recently some Lyapunov-like functions, called filtered Lyapunov functions, have been used in [1] (and previous ones). Filtered Lyapunov functions, like Lyapunov functions, are tools for ascertaining stability but, unlike Lyapunov functions, may depend on time-varying parameters which are the output of certain dynamical filters. The flexibility of filtered Lyapunov functions can be seen in the design of composite Lyapunov functions for interconnected systems $\Sigma_1 : \dot{\mathbf{x}} = f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z}),\ \ \Sigma_2 : \dot{\mathbf{z}} = a(\mathbf{z})$ consisting of a "filtered" combination of the Lyapunov functions W_1 and W_2 for Σ_1 and, respectively, Σ_2, viz. $p[d_1 W_1 + d_2 W_2]$ where $d_1, d_2 > 0$ and p is the output of a filter implemented by using the terms appearing in the time derivatives $\dot{W}_1$ and $\dot{W}_2$. More recently, dynamically scaled Lyapunov functions $W_1 + pW_2$ have been studied in [12] for the more restricted class of interconnections $\dot{\mathbf{x}} = h(\mathbf{z})$, $\dot{\mathbf{z}} = a(\mathbf{z})$. However, filtered Lyapunov function are more general than dynamically scaled Lyapunov functions and have the same stability properties as classical Lyapunov functions. Following the preliminary work [1], we want to give a twofold contribution in this paper. First, we define a general class of filtered Lyapunov functions and prove that these functions have the same stability properties as Lyapunov functions. The advantage of introducing filtered Lyapunov functions is that it is very easy to combine the filtered Lyapunov functions of two systems

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Σ_1 and Σ_2 and to obtain a new filtered Lyapunov function for the interconnection of Σ_1 and Σ_2. For example, if the filtered Lyapunov functions of Σ_1 and Σ_2 are quadratic also the filtered Lyapunov function for the interconnection of Σ_1 and Σ_2 is quadratic. We illustrate this in the case of triangular systems

$$\dot{\mathbf{x}} = f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z}, \mathbf{p}) \tag{1}$$
$$\dot{\mathbf{z}} = a(\mathbf{z}, \mathbf{p}) \tag{2}$$

with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ and $p \in \mathbb{R}$ is a time-varying parameter. For these triangular systems, under the same assumptions of [9] we show how to construct a filtered Lyapunov function by avoiding the calculation of a cross term along the system trajectories.

Next, we want to study the stabilization problem for systems (1)-(2) with controls, using the design of filtered Lyapunov functions. To this aim we consider

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z}, \mathbf{p}) + g(\mathbf{x}, \mathbf{z}, \mathbf{p}, \mathbf{v})\mathbf{v} \\ \dot{\mathbf{z}} &= a(\mathbf{z}, \mathbf{p}) + k(\mathbf{x}, \mathbf{z}, \mathbf{p}, \mathbf{v})\mathbf{v}\end{aligned} \tag{3}$$

with $v \in \mathbb{R}$ the control input. Our feedback stabilizing law is designed with the filtered Lyapunov function for (1)-(2) and according to the feedback passivation approach ([4]). By stability here we mean partial stability with respect to x and z (in a sense which will be specified later). Finally, we consider systems

$$\begin{aligned}&\dot{\mathbf{y}}_j = s_j(\mathbf{y}_j) + q_j(\mathbf{y}_j, \ldots, \mathbf{y}_s) + r_j(\mathbf{y}_j, \ldots, \mathbf{y}_s, \mathbf{u})\mathbf{u}, \\ &j = 1, \ldots, s,\end{aligned} \tag{4}$$

with $y_j \in \mathbb{R}^{d_j}$ and control input $u \in \mathbb{R}$. Our stabilization approach consists of iterating a certain number of steps on systems of the form (3). The novelty is that to obtain a stabilizing feedback for (4) it is not necessary to define a Lyapunov function at each step which involves a line integral along the system trajectories as suggested in [9] or the sum of nonlinear scalings of two Lyapunov functions.

II. NOTATION

- $\mathbb{R}^n$ is the set of n-dimensional column vectors of real numbers. $\mathbb{R}_+$ denotes the set of real non-negative numbers. $\mathbb{R}_>$ denotes the set of real positive numbers. $\mathbb{R}^{n\times m}$ denotes the set of real matrices $n \times m$. For any vector $x \in \mathbb{R}^n$ we denote by x_i the i–th element of x. For any matrix $G \in \mathbb{R}^{n\times m}$ we denote by $[G]_{ij}$ the (i, j)–th element of G and by G_i the i–th row of G.
- for any $\mathscr{R} \subset \mathbb{R}^n$, $\mathrm{clos}(\mathscr{R})$ denotes the closure of $\mathscr{R}$ in $\mathbb{R}^n$. We denote by $\mathbf{C}^0(\mathscr{X}, \mathscr{Y})$, $\mathscr{X} \subset \mathbb{R}^q$ and $\mathscr{Y} \subset \mathbb{R}^s$, the set of continuous functions $f : \mathscr{X} \to \mathscr{Y}$, $\mathbf{C}^j(\mathscr{X}, \mathscr{Y})$, $j = 1, \ldots, +\infty$, the set of j-times continuously differentiable functions $f : \mathscr{X} \to \mathscr{Y}$. Moreover, by $\mathbf{L}^1(\mathscr{X}, \mathscr{Y})$ the set of functions $f \in \mathbf{C}^0(\mathscr{X}, \mathscr{Y})$ such that $\int_{\mathscr{X}} \|f(s)\| ds < +\infty$ and by $\mathbf{L}^\infty(\mathscr{X}, \mathscr{Y})$ the set of functions $f \in \mathbf{C}^0(\mathscr{X}, \mathscr{Y})$ such that $\sup_{s\geqslant 0} \|f(s)\| < +\infty$.
- let $\dot{W}|_{\dot{\mathbf{x}}=f(\mathbf{x})}$ denote the time derivative of $W \in \mathbf{C}^1(\mathbb{R}^n, \mathbb{R})$ along the trajectories of $\dot{\mathbf{x}} = f(\mathbf{x})$, viz. $\dot{W}|_{\dot{\mathbf{x}}=f(\mathbf{x})} = \frac{\partial W}{\partial x}(\mathbf{x})f(\mathbf{x})$. We will also use bold letters $\mathbf{x}, \mathbf{y}, \mathbf{z} \ldots$ to denote trajectories and $x, y, z \ldots$ to denote their values. For any $f \in \mathbf{C}^0(\mathscr{X}, \mathscr{Y})$ we denote by $f|_{x=\tilde{x}}$ the function f evaluated for $x = \tilde{x}$.
- A function $\alpha \in \mathbf{C}^0([0, r), \mathbb{R}_+)$, $r \in (0, +\infty]$, is said to be of class $\mathscr{K}_+$ (or $\alpha \in \mathscr{K}_+$) if $\alpha(0) \geqslant 0$ and it is non-decreasing. A function $\alpha \in \mathbf{C}^0([0, r), \mathbb{R}_+)$, $r \in (0, +\infty]$, is said to be of class $\mathscr{K}$ (or $\alpha \in \mathscr{K}$) if $\alpha(0) = 0$ and it is strictly increasing; a function $\alpha \in \mathbf{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ is said to be of class $\mathscr{K}_\infty$ (or $\alpha \in \mathscr{K}_\infty$) if $\alpha \in \mathscr{K}$ and $\lim_{s\to+\infty} \alpha(s) = +\infty$. We say that a function $\alpha \in \mathscr{K}$ (resp. $\alpha \in \mathscr{K}_\infty$) is of class $\mathscr{K}_{\mathscr{O}(j)}$ or $\alpha \in \mathscr{K}_{\mathscr{O}(j)}$, $j = 1, 2$, (resp. of class $\mathscr{K}_{\infty,\mathscr{O}(j)}$ or $\alpha \in \mathscr{K}_{\infty,\mathscr{O}(j)}$) if there exist $a > 0$ such that $\alpha(s) = as^j$ for all $s \in [0, 1]$.

III. FILTERED LYAPUNOV FUNCTIONS

3.1. DEFINITIONS

In this section we introduce the definition of filtered Lyapunov functions and show that filtered Lyapunov functions and Lyapunov functions have similar properties. Let $\mathscr{P} := (0, \overline{p})$ and $f \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathrm{clos}(\mathscr{P}), \mathbb{R}^n)$ locally Lipschitz continuous with respect to the first argument uniformly on $\mathrm{clos}(\mathscr{P})$, that is for each compact set $\mathscr{R} \subset \mathbb{R}^n$ there exists a constant c such that $\|f(z_1, p) - f(z_2, p)\| \leqslant c\|z_1 - z_2\|$ for all $z_1, z_2 \in \mathscr{R}$ and $p \in \mathrm{clos}(\mathscr{P})$. Moreover, $f(0, p) = 0$ for each $p \in \mathscr{P}$.

Definition III.1 *We say that* $W \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathrm{clos}(\mathscr{P}), \mathbb{R}_+)$ *is a smooth filtered Lyapunov function (FLP) for*

$$\dot{\mathbf{z}} = f(\mathbf{z}, \mathbf{p}) \tag{5}$$

if

(i) *there exist* $k_2, k_1 > 0$ *and* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ *such that* $p^{k_1}\alpha_1(\|z\|) \leqslant W(z, p) \leqslant p^{k_2}\alpha_2(\|z\|)$ *for all* z *and* $p \in \mathscr{P}$,

(ii) *there exists a positive semi-definite* $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$, *locally Lipschitz at zero, such* *that for each* $z_0 \in \mathbb{R}^n$ *and* $p_0 \in \mathscr{P}$ *the trajectory* $(\mathbf{z}(t, z_0, p_0), \mathbf{p}(t, z_0, p_0))$ *of (5) with*

$$\dot{\mathbf{p}} = -\Gamma(\mathbf{z})\mathbf{p} \tag{6}$$

ensuing from (z_0, p_0) *at* $t = 0$ *is defined for all* $t \geqslant 0$ *and* $\Gamma(\mathbf{z}(\cdot, z_0, p_0)) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$*,*
(iii) *there exists* $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \mathrm{clos}(\mathscr{P}), \mathbb{R}_+)$ *such that* $\phi(\cdot, p)$ *is positive semi-definite for each* $p \in \mathscr{P}$ *and* $\dot{W}|_{(\mathbf{5})-(\mathbf{6})} \leqslant -\phi(\mathbf{z}, \mathbf{p})$*.*

If $\alpha_j \in \mathscr{K}_{\mathscr{O}(2)}$, $j = 1, 2$, *we say that* W *is locally quadratic at zero.* If W is quadratic for each $p \in \mathscr{P}$ *we say that* W *is quadratic.* If $\phi(\cdot, p)$ is positive definite for each $p \in \mathscr{P}$ *we say that* W *is strict. We say that* W *is strongly convex* if it is locally quadratic at zero and there exists $\mu > 0$ such that $\alpha_1(\|z\|) \geqslant \mu\|z\|^2$ for all z. Finally, if $\phi(\cdot, p)$ is for each $p \in \mathscr{P}$ locally quadratic at zero *we say that* W *has locally quadratic at zero stability margin.*

3.2. PROPERTIES, DIRECT AND CONVERSE THEOREMS

As a Lyapunov function, the existence of a FLP implies some stability properties for (5). In this context it is natural to refer to the stability of a system of the form

$$\dot{\mathbf{z}} = f(\mathbf{z}, \mathbf{p}),\ \dot{\mathbf{p}} = g(\mathbf{z}, \mathbf{p}) \tag{7}$$

with respect to the state z. Here $p \in \mathscr{P} \subset \mathbb{R}^p$. Similar definitions and results can be stated for $\dot{\mathbf{p}} = g(\mathbf{z}, \mathbf{p}, \mathbf{x})$ with some exogenous input $\mathbf{x}$. Let $\mathbf{w}(t, w_0) := (\mathbf{z}(t, w_0), \mathbf{p}(t, w_0))$ denote the trajectory of (7) ensuing from $w_0 := (z_0, p_0)$ at $t = 0$ (and defined over its maximal extension interval).

Definition III.2 *The equilibrium* $z = 0$ *of (7) (or simply (7)) is*
- stable with respect to z *if for each* $\varepsilon > 0$ *and* $p_0 \in \mathscr{P}$ *there exists* $\delta > 0$ *such that* $\|\mathbf{z}(t, w_0)\| < \varepsilon$ *for all* $t \geqslant 0$ *provided* $\|z_0\| < \delta$*; if, in addition, for each* $p_0 \in \mathscr{P}$ *there exists* $\eta > 0$ *such that* $\lim_{t\to+\infty} \mathbf{z}(t, w_0) = 0$ *provided* $\|z_0\| < \eta$ *we say that (7) is locally asymptotically stable with respect to* z*;*
- globally stable with respect to z *if for each* $p_0 \in \mathscr{P}$ *there exists* $\delta \in \mathscr{K}_\infty$ *such that* $\|\mathbf{z}(t, w_0)\| < \varepsilon$ *for all* $t \geqslant 0$ *and* $\varepsilon > 0$ *provided* $\|z_0\| < \delta(\varepsilon)$*; if, in addition,* $\lim_{t\to+\infty} \mathbf{z}(t, w_0) = 0$ *for all* z_0 *and* $p_0 \in \mathscr{P}$*, we say that (7) is globally asymptotically stable with respect to* z*.*
- locally exponentially stable with respect to z *if for each* $p_0 \in \mathscr{P}$ *there exist* $\eta, \lambda, \mu > 0$ *such that* $\|\mathbf{z}(t, w_0)\| \leqslant \mu \exp^{-\lambda t}$ *for all* $t \geqslant 0$ *provided* $\|z_0\| < \eta$*.*

The notion of *partial stability with respect to* z has been introduced in [5], def. 55.2, and requires that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{z}(t, w_0)\| < \varepsilon$ for all $t \geqslant 0$ provided $\|w_0\| < \delta$. This notion is stronger than ours in the sense that we do not require that p_0 be "small" enough. Our notion is closer to the notion of *partial stability with arbitrary initial* p*-perturbations* introduced in [15], which requires that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{z}(t, w_0)\| < \varepsilon$ for all $t \geqslant 0$ provided $\|z_0\| < \delta$ (see also [14]). Also this notion is stronger than ours in the sense that in our case $\delta > 0$ may depend on $p_0 \in \mathscr{P}$. On the other hand, the two notions are equivalent when $\mathscr{P}$ is compact. In what follows we give (without proof for lack of space) sufficient conditions for partial stability based on FLP's and instrumental for the design of feedback controllers in the next sections. It is worth noting that these conditions are weaker than those given in [5] and [15] since FLP's $W(z, p)$ are not bounded below by some $\mathcal{K}$-class function $\alpha(\|z\|)$ uniformly on $\mathscr{P}$.

Theorem III.1 *If there exists a smooth (resp. strict) filtered Lyapunov function* W *for (5) then there exists* $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ *such that (5)-(6) is globally (resp. asymptotically) stable with respect to* z*. If in addition* W *is locally quadratic at zero and has locally quadratic at zero stability margin, (5)-(6) is also locally exponentially stable with respect to* z*.*

Proof: First, we show that if W is a smooth filtered Lyapunov function for (5) then there exists $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ such that (5)-(6) is globally stable with respect to z. Fix $p_0 \in \mathscr{P}$ and let $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \mathrm{clos}(\mathscr{P}), \mathbb{R}_+)$, $k_2, k_1 > 0$, locally Lipschitz at zero $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ and $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ be as in definition III.1. The trajectories $\mathbf{z}(t, w_0)$ and $\mathbf{p}(t, w_0)$ of (5)-(6) are defined for all $t \geqslant 0$ and $\int_0^\infty \Gamma(\mathbf{z}(r, w_0))dr < +\infty$ and $p_0 \geqslant \mathbf{p}(t, w_0) = \exp\{-\int_0^\infty \Gamma(\mathbf{z}(r, w_0))dr\}p_0 > 0$ for all $t \geqslant 0$ and z_0. On the other hand, since $\dot{W}|_{(\mathbf{5})-(\mathbf{6})} \leqslant 0$,

$$\begin{aligned} &\mathbf{p}^{k_1}(t, w_0)\alpha_1(\|\mathbf{z}(t, w_0)\|) \leqslant W(\mathbf{z}(t, w_0), \mathbf{p}(t, w_0)) \\ &\leqslant W(z_0, p_0) \leqslant p_0^{k_2}\alpha_2(\|z_0\|) \end{aligned} \tag{8}$$

and therefore

$$\begin{aligned} &\|\mathbf{z}(t, z_0)\| \\ &\leqslant \alpha_1^{-1}\Big(p_0^{k_2-k_1}\alpha_2(\|z_0\|)\exp\{k_1\int_0^\infty \Gamma(\mathbf{z}(r, w_0))dr\}\Big) \end{aligned}$$

for all $t \geqslant 0$ and z_0. Moreover, reasoning as in the proof of theorem 1 of [9] we prove that $\Psi : w_0 \mapsto \Psi(w_0) :=$

$\exp\{\int_0^\infty \Gamma(\mathbf{z}(r, w_0))dr\}$ is in $\mathbf{C}^0(\mathbb{R}^n \times \mathscr{P}, \mathbb{R}_>)$. We conclude that there exists $\sigma \in \mathscr{K}_+$ such that

$$\begin{aligned}\|\mathbf{z}(t, w_0)\| &\leqslant \alpha_1^{-1}\left(p_0^{k_2-k_1}\alpha_2(\|z_0\|)\sigma^{k_1}(\|z_0\| + p_0)\right) \\ &:= \mu(\|z_0\|) \end{aligned} \tag{9}$$

for all $t \geqslant 0$ and z_0 with $\mu \in \mathscr{K}_\infty$ (define $\sigma(r) := \max_{\|w\| \leqslant r} \Psi(w)$). Therefore, $\delta := \mu^{-1} \in \mathscr{K}_\infty$ and $\|\mathbf{z}(t, w_0)\| < \varepsilon$ for all $\varepsilon > 0$ and $t \geqslant 0$ provided $\|z_0\| \leqslant \delta(\varepsilon)$, viz. (5)-(6) is globally stable with respect to z.

Next, we show that if W is also strict then (5)-(6) is globally asymptotically stable with respect to z. To this aim, fix $p_0 \in \mathscr{P}$. It is sufficient to show that $\lim_{t\to+\infty} \mathbf{z}(t, w_0) = 0$ for each z_0. We have already established that $\mathbf{p}(\cdot, w_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}_+)$ and $\mathbf{z}(\cdot, w_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for each z_0. Moreover, $\mathbf{p}(t, w_0) \downarrow \overline{p}_0(w_0) := p_0 \exp\{-\int_0^{+\infty} \Gamma(\mathbf{z}(r, w_0))dr\} > 0$ for each z_0. Since $\dot{W}|_{(\mathbf{5})-(\mathbf{6})} \leqslant 0$ and W is strict, by LaSalle's invariance principle we conclude that $\lim_{t\to+\infty} \mathbf{z}(t, w_0) = 0$ for each z_0.

Finally, if W is also locally quadratic at zero and has locally quadratic at zero stability margin, we show that (5)-(6) is also locally exponentially stable with respect to z. Fix $p_0 \in \mathscr{P}$. Since $\overline{p}_0 : w_0 \mapsto \overline{p}_0(w_0)$ is in $\mathbf{C}^0(\mathbb{R}^n \times \mathscr{P}, \mathbb{R}_+)$ and $\overline{p}_0|_{z_0=0} = p_0$ since $\mathbf{z} = 0$ is a trajectory of (5)-(6), there exists $c, d > 0$ such that $\overline{p}_0(w_0) \geqslant d > 0$ as long as $\|z_0\| \leqslant c$. Since $\phi(\cdot, p)$ is for each $p \in \mathscr{P}$ locally quadratic at zero, we can find $\lambda > 0$ such that $\phi(z, p) \geqslant \lambda\|z\|^2$ for all $z : \|z\| \leqslant c'$ (possibly smaller than c) and $p \in [d, \overline{p}]$. Moreover, since W is locally quadratic at zero, there also exists $h > 0$ such that $\phi(z, p) \geqslant \frac{\lambda W(z,p)}{\overline{p}^{k_2} h}$ for all $z : \|z\| \leqslant c''$ (possibly smaller than c') and $p \in [d, \overline{p}]$. By integration of $\dot{W}|_{(\mathbf{5})-(\mathbf{6})} \leqslant -\phi(\mathbf{z}, \mathbf{p})$ over $[0, t]$ and since $\mathbf{p}(t, w_0) \in [d, \overline{p}]$ for all $t \geqslant 0$ and $z_0 : \|z_0\| \leqslant c''$, $W(\mathbf{z}(t, w_0), \mathbf{p}(t, w_0)) \leqslant \exp\{-\frac{\lambda t}{\overline{p}^{k_2} h}\} W(z_0, p_0)$ provided $\|\mathbf{z}(t, w_0)\| \leqslant c''$. On account of the fact that W is locally quadratic at zero, this implies that (5)-(6) is also locally exponentially stable with respect to z. □

Converse theorems for strict Lyapunov functions can be obviously used for strict FLP's, since any strict Lyapunov function (in the classical sense) is also a strict FLP. However, strict FLP's design is much easier.

Theorem III.2 *Let $\dot{\mathbf{z}} = f(\mathbf{z})$, locally Lipschitz $f \in \mathbf{C}^\infty(\mathbb{R}^m, \mathbb{R}^m)$ with $f(0) = 0$, be a globally asymptotically stable system. If there exist a positive definite and radially unbounded $V \in \mathbf{C}^\infty(\mathbb{R}^m, \mathbb{R}_+)$, positive definite $\phi \in \mathbf{C}^0(\mathbb{R}^m, \mathbb{R}_+)$, $f_0 \in \mathbf{C}^0(\mathbb{R}^m, \mathbb{R}^m)$ and $\gamma \in \mathscr{K}_{\infty, \mathscr{O}(1)}$ such that*
(i) $\frac{\partial V}{\partial z}(z) f_0(z) \leqslant -\phi(z)$ *for all* z,
(ii) $\frac{1}{V(z)}|\frac{\partial V}{\partial z}(z)[f(z) - f_0(z)]| \leqslant \gamma(\|z\|)$ *for all* z.
then V is a smooth strict FLP for $\dot{\mathbf{z}} = f(\mathbf{z})$.

Proof: Note that $\frac{\partial V}{\partial z}(z) f(z) \leqslant -\phi(z) + |\frac{\partial V}{\partial z}(z)[f(z) - f_0(z)]|$. Define $\dot{\mathbf{p}} = -\frac{1}{V(z)}|\frac{\partial V}{\partial z}(z)[f(z) - f_0(z)]|\mathbf{p}$ and $W(x, p) = pV(z)$. W is a strict FLP for $\dot{\mathbf{z}} = f(\mathbf{z})$, since $\frac{\partial W}{\partial z}(z, p) f(z) - \frac{\partial W}{\partial p}(z, p)\frac{1}{V(z)}|\frac{\partial V}{\partial z}(z)[f(z) - f_0(z)]|\mathbf{p} \leqslant -p\phi(z)$ for all z and $p \in [0, 1]$ and $\Gamma(\mathbf{z}) := \frac{1}{V(\mathbf{z})}|\frac{\partial V}{\partial z}(\mathbf{z})[f(\mathbf{z}) - f_0(\mathbf{z})]|$ is integrable along the trajectories of $\dot{\mathbf{z}} = f(\mathbf{z})$ on account of its global asymptotic and local exponential stability and on account of (**ii**) and $\gamma \in \mathscr{K}_{\infty, \mathscr{O}(1)}$.□

An important consequence of theorem III.2 is that any globally asymptotically and locally exponentially stable system admits a strict quadratic FLP.

Corollary III.1 *Let $\dot{\mathbf{z}} = f(\mathbf{z})$, locally Lipschitz $f \in \mathbf{C}^\infty(\mathbb{R}^m, \mathbb{R}^m)$ with $f(0) = 0$, be a globally asymptotically and locally exponentially stable system. If $V(z) = (1/2)z^T P z$ is a strict Lyapunov function for the linear approximation $\dot{\mathbf{z}} = \frac{\partial f}{\partial z}|_{z=0}\mathbf{z}$ of $\dot{\mathbf{z}} = f(\mathbf{z})$ around the origin, then $W(x, p) = pV(z)$ is a strict quadratic FLP for $\dot{\mathbf{z}} = f(\mathbf{z})$.*

Proof: On account of local exponential stability of $\dot{\mathbf{z}} = f(\mathbf{z})$, $\frac{\partial f}{\partial z}|_{z=0}$ is Hurwitz. Let $P \in \mathbb{R}^{m\times m}$ be the unique symmetric positive definite matrix $P\frac{\partial f}{\partial z}|_{z=0} + [\frac{\partial f}{\partial z}|_{z=0}]^T P = -I$. Note that $\frac{\partial V}{\partial z}(z) f(z) \leqslant -\|z\|^2 + |\frac{\partial V}{\partial z}(z)[f(z) - \frac{\partial f}{\partial z}|_{z=0} z]|$. There clearly exists $\gamma \in \mathscr{K}_{\infty, \mathscr{O}(1)}$ such that $\frac{1}{V(z)}|\frac{\partial V}{\partial z}(z)[f(z) - f_0(z)] \leqslant \gamma(\|z\|)$ for all z. Application of theorem III.2 with $f_0(z) := \frac{\partial f}{\partial z}|_{z=0} z$ gives the claimed result.□

Corollary III.1 (as well as theorem III.2) can be used to easily construct FLP's.

Example III.1 Consider

$$\dot{\mathbf{z}}_1 = \mathbf{z}_2,\ \dot{\mathbf{z}}_2 = -h(\mathbf{z}_1) - k(\mathbf{z}_2) \tag{10}$$

with locally Lipschitz h, k such that $h(0) = 0$, $k(0) = 0$, $\frac{\partial h}{\partial s}|_{s=0} := H > 0$, $\frac{\partial k}{\partial s}|_{s=0} := K > 0$, $h(s)s > 0$ and $k(s)s > 0$ for all s (many physical systems have the simple model (10): see [5]). The Lyapunov energy function $V(z) = (1/2)z_2^2 + \int_0^{z_1} h(s)ds$ is such that $\dot{V}|_{(\mathbf{10})} = -\mathbf{z}_2 k(\mathbf{z}_2)$ and from application of LaSalle's principle we prove global asymptotic stability of (10). Local exponential stability of (10) follows from the linear approximation of (10) around the origin being

Hurwitz. The linear approximation of (10) at the origin has a strict quadratic Lyapunov function $V(z) = (1/2)\begin{pmatrix} z_1^T & z_2^T \end{pmatrix}\begin{pmatrix} H+\frac{aK}{H} & 1 \\ 1 & \frac{a}{H} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, with $a > 0 : \frac{a}{H}(H+\frac{aK}{H}) > 1$, and by corollary III.1 $W(z,p) = pV(z)$ is a strict quadratic FLP for (10) with $\dot{\mathbf{p}} := -\mathbf{p}\frac{1}{V(\mathbf{z})}|\frac{\partial V}{\partial z}(\mathbf{z})[f(\mathbf{z}) - \begin{pmatrix} 0 & 1 \\ -H & -K \end{pmatrix}\mathbf{z}]|$.

Example III.2 Consider the model of a flexible joint robot with states $z_1 = q_l - y_d$, $z_2 = \dot{q}_l$, $z_3 = q_m - y_d - K^{-1}h(y_d)$ and $z_4 = q_m$ ($q_l, \dot{q}_l$ link coordinates, $q_m, \dot{q}_m$ motor coordinates, h the gravity term and $(y_d, y_d + K^{-1}h(y_d))$ the desired set point for (q_l, q_m))

$$\begin{aligned} &\dot{\mathbf{z}}_1 = \mathbf{z}_2,\ \dot{\mathbf{z}}_2 = -B_1^{-1}(\mathbf{z}_1 + y_d)[C(\mathbf{z}_1 + y_d, \mathbf{z}_2)\mathbf{z}_2 \\ &+K(\mathbf{z}_3 - \mathbf{z}_3) + h(\mathbf{z}_1 + y_d) - h(y_d)] \\ &\dot{\mathbf{z}}_3 = \mathbf{z}_4,\ \dot{\mathbf{z}}_3 = B_2^{-1}[K\mathbf{z}_3 - h(y_d) + \mathbf{v}] \end{aligned} \tag{11}$$

If only measurements of $\mathbf{z}_1$ are available, as shown in [3] a linear dynamic output feedback controller

$$\mathbf{v} = K\hat{\mathbf{z}} + h(y_d),\ \dot{\hat{\mathbf{z}}} = \mathbf{L}\hat{\mathbf{z}} + \mathbf{M}\hat{\mathbf{z}}_{\mathbf{1}} \tag{12}$$

globally asymptotically and locally exponentially stabilizes (11). Global stability follows from a non-strict Lyapunov energy function $V(z,\hat{x})$ for (11)-(12). Asymptotic stability of (11)-(12) follows from LaSalle's principle and local exponential stability from the linear approximation of (11)-(12) around the origin being Hurwitz. The linear approximation of (11)-(12) at the origin has a strict quadratic Lyapunov function $(1/2)V(z,\hat{x})$ and by corollary III.1 $W(z,\hat{x},p) = pV(z,\hat{x})$ is a strict quadratic FLP for (11)-(12).

Another important fact is that any FLP's (as well as Lyapunov function) which is locally quadratic at zero with locally quadratic at zero stability margin can be re-designed in such a way to be strongly convex.

Theorem III.3 *If there exists a smooth strict filtered Lyapunov function W for (5), locally quadratic at zero with locally quadratic at zero stability margin, there also exists a smooth strongly convex filtered Lyapunov function $\overline{W}$ for (5), locally quadratic at zero with locally quadratic at zero stability margin. If W is such that $\frac{\partial W}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$, also $\overline{W}$ is such that $\frac{\partial \overline{W}}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$.*

Proof: Let W be a smooth strict filtered Lyapunov function for (5) with locally quadratic at zero stability margin. Let $k_2, k_1 > 0$, $\alpha_1, \alpha_2 \in \mathscr{K}_{\infty,\mathscr{O}(2)}$, $j = 1, 2,$, $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ and $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \mathrm{clos}(\mathscr{P}), \mathbb{R}_+)$ as in definition III.1. Let $d > 0$ be such that $\alpha_1(s) = ds^2$ for all $s \in [0,1]$ (which always exists since $\alpha_1 \in \mathscr{K}_{\infty,\mathscr{O}(2)}$). Define $\overline{W}(z,p) = \gamma(\|z\|)W(z,p)$ with smooth $\gamma \in \mathscr{K}_+$ such that $\gamma(s) = 1$ for all $s \in [0,1]$ and $\gamma(s)\alpha_1(s) \geqslant s^2$ for all $s \geqslant 0$. By construction,

$$p^{k_1} d\|z\|^2 \leqslant \overline{W}(z,p) \leqslant p^{k_2}\gamma(\|z\|)\alpha_2(\|z\|)) \tag{13}$$

for all z and $p \in \mathscr{P}$. Moreover, by defining

$$\dot{\mathbf{p}} = -\Big[\Gamma(\mathbf{z}) + \overline{p}^{k_2}\|\frac{\partial}{\partial z}\gamma(\|z\|)\|\alpha_2(\|z\|)\Big]\mathbf{p} \tag{14}$$

we have $\dot{\overline{W}}|_{(\mathbf{5})-(\mathbf{14})} \leqslant -\mathbf{p}\gamma(\|\mathbf{z}\|)\phi(\mathbf{z},\mathbf{p})$. Therefore, $\overline{W}$ is a smooth strongly convex filtered Lyapunov function for (5). Since $\gamma(s) = 1$ for all $s \in [0,1]$, $\overline{W}$ is strict and has locally quadratic at zero stability margin. Moreover, if $\frac{\partial W}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$, also $\frac{\partial \overline{W}}{\partial p}(z,p) = \gamma(\|z\|)\frac{\partial W}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$. □

We conclude this section with a general tool for designing simple FLP'sfor a wide class of systems, which can be used in designing composite filtered Laypunov functions for interconnected systems. This result can be proved along the same lines of the proof of theorem 1.1 of [2].

Theorem III.4 *Let*

$$\dot{\mathbf{z}} = J\mathbf{z} + B\mathbf{v} + f(\mathbf{z},\mathbf{v}) \tag{15}$$

be a given system with $z \in \mathbb{R}^m$ and control $v \in \mathbb{R}$, (J,B) a Brunowski pair. Assume the existence of $\psi \in \mathbf{C}^1(\mathbb{R}^m, [1,+\infty))$ and $\Phi_j \in \mathbf{C}^0(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}_+)$, $j = 1,\ldots,m$, and $\beta_{ji} \in \mathbf{C}^0(\mathbb{R}^{m-j+2}, \mathbb{R}_+)$, $j = 1,\ldots,m$ and $i = j+1,\ldots,m+1$, such that
(i) *$\|\frac{\partial\psi}{\partial z}(z)[Jz + B\frac{z_{m+1}}{\psi(z)} + f(z, \frac{z_{m+1}}{\psi(z)})]\| \leqslant K\psi(z)$ for all z and $z_{m+1} : |z_{m+1}| \leqslant 1$ and for some $K > 0$,*
(ii) *$f_j(z, \frac{z_{m+1}}{\psi(z)})| \leqslant \Phi_j(z, z_{m+1})$ and*

$$\begin{aligned} &[\Phi_j - \Phi_j|_{z_i=\overline{z}_i}]^2 \leqslant [z_i - \overline{z}_i]^2\beta_{ji}(\overline{z}_i, z_{j+1},\ldots,z_{m+1}), \\ &\beta_{ji}(0,0,\ldots,0) = 0,\ \Phi_j|_{z_i=0,i=j+1,\ldots,m+1} = 0 \end{aligned} \tag{16}$$

for $j = 1,\ldots,m$, $i = j+1,\ldots,m+1$ and for all $z, \overline{z} \in \mathbb{R}^m$ and $z_{m+1} \in \mathbb{R}$.

There exist $R_1,\ldots,R_m \geqslant 1$ such that the nested feedback controller

$$\begin{aligned} &v = -\sigma_m(z_m - \tilde{z}_m(z_1,\ldots,z_{m-1})), \\ &\tilde{z}_{j+1}(z_1,\ldots,z_j) := \sigma_j(z_j - \tilde{z}_j(z_1,\ldots,z_{j-1})), \\ &j = 1,\ldots,m-1,\ \tilde{z}_1 := 0, \\ &\sigma_j(s) := \frac{1}{2R_j\psi(z)}\frac{s}{\sqrt{1+s^2}}, \end{aligned} \tag{17}$$

renders the origin $z = 0$ of (15) globally asymptotically stable and locally exponentially stable. Moreover, there exist $d_1, \ldots, d_m > 0$ such that

$$W(z,p) = p \sum_{j=1}^{m} d_j W_j(z_j - \tilde{z}_j(z_1, \ldots, z_{j-1})),$$
$$W_j(s) := \sqrt{1+s^2} - 1,$$

is a strict FLP for (15)-(17) locally quadratic at zero with locally quadratic at zero stability margin.

Example III.3 Consider the first three states of the ball and beam system without friction

$$\dot{\mathbf{z}}_1 = \mathbf{z}_2,\ \dot{\mathbf{z}}_2 = \sin(\mathbf{z}_3) + \mathbf{z}_1\mathbf{v}^2,\ \dot{\mathbf{z}}_3 = \mathbf{v} \tag{18}$$

where z_1 is the position of the ball, z_2 its velocity, z_3 is the angle of the beam and $\mathbf{v}$ its velocity. The forwarding approach of [11] and [9] for designing Lyapunov fuctions cannot be applied to (18) for the absence of the friction term $-\beta\mathbf{z}_2$ in $\dot{\mathbf{z}}_2$. On the other hand, (18) has the form (15) with $f(z,v) = (z_2, \sin z_3 - z_3 + z_1 v^2, 0)^T$ and satisfies (**i**)-(**iii**) with $\psi(z) = \sqrt{1+\|z\|^2}$ and $K = 3$.

IV. FILTERED LYAPUNOV FUNCTIONS FOR CASCADED SYSTEMS

Before focusing on the stabilization problem of (4), consider the problem of constructing filtered Lyapunov functions for the cascaded system (1)-(2) with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and $p \in \mathscr{P} := (0, \overline{p})$, $f(0) = 0$, $h(x, 0, p) = 0$, $a(0,p) = 0$ for all x and $p \in \operatorname{clos}(\mathscr{P})$, $\frac{\partial f}{\partial x}|_{x=0} := F$, $\frac{\partial h}{\partial z}|_{(x,z)=(0,0)} := H(p)$, $\frac{\partial a}{\partial z}|_{z=0} := A(p)$ for all $p \in \operatorname{clos}(\mathscr{P})$, with rational $A \in \mathbf{C}^\infty(\operatorname{clos}(\mathscr{P}), \mathbb{R}^{m\times m})$ and $H \in \mathbf{C}^\infty(\operatorname{clos}(\mathscr{P}), \mathbb{R}^{n\times m})$ (as it will be seen in section VII A and H being rational is not a restriction, since it comes naturally from the step-by-step stabilization design for (4) and the structure of the filtered Lyapunov functions). Moreover, $F \in \mathbb{R}^{n\times n}$, $a \in \mathbf{C}^\infty(\mathbb{R}^m \times \operatorname{clos}(\mathscr{P}), \mathbb{R}^m)$ and $h \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \operatorname{clos}(\mathscr{P}), \mathbb{R}^n)$. Without loss of generality, $\overline{p} \in (0,1]$ (otherwise consider $p' = p/\overline{p}$ as the new parameter). Assume that

(**A1**) *there exists a symmetric positive definite $W_0 \in \mathbb{R}^{n\times n}$ such that $W_0F + F^TW_0 \leqslant 0$ and a partition of $x = (x_1^T, x_2^T)^T$ such that (1)-(2) is globally asymptotically and locally exponentially stable with respect to x_2,*

(**A2**) *there exist $\gamma_2 \in \mathscr{K}_{\mathscr{O}(1)}$ and $\gamma_0, \gamma_1, \gamma_3 \in \mathscr{K}_{\mathscr{O}(2)}$ such that*

$$\begin{aligned}
&\|f(x) - Fx\| \leqslant \gamma_0(\|x_2\|),\\
&\|h(x,z,p) - H(p)z\| \leqslant \gamma_1(\|z\|) + \gamma_2(\|z\|)\|x\|,\\
&\|a(z,p) - A(p)z\| \leqslant \gamma_3(\|z\|)
\end{aligned} \tag{19}$$

for all x, z and $p \in \operatorname{clos}(\mathscr{P})$,

(**A3**) *there exists a smooth strict FLP V for (2) locally quadratic at zero with locally quadratic at zero stability margin and such that $\frac{\partial V}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$.*

Assumptions (**A1**) and (**A3**) are, to some extent, a minimal set of conditions for guaranteeing boundedness of $\mathbf{x}(t, w_0)$ and local exponential and global convergence to zero of $\mathbf{x}_2(t, w_0)$ and $\mathbf{z}(t, w_0)$. Moreover, (**A2**) together with the assumptions on h and a require that $\dot{\mathbf{x}} = F\mathbf{x} + H(p)\mathbf{z}$, $\dot{\mathbf{z}} = A(p)\mathbf{z}$ is the linear approximation of (1)-(2) around $(x,z) = (0,0)$ for fixed $p \in \mathscr{P}$.

On account of (**A3**) and by theorem III.3 V can be assumed strongly convex without loss of generality and we will do this throughout this section. The requirement $\frac{\partial V}{\partial p}(z,p) \geqslant 0$ for all z and $p \in \mathscr{P}$ is natural for a FLP and can be always satisfied as it will be seen in the recursive procedure based on sections VI and VII.

Under similar assumptions a composite Lyapunov function $U(x,z) := W(x) + \Psi(x,z) + V(z)$ for $\dot{\mathbf{x}} = f(\mathbf{x}) + h(\mathbf{x}, \mathbf{z})$, $\dot{\mathbf{z}} = a(\mathbf{z})$ has been defined in theorem 1 of [9] with Ψ calculated along the system trajectories. We will show how it is possible to avoid the calculation of a cross term Ψ along the state trajectories and obtain a FLP which is a simple combination of FLP's for the subsystems of the cascade. The main result of this section is the following.

Theorem IV.1 *Under assumptions (**A2**)-(**A3**), there exist $\Psi_0 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times m})$, $\Psi_1 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times n})$, $c \geqslant 1$ and integers j_0, k_1 such that*

$$\begin{aligned}
U(x,z,p) &:= p^{k_1+j_0+1}[x^TW_0x\\
&+z^T\Psi_0(p)z + z^T\Psi_1(p)x + \frac{c}{p^{k_1+j_0}}V(z,p)]
\end{aligned} \tag{20}$$

is a smooth strongly convex FLP for (1)-(2).

Proof: By (**A3**) there exist $k_2, k_1, \alpha_1 > 0$ and $\alpha_2 \in \mathscr{K}_{\infty,\mathscr{O}(2)}$ such that

$$\alpha_1 p^{k_1}\|z\|^2 \leqslant V(z,p) \leqslant p^{k_2}\alpha_2(\|z\|) \tag{21}$$

for all z and $p \in \mathscr{P}$. Moreover, there exist a positive semi-definite $\Gamma \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}_+)$, locally Lipschitz at zero, and $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \operatorname{clos}(\mathscr{P}), \mathbb{R}_+)$, with $\phi(\cdot, p)$

positive definite for each $p \in \mathscr{P}$, such that for each z_0 and $p_0 \in \mathscr{P}$ the trajectory $\mathbf{w}(t, w_0) := (\mathbf{z}(t, w_0), \mathbf{p}(t, w_0))$ of (2)-(6) ensuing from $w_0 := (z_0, p_0)$ at $t = 0$ is defined for all $t \geqslant 0$ with $\Gamma(\mathbf{z}(\cdot, w_0)) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\dot{V}|_{(\mathbf{2})-(\mathbf{6})} \leqslant -\phi(\mathbf{z}, \mathbf{p})$. Moreover, let $W(x) := x^T W_0 x$ and pick $b, \beta > 0$ such that

$$b\|x\|^2 \leqslant W(x) \leqslant \beta\|x\|^2 \tag{22}$$

for all x (say, $b :=$minimal eigenvalue of W_0 and $\beta :=$maximal eigenvalue of W_0). We need the following key facts.

Fact IV.1 *$P(p)A(p) + A^T(p)P(p) < 0$ for each $p \in \mathscr{P}$, with $P(p) := \frac{1}{2}\frac{\partial^2 V}{\partial z^2}|_{z=0} > 0$. Therefore, $A(p)$ has its eigenvalues in the left-half complex plane for each $p \in \mathscr{P}$.*

The proof is a direct consequence of $\dot{V}|_{(\mathbf{2})-(\mathbf{6})} \leqslant -\phi(\mathbf{z}, \mathbf{p})$ by considering the quadratic approximation of V and $\dot{V}$.

Fact IV.2 *There exist $\tau_1, \tau_2, \tau_3 \in \mathscr{K}_{\mathscr{O}(1)}$ and rational $M \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}), \mathbb{R}^{n\times m})$ such that for all z and $p \in \mathrm{clos}(\mathscr{P})$*

$$\begin{aligned}&2x^T W_0[f(x) + h(x, z, p)] \leqslant x^T M(p)z\\&+\tau_1(\|z\|)\|z\|^2 + [\tau_2(\|z\|) + \tau_3(\|x_2\|)]W(x).\end{aligned} \tag{23}$$

Indeed, by virtue of (**A1**) and (**A2**), for all x, z and $p \in \mathscr{P}$

$$\begin{aligned}&2x^T W_0[f(x) + h(x, z, p)] \leqslant 2x^T W_0 H(p)z\\&+2\|W_0 x\|[\gamma_0(\|x_2\|) + \gamma_1(\|z\|) + \gamma_2(\|z\|)\|x\|]\\&\leqslant \frac{\|W_0\|W(x)}{b}[\frac{\gamma_0(\|x_2\|)}{\|x_2\|} + \gamma_2(\|z\|) + \|z\|] + \frac{\|W_0\|\gamma_1^2(\|z\|)}{\|z\|}\end{aligned}$$

Therefore, set $M(p) := 2W_0 H(p)$, $\tau_1(s) := \|W_0\|\frac{\gamma_1^2(s)}{s^3}$, $\tau_2(s) := \frac{\|W_0\|}{b}[\gamma_2(s) + s]$ and $\tau_3(s) := \frac{\|W_0\|}{b}\frac{\gamma_0(s)}{s}$.

Fact IV.3 *There exist $\Psi_0 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times m})$, $\Psi_0(p)$ symmetric and positive definite for each $p \in \mathscr{P}$, and $\Psi_1 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times n})$ such that for all $p \in \mathscr{P}$*

$$\begin{aligned}&\Psi_0(p)A(p) + A^T(p)\Psi_0(p)\\&\leqslant -\frac{1}{2}[\Psi_1(p)H(p) + H^T(p)\Psi_1^T(p)],\\&\Psi_1(p)F + A^T(p)\Psi_1(p) = -M^T(p).\end{aligned} \tag{24}$$

There also exist $\bar{p}^* \in \mathscr{P}$ with $\mathscr{P}^* := (0, \bar{p}^*)$, integers $j_1 \geqslant 0$, $j_0 \geqslant 2j_1$, $\tilde{\Psi}_0 \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}^*), \mathbb{R}^{m\times m})$ with $\tilde{\Psi}_0(p)$ symmetric and positive definite for each $p \in \mathrm{clos}(\mathscr{P}^*)$, $\tilde{\Psi}_1 \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}^*), \mathbb{R}^{m\times n})$ and symmetric and positive semi-definite $A_2, B_2 \in \mathbb{R}^{m\times m}$ such that $\Psi_1(p) = \frac{1}{p^{j_1}}\tilde{\Psi}_1(p)$ and $\Psi_0(p) = \frac{1}{p^{j_0}}\tilde{\Psi}_0(p)$ for all $p \in \mathscr{P}^*$ and $\tilde{\Psi}_1(p)\tilde{\Psi}_1^T(p) \leqslant A_2$, $\tilde{\Psi}_0(p) \leqslant B_2$, $\forall p \in \mathrm{clos}(\mathscr{P}^*)$.

The proof is as follows. Let $\Psi_0 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times m})$, $\Psi_0(p)$ symmetric and positive definite for each $p \in \mathscr{P}$, and $\Psi_1 \in \mathbf{C}^\infty(\mathscr{P}, \mathbb{R}^{m\times n})$ be such that (24) are satisfied with equalities. For each $p \in \mathscr{P}$ the matrix equations (24) (with equalities) admit a unique solution $(\Psi_1(p), \Psi_0(p))$ since $\sigma(A(p)) \cap \sigma(-F) = \{\varnothing\}$ on account of fact #1. Moreover, $\Psi_1(p)$ and $\Psi_0(p)$ are rational matrices, therefore there exists $\bar{p}^* \in \mathscr{P}$ with $\mathscr{P}^* := (0, \bar{p}^*)$, integers $j_1 \geqslant 0$, $j_0 \geqslant 2j_1$, $\tilde{\Psi}_0 \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}^*), \mathbb{R}^{m\times m})$ with $\tilde{\Psi}_0(p)$ symmetric and positive definite for each $p \in \mathrm{clos}(\mathscr{P}^*)$, $\tilde{\Psi}_1 \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}^*), \mathbb{R}^{m\times n})$ and symmetric and positive semi-definite $A_2, B_2 \in \mathbb{R}^{m\times m}$ such that $\Psi_1(p) = \frac{1}{p^{j_1}}\tilde{\Psi}_1(p)$ and $\Psi_0(p) = \frac{1}{p^{j_0}}\tilde{\Psi}_0(p)$ and (24) holds true for all $p \in \mathscr{P}^*$. In other words Ψ_0 and Ψ_1 have at least j_0 and, respectively, j_1 poles in $p = 0$.

Fact IV.4 *There exists $\tilde{\tau} \in \mathscr{K}_{\mathscr{O}(1)}$ such that for all x, z and $p \in \mathscr{P}^*$*

$$\begin{aligned}&\tau_1(\|z\|)\|z\|^2 + (\tau_2(\|z\|) + \tau_3(\|x_2\|))W(x)\\&+\|z^T\Psi_1(p)[h(x, z, p) - H(p)z]\\&+(2z^T\Psi_0(p) + x^T\Psi_1^T(p))(a(z, p) - A(p)z)\|\\&\leqslant (\tilde{\tau}(\|z\|) + \tau_3(\|x_2\|))[\frac{1}{p^{k_1+j_0}}V(z, p) + W(x)].\end{aligned} \tag{25}$$

Indeed, by virtue of (**A2**), (21) and (22) since $j_0 \geqslant j_1$, for all x, z and $p \in \mathscr{P}^*$

$$\begin{aligned}&\|z^T\Psi_1(p)[h(x, z, p) - H(p)z]\|\\&+\|(2z^T\Psi_0(p) + x^T\Psi_1^T(p))[a(z, p) - A(p)z]\|\\&\leqslant \frac{1}{p^{j_0}}\Big[\|z\|\Big(\gamma_1(\|z\|) + \gamma_2^2(\|z\|) + \gamma_3(\|z\|) + \frac{\gamma_3(\|z\|)}{\|z\|^2}\Big)\\&+2\|x\|^2\|z\| \leqslant \Big[\frac{1}{p^{k_1+j_0}}V(z, p) + W(x)\Big]\Big[\Big(\frac{\gamma_1(\|z\|)}{\|z\|}\\&+\frac{\gamma_2^2(\|z\|)}{\|z\|} + \frac{\gamma_3(\|z\|)}{\|z\|} + \frac{\gamma_3^2(\|z\|)}{\|z\|^3}\Big)\frac{1}{\alpha_1} + \frac{2}{b}\|z\|\Big]\end{aligned} \tag{26}$$

Therefore, set $\tilde{\tau}(s) := \Big(\frac{\gamma_1(s)}{s} + \frac{\gamma_2^2(s)}{s} + \frac{\gamma_3(s)}{s} + \frac{\gamma_3^2(s)}{s^3} + \tau_1(s)\Big)\frac{1}{\alpha_1} + \frac{2s}{b} + \tau_2(s)$ which is in $\mathscr{K}_{\mathscr{O}(1)}$ on account of (**A2**).

For the rest of the proof of theorem IV.1, let $\mathbf{w}(t, x_0, w_0) := (\mathbf{z}(t, x_0, w_0), \mathbf{p}(t, x_0, w_0))$ denote the trajectory of (2) with

$$\dot{\mathbf{p}} = -\Lambda(\mathbf{x}_2, \mathbf{z})\mathbf{p}, \quad \Lambda(x_2, z) := \Gamma(z) + 2(\tilde{\tau}(\|z\|) + \tau_2(\|x_2\|)), \tag{27}$$

ensuing from $w_0 := (z_0, p_0)$, and defined over its maximal extension interval. Also, let $\mathbf{x}(t, x_0, w_0)$ denote the trajectory of (1) ensuing from x_0 at $t = 0$ and defined over its maximal extension interval. The function U, defined in (20), is $\mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}^*), \mathbb{R}_+)$ on account of the smoothness of $\tilde{\Psi}_1$ and $\tilde{\Psi}_0$ on $\mathrm{clos}(\mathscr{P}^*)$ and $j_0 \geqslant 2j_1$ (fact IV.3) whatever is $c \geqslant 1$. Pick $c \geqslant 1$ such that

$$\begin{aligned}
&\frac{1}{2}[b\|x\|^2 + \frac{\alpha_1 c}{p^{j_0}}\|z\|^2] \\
&\geqslant \Big|\frac{1}{p^{j_0}} z^T\Big\{[(k_1+1)\tilde{\Psi}_0(p) + \frac{\partial \tilde{\Psi}_0}{\partial p}(p)p\Big\}z \\
&+ \frac{1}{p^{j_1}} z^T[(k_1 + j_0 - j_1 + 1)\tilde{\Psi}_1(p) + \frac{\partial \tilde{\Psi}_1}{\partial p}(p)p]x\Big| \\
&+ |z^T \Psi_0(p) z + z^T \Psi_1(p) x|
\end{aligned} \tag{28}$$

for all x, z and $p \in \mathrm{clos}(\mathscr{P}^*)$. The choice of c is possible on account of the smoothness of $\tilde{\Psi}_1$ and $\tilde{\Psi}_0$ on $\mathrm{clos}(\mathscr{P}^*)$ and $j_0 \geqslant 2j_1$ (fact IV.3).

By (21) and (28) it is easy to establish the existence of $\delta \in \mathscr{K}_{\infty,\mathscr{O}(2)}$ and $h > 0$ such that for all x, z and $p \in \mathrm{clos}(\mathscr{P}^*)$

$$hp^{k_1+j_0+1}\|(x,z)\|^2 \leqslant U(x,z,p) \leqslant \delta(\|(x,z)\|) \tag{29}$$

(define $h := (1/2)\min\{b, c\alpha_1\}$ and $\delta(r) := (\beta + \|B_2 + A_2\| + 1)r^2 + c\alpha_2(r)$). Using (21) and (28) and facts IV.2, IV.3 and IV.4, we obtain $\dot{U}|_{(\mathbf{1})-(\mathbf{2})-(\mathbf{27})} \leqslant -\psi(\mathbf{x}, \mathbf{z}, \mathbf{p})$ with $\psi(x, z, p) := c\phi(z, p)p$ and $\psi(\cdot, \cdot, p)$ positive semi-definite for each $p \in \mathscr{P}^*$.

We are left with proving that $\mathbf{x}(t, x_0, w_0)$ and $\mathbf{w}(t, x_0, w_0)$ are defined for all $t \geqslant 0$, $\Lambda(\mathbf{x}_2(\cdot, x_0, w_0), \mathbf{z}(\cdot, x_0, w_0)) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$ and Λ is locally Lipschitz at zero. By **(A3)** V is locally quadratic at zero with locally quadratic at zero stability margin. Since, in addition, $\frac{\partial V}{\partial p}(z, p) \geqslant 0$ for all z and $p \in \mathscr{P}$, $\dot{V}|_{(\mathbf{1})-(\mathbf{2})-(\mathbf{27})} \leqslant -\phi(\mathbf{z}, \mathbf{p}) - 2\frac{\partial V}{\partial p}(\mathbf{z}, \mathbf{p})\mathbf{p}(\tilde{\tau}(\|\mathbf{z}\|) + \tau_2(\|\mathbf{x}_2\|)) \leqslant -\phi(\mathbf{z}, \mathbf{p})$. Therefore, by theorem III.1 (2)-(27) is globally asymptotically and locally exponentially stable with respect to z. Fix $p_0 \in \mathscr{P}^*$. By fact IV.2 there exists $\nu \in \mathscr{K}_{\mathscr{O}(1)}$ such that $\dot{W}|_{(\mathbf{1})-(\mathbf{2})-(\mathbf{27})} \leqslant \nu(\|\mathbf{z}\|)(W(\mathbf{x}) + 1)$ and, since (2)-(27) is locally exponentially stable with respect to z, $\nu(\|\mathbf{z}(\cdot, x_0, w_0)\|) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $N(x_0, w_0) := \int_0^{+\infty} \nu(\|\mathbf{z}(s, x_0, w_0)\|)ds < +\infty$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$. Therefore, $W(\mathbf{x}(t, x_0, w_0)) \leqslant (W(x_0) + 1)e^{N(x_0, w_0)} < +\infty$ for all $t \geqslant 0$ and x_0 and z_0 and $\mathbf{x}(t, x_0, w_0)$ is defined (and even bounded) for all $t \geqslant 0$, x_0, z_0 and $p_0 \in \mathscr{P}^*$. Once again since (2)-(27) is locally exponentially stable with respect to z and $\tilde{\tau} \in \mathscr{K}_{\mathscr{O}(1)}$, it follows that $\tilde{\tau}(\|\mathbf{z}(\cdot, x_0, w_0)\|) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$. For similar reasons, on account of **(A1)** $\tau_2(\|\mathbf{x}_2(\cdot, x_0, w_0)\|) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$. Since Γ is locally Lipschitz at zero, also Λ is locally Lipschitz at zero and $\Lambda(\mathbf{x}_2(\cdot, w_0), \mathbf{z}(\cdot, x_0, w_0)) \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$.□

Example IV.1 Consider the first three states of the cart-pendulum system (see (186) of [11]) $\dot{\mathbf{x}}_1 = \mathbf{v}$, $\dot{\mathbf{x}}_2 = \mathbf{x}_3$, $\dot{\mathbf{x}}_3 = \frac{2\mathbf{x}_2\mathbf{x}_3^2}{1+\mathbf{x}_2^2} + \mathbf{x}_2\sqrt{1+\mathbf{x}_2^2} - \mathbf{v}\sqrt{1+\mathbf{x}_2^2}$. Only approximations of Lyapunov functions can be designed for this system as shown in [9] and [11]. Let us design a FLP. Set $\mathbf{v} := \frac{1}{\sqrt{1+\mathbf{x}_2^2}}[\frac{2\mathbf{x}_2\mathbf{x}_3^2}{1+\mathbf{x}_2^2} + \mathbf{x}_2\sqrt{1+\mathbf{x}_2^2} + \mathbf{x}_2 + \mathbf{x}_3]$ to globally asymptotically and locally exponentially stabilize the (x_2, x_3)-dynamics so that the resulting system is in the form (1)-(2) with $x := x_1$, $z := (x_2, x_3)^T$ and satisfies **(A1)**-(**A3**) with $W_0 = I$, $V(z) = \frac{1}{2}\begin{pmatrix} x_2^T & x_3^T \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$ and $\gamma_1(s) = 2s^2 + s\frac{\sqrt{1+s^2}-1}{\sqrt{1+s^2}}$ (the other functions are all zero). Direct calculations give $\Psi_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $\Psi_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\tilde{\tau}(s) := \frac{3}{s^3}[2s^2 + s\frac{\sqrt{1+s^2}-1}{\sqrt{1+s^2}}]^2$. Therefore, a FLP is

$$\begin{aligned}
U(x, z, p) &:= p[x_1^2 + \begin{pmatrix} x_2^T & x_3^T \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} x_1 \\
&+ 16\begin{pmatrix} x_2^T & x_3^T \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}]
\end{aligned} \tag{30}$$

with $\dot{\mathbf{p}} = 2\tilde{\tau}(\|z\|)\mathbf{p}$. This quadratic FLP can be compared with the Lyapunov functions resulting in [9] and [11]. The simplicity of the FLP's is a key feature to which corresponds a major complexity for the $\dot{\mathbf{p}}$ equation, which however does not complicate the overall design especially when using the FLP's in an iterated procedure design.

A FLP can be used for designing *dynamic state feedback* stabilizing control laws exactly in the same way as a Lyapunov function can be used for designing *static state feedback* stabilizing control laws by feedback passivation approach ([4]). The main idea is illustrated in the following section.

V. ASYMPTOTIC STABILIZATION OF CASCADED SYSTEMS

Consider the controlled system (3) with $g \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}) \times \mathbb{R}, \mathbb{R}^n)$, $k \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}) \times \mathbb{R}, \mathbb{R}^m)$, $g|_{z=0} = g_0 \in \mathbb{R}^n$ and $k|_{z=0} = k_0 \in \mathbb{R}^m$ for all x, v and $p \in \mathrm{clos}(\mathscr{P})$. Assume (**A2**)-(**A3**) and, in addition,
(**A4**) *the pair*

$$\begin{pmatrix} F & H(p) \\ 0 & A(p) \end{pmatrix}, \begin{pmatrix} g_0 \\ k_0 \end{pmatrix} \tag{31}$$

is stabilizable for each $p \in \mathscr{P}$.

We want to find a *dynamic* state feedback law which globally asymptotically stabilizes (3) with respect to (x, z), using the filtered Lyapunov function $U(x, z, p)$ we have constructed for (1)-(2). The Lyapunov-based controller proposed in [9] under the additional assumptions (**A4**)-(**A5**) inherits the drawbacks from the Lyapunov function constructed under the assumptions (**A2**)-(**A3**). By smoothness of g and k and since $g|_{z=0} = g_0$ and $k|_{z=0} = k_0$ for all x, v and $p \in \mathrm{clos}(\mathscr{P})$, we can assume without loss of generality that

$$\begin{aligned} g(x, z, p, v)v &= g|_{v=0}v + \tilde{g}(x, z, p, v)v^2, \\ k(x, z, p, v)v &= k|_{v=0}v + \tilde{k}(x, z, p, v)v^2 \end{aligned} \tag{32}$$

with $\tilde{g} \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}), \mathbb{R}^n)$ and $\tilde{k} \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}), \mathbb{R}^m)$ such that $\tilde{g}|_{z=0} = 0$ and $\tilde{k}|_{z=0} = 0$ for all x, v and $p \in \mathrm{clos}(\mathscr{P})$. Using this fact, find $\pi \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}_+)$ such that $\pi|_{z=0} = \pi_0 > 0$ and

$$\begin{aligned} \pi(x, z) \geqslant 1 + |\tilde{g}^T(x, z, p, v)\Big(\frac{\partial U}{\partial x}(x, z, p)\Big)^T \\ + \tilde{k}^T(x, z, p, v)\Big(\frac{\partial U}{\partial z}(x, z, p)\Big)^T| \end{aligned} \tag{33}$$

for all x, z, for all $p \in \mathrm{clos}(\mathscr{P})$ and $v : |v| \leqslant |\tilde{w}(x, z, p)|$, with

$$\begin{aligned} \tilde{w}(x, z, p) := g^T|_{v=0}\Big(\frac{\partial U}{\partial x}(x, z, p)\Big)^T \\ + k^T|_{v=0}\Big(\frac{\partial U}{\partial z}(x, z, p)\Big)^T \end{aligned} \tag{34}$$

and U as in theorem IV.1. Define a control law for (3) as follows

$$v = \tilde{v}(x, z, p) := -\frac{\tilde{w}(x, z, p)}{\pi(x, z)}. \tag{35}$$

We want to prove the following stabilization result.

Theorem V.1 *Under assumptions* (**A2**)*-*(**A5**) *the closed-loop system (3)-(27)-(35) is globally asymptotically stable and locally exponentially stable with respect to* (x, z).

We will prove the theorem into two steps: first, we prove that the trajectories of (3)-(27)-(35) are defined and bounded for all $t \geqslant 0$, secondly we prove that (3)-(27)-(35) is locally exponentially and $(x, z) = (0, 0)$ is globally attractive with respect to (x, z) and finally that it is globally stable with respect to (x, z). Let $\boldsymbol{\zeta}(t, \zeta_0) := (\mathbf{x}(t, \zeta_0), \mathbf{z}(t, \zeta_0), \mathbf{p}(t, \zeta_0))$ denote the trajectory of (3)-(27)-(35) ensuing from $\zeta_0 := (x_0, z_0, p_0)$ at $t = 0$ and defined over its maximal extension interval. Also, let p^* and $\mathscr{P}^*$ be as in fact IV.3 of theorem IV.1. We establish the following fact.

Fact V.1 $\boldsymbol{\zeta}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{P}^*)$ *for each* x_0, z_0 *and* $p_0 \in \mathscr{P}^*$*. Moreover,* $\lim_{t\to+\infty} \mathbf{z}(t, \zeta_0) = 0$ *and* $\lim_{t\to+\infty} \mathbf{p}(t, \zeta_0) = \overline{p}_0(\zeta_0) := p_0 e^{-\int_0^{+\infty} \Gamma(\mathbf{z}(s,\zeta_0))ds} \in \mathscr{P}^*$.

The proof is as follows. Fix x_0, z_0 and $p_0 \in \mathscr{P}^*$. By direct calculations and on account of (33) we obtain

$$\dot{U}|_{(\mathbf{3})-(\mathbf{27})-(\mathbf{35})} \leqslant -c\phi(\mathbf{z}, \mathbf{p})\mathbf{p} - \tilde{v}^2(\mathbf{x}, \mathbf{z}, \mathbf{p}) \leqslant 0. \tag{36}$$

Moreover, by (21) and (28), $\frac{1}{2}[bp^{k_1+j_0+1}\|x\|^2 + pV(z, p)] \leqslant U(x, z, p)$ for all x, z and $p \in \mathrm{clos}(\mathscr{P}^*)$. Therefore $U(\boldsymbol{\zeta}(\cdot, \zeta_0)) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}_+)$ and, on account of (29), also

$$\begin{aligned} &\mathbf{x}(\cdot, \zeta_0)\mathbf{p}^{(k_1+j_0+1)/2}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^n), \\ &V(\mathbf{z}(\cdot, \zeta_0), \mathbf{p}(\cdot, \zeta_0))\mathbf{p}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}_+), \\ &\mathbf{z}(\cdot, \zeta_0)\mathbf{p}^{(k_1+1)/2}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^m), \\ &\mathbf{p}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathrm{clos}(\mathscr{P}^*)). \end{aligned} \tag{37}$$

Note that $U(x, z, p) = Y(xp^{(k_1+j_0+1)/2}, zp^{(k_1+1)/2}, V(z)p, p)$ for all x, z and p with $Y \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathrm{clos}(\mathscr{P}^*), \mathbb{R}_+)$. On account of (36) $\dot{Y}|_{(\mathbf{3})-(\mathbf{27})-(\mathbf{35})} \leqslant -c\phi(\mathbf{z}, \mathbf{p})\mathbf{p} - \tilde{v}^2(\mathbf{x}, \mathbf{z}, \mathbf{p})$ and since the uncontrolled system (3)-(27) is equal to (1)-(2)-(27) and (2)-(27) is locally exponentially and globally asymptotically stable with respect to z, we conclude by virtue of LaSalle's invariance principle that $\lim_{t\to+\infty} \mathbf{p}(t, \zeta_0) = \overline{p}_0(\zeta_0) \in \mathscr{P}^*$ and $\lim_{t\to+\infty} \mathbf{z}(t, \zeta_0)\mathbf{p}^{(k_1+1)/2}(t, \zeta_0) = 0$. As a consequence of (37), $\mathbf{x}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathbf{z}(\cdot, \zeta_0) \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$, with $\lim_{t\to+\infty} \mathbf{z}(t, \zeta_0) = 0$ which proves the claimed fact.

We establish the second following fact.

Fact V.2 *$\lim_{t\to+\infty} \mathbf{x}(t,\zeta_0) = 0$ and $\lim_{t\to+\infty} \mathbf{z}(t,\zeta_0) = 0$ for each x_0, z_0 and $p_0 \in \mathscr{P}^*$. Moreover, for each $p_0 \in \mathscr{P}^*$ there exist $\eta, \lambda, \mu > 0$ such that $\|(\mathbf{x}(t,\zeta_0), \mathbf{z}(t,\zeta_0))\| \leqslant \mu e^{-\lambda t}$ for all $t \geqslant 0$ provided $\|(x_0, z_0)\| \leqslant \eta$.*

The proof is as follows. Fix x_0, z_0 and $p_0 \in \mathscr{P}^*$. By fact V.1, (36) and LaSalle's invariance principle, $\zeta(t,\zeta_0)$ tends to the largest invariant set of (3)-(27)-(35) contained in $\{(x,z,p) : \phi(z,p) = 0, \tilde{v}(x,z,p) = 0, p \in \mathscr{P}^*\}$ and denoted by $\mathscr{N}$. Let $\overline{P}(p) := \frac{1}{2}\frac{\partial^2 U}{\partial(x,z)^2}\Big|_{(x,z)=(0,0)}$ for each $p \in \mathscr{P}^*$. Since $h|_{z=0} = 0$, $g|_{z=0} = g_0$, $k|_{z=0} = k_0$, $\frac{\partial h}{\partial z}|_{(x,z)=(0,0)} = H(p)$ for all x, v and $p \in \mathscr{P}^*$, the trajectory $\zeta(t,\zeta_0)$ of (3)-(27)-(35) and, respectively, the trajectory $\zeta'(t,\zeta_0)$ of

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{pmatrix} = L(p) \begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix}, \ \dot{\mathbf{p}} = 0, \tag{38}$$

$$L(p) := \begin{pmatrix} F & H(p) \\ 0 & A(p) \end{pmatrix} - \frac{1}{\pi_0} \begin{pmatrix} g_0 \\ k_0 \end{pmatrix} \begin{pmatrix} g_0 \\ k_0 \end{pmatrix}^T \overline{P}(p), \tag{39}$$

ensuing from any $\zeta_0 \in \mathscr{N}$ are such that $\zeta(t,\zeta_0) = \zeta'(t,\zeta_0)$ for all $t \geqslant 0$. The matrix $\overline{P}(p)$ is symmetric and positive definite for each $p \in \mathscr{P}^*$ on account of the strong convexity of V and (28). Also, since U is a FLP for (1)-(2), locally quadratic at zero with locally quadratic stability margin, $\overline{P}(p)\begin{pmatrix} F & H(p) \\ 0 & A(p) \end{pmatrix} + \begin{pmatrix} F & H(p) \\ 0 & A(p) \end{pmatrix}^T (p)\overline{P}(p) \leqslant 0$ for each $p \in \mathscr{P}^*$. Therefore, since $\begin{pmatrix} g_0^T & k_0^T \end{pmatrix}, \begin{pmatrix} F & H(p) \\ 0 & A(p) \end{pmatrix}^T$ is detectable by virtue of (**A4**), it follows by standard arguments on Riccati equations that $L(p)$ is Hurwitz for each $p \in \mathscr{P}^*$. We conclude that any trajectory $\zeta(t,\zeta_0)$ of (3)-(27)-(35) ensuing from $\mathscr{N}$ tends to zero as $t \to +\infty$. Therefore, by LaSalle's invariance principle the trajectories of (3)-(27)-(35) tend to zero as $t \to +\infty$ which proves the claimed fact.

We use facts V.1 and V.2 to prove theorem V.1. First, we prove for each $p_0 \in \mathscr{P}^*$ the existence of $\eta, \lambda, \mu > 0$ such that $\|(\mathbf{x}(t,\zeta_0), \mathbf{z}(t,\zeta_0))\| \leqslant \mu e^{-\lambda t}$ for all $t \geqslant 0$ provided $\|(x_0, z_0)\| \leqslant \eta$. Fix $p_0 \in \mathscr{P}^*$. Note that the (x,z)-subsystem of (38) is exactly the Jacobian linearization of (3)-(35) around $(x,z) = (0,0)$. This system can be seen as a time-varying system $\dot{\mathbf{w}} = L(t)\mathbf{w}$. Clearly, the eigenvalues of $L(t)$ are in the left half complex plane for each $t \geqslant 0$ (since the matrix between the square brackets in (38) is Hurwitz for each $p \in \mathscr{P}^*$) and (2.7) of [7] is satisfied on account of fact V.1 and the smoothness of A and H on $\mathrm{clos}(\mathscr{P}^*)$ and, respectively, of $\overline{P}$ on $\mathrm{clos}(\mathscr{P}^*)$. As remarked in [7] we conclude that (3)-(35) is locally exponentially with respect to (x,z).

The final step of the proof is to prove the global stability of (3)-(27)-(35) with respect to (x,z). Fix $p_0 \in \mathscr{P}^*$. By local exponential stability of (3)-(27)-(35) with respect to (x,z) and since Λ is locally Lipschitz at zero, $\Lambda(\mathbf{z}(\cdot,\zeta_0)) \in \mathbf{L}^1(\mathbb{R}_+,\mathbb{R}_+)$ for each x_0 and z_0. Moreover, on account of (29), (36) and fact V.1, there exist $\sigma \in \mathscr{K}_+$, $h > 0$ and $\delta \in \mathscr{K}_\infty$ such that

$$h\|(\mathbf{x}(t,\zeta_0), \mathbf{z}(t,\zeta_0))\|^2 \leqslant \frac{U(\mathbf{x}(t,\zeta_0)), \mathbf{z}(t,\zeta_0), \mathbf{p}(t,\zeta_0))}{\mathbf{p}^{j_0+k_1+1}(t,\zeta_0)}$$
$$\leqslant \frac{\sigma^{j_0+k_1+1}(\|(x_0,z_0)\| + p_0)}{p_0^{j_0+k_1+1}} \delta(\|(x_0,z_0)\|)$$

for all $t \geqslant 0$ and x_0, z_0. Since $\mu : r \in \mathbb{R}_+ \mapsto \mu(r) := \frac{\sigma^{j_0+k_1+1}(r+p_0)}{h p_0^{j_0+k_1+1}}\delta(r)$ is a $\mathscr{K}_\infty$-class function, this proves that (3)-(27)-(35) is globally stable with respect to (x,z).□

Example V.1 (Example (IV.1) cont'ed). Let us design a feedback stabilizer for (IV.1) using the FLP (30). Set $\mathbf{v} := \frac{1}{\sqrt{1+\mathbf{x}_2^2}}[\frac{2\mathbf{x}_2\mathbf{x}_3^2}{1+\mathbf{x}_2^2} + \mathbf{x}_2\sqrt{1+\mathbf{x}_2^2} + \mathbf{x}_2 + \mathbf{x}_3] + \mathbf{w}$ and design $\mathbf{w}$ according to the steps (33) and (35) so to obtain

$$\mathbf{w} := -[\frac{\partial U}{\partial \mathbf{x}_1} + \frac{\partial U}{\partial \mathbf{x}_3}\sqrt{1+\mathbf{x}_3^2}].$$

VI. STRICT FILTERED LYAPUNOV FUNCTIONS

In this section we state a key fact for the stabilization of systems obtained from (3) by adding blocks on top as in the case of (4): from the smooth strongly convex FLP U it is always possible to design a smooth strict strongly convex FLP $\tilde{U}$ for the closed-loop system (3)-(35). Moreover, $\tilde{U}$ has locally quadratic at zero stability margin and $\frac{\partial \tilde{U}}{\partial p}(x,z,p) \geqslant 0$ for all x, z and $p \in \mathscr{P}^*$. This proves that assumption (**A3**), which has been previously stated on (2), is satisfied for the augmented and closed-loop system (3)-(35).

Theorem VI.1 *Under assumptions (**A2**)-(**A5**), there exist $\overline{p}^{**} \in \mathscr{P}^*$, integer $j_2 \geqslant j_0 + k_1 + 1$ and $\overline{P}_0 \in \mathbf{C}^\infty(\mathscr{P}^{**}, \mathbb{R}^{(n+m)\times(n+m)})$, with $\mathscr{P}^{**} := (0, \overline{p}^{**})$, such that*

$$\tilde{U}(x,z,p) = U(x,z,p) + p^{j_2} \begin{pmatrix} x^T & z^T \end{pmatrix} \overline{P}_0(p) \begin{pmatrix} x \\ z \end{pmatrix} \tag{40}$$

is a smooth strict strongly convex FLP for (3)-(35) with locally quadratic at zero stability margin and $\frac{\partial \tilde U}{\partial p}(x,z,p) \geqslant 0$ *for all* x, z *and* $p \in \mathscr{P}^{**}$.

Proof: (Sketch). Let $\tilde U$ be as in (20), $b, \alpha_1, c, \tilde\Psi_0, \tilde\Psi_1$ as in (28) and $\tilde v$ as in (35). For each $p \in \mathscr{P}^*$ let $\overline{P}_0(p)$ be the symmetric and positive definitematrix such that

$$\overline{P}_0(p)L(p) + L^T(p)\overline{P}_0(p) = -I \qquad (41)$$

with $L(p)$ as in (39). Also, set $X(x,z,p,v) := Fx + h(x,z,p) + g(x,z,p,v)v$ and $Y(x,z,p,v) := a(z,p) + k(x,z,p,v)v$. Find $\overline{p}^{**} \in \mathscr{P}^*$, $\rho_1 \in \mathscr{K}_{\mathscr{O}(1)}$, $\rho_2 \in \mathscr{K}_+$, locally Lipschitz at zero, and integer $j_2 \geqslant j_0 + k_1 + 1$ such that $G : p \mapsto G(p) := p^{j_2-(j_0+k_1+1)}\overline{P}_0(p) \in \mathbf{C}^\infty(\mathrm{clos}(\mathscr{P}^{**}), \mathbb{R}^{(n+m)\times(n+m)}$, with $\mathscr{P}^{**} := (0, \overline{p}^{**})$ and, in addition,

$$\begin{aligned}&2p^{j_2-(j_0+k_1+1)}\overline{P}_0(p)\begin{pmatrix}X(x,z,p,\tilde v(x,z,p))\\ Y(x,z,p,\tilde v(x,z,p))\end{pmatrix}\\ &\leqslant -\|(x,z)\|^2 p^{j_2-(j_0+k_1+1)}\\ &+\rho_1(\|z\|)\rho_2(\|(x,z)\|)[\frac{V(z,p)}{p^{j_0+k_1}} + W(x)] \qquad (42)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}[b\|x\|^2 + \frac{\alpha_1 c}{p^{j_0}}\|z\|^2] \geqslant \Big|\frac{1}{p^{j_0}}z^T\Big\{[(k_1+1)\tilde\Psi_0(p)\\ &+\frac{d\tilde\Psi_0}{dp}(p)p\Big\}z + \frac{1}{p^{j_1}}z^T[(k_1+j_0-j_1+1)\tilde\Psi_1(p)\\ &+\frac{d\tilde\Psi_1}{dp}(p)p]x\Big| + |z^T\Psi_0(p)z + z^T\Psi_1(p)x|,\\ &\frac{1}{2}[b\|x\|^2 + \frac{\alpha_1 c}{p^{j_0}}\|z\|^2]\\ &\geqslant \Big|\begin{pmatrix}x^T & z^T\end{pmatrix}\frac{d}{dp}(p^{j_2-(j_0+k_1+1)}\overline{P}_0(p))\begin{pmatrix}x^T\\ z^T\end{pmatrix}\Big| \qquad (43)\end{aligned}$$

for all x, z and $p \in \mathrm{clos}(\mathscr{P}^{**})$. Define a smooth strict strongly convex FLP $\tilde U \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}^{**}), \mathbb{R}_+)$ for (3)-(35) as in (40) with

$$\dot{\mathbf{p}} = -\tilde\Lambda(\mathbf{x},\mathbf{z})\mathbf{p},\ \tilde\Lambda(x,z) := \Lambda(z) + 2\rho_1(\|z\|)\rho_2(\|(x,z)\|).$$

Since $\dot{\tilde U}_{(\mathbf{3})-(\mathbf{35})-(\mathbf{44})} \leqslant -\mathbf{p}^{j_2}\|(\mathbf{x},\mathbf{z}\|^2$ on account of (41) and (43), $\tilde U$ has locally quadratic at zero stability margin. Moreover, by (43) we also have $\frac{\partial \tilde U}{\partial p} \geqslant 0$ for all z and $p \in \mathrm{clos}(\mathscr{P}^{**})$.□

VII. ADDING BLOCKS

In this section we want to determine how a stabilizing feedback law can be designed for (4). To this aim, it is sufficient to design a stabilizing feedback law for the simpler system

$$\begin{aligned}\dot{\mathbf{y}} &= s(\mathbf{y}) + q(\mathbf{y},\mathbf{x},\mathbf{z}) + r(\mathbf{y},\mathbf{x},\mathbf{z},\mathbf{u})\mathbf{u} \quad (44)\\ \dot{\mathbf{x}} &= f(\mathbf{x}) + h(\mathbf{x},\mathbf{z}) + g(\mathbf{x},\mathbf{z},\mathbf{u})\mathbf{u},\\ \dot{\mathbf{z}} &= a(\mathbf{z}) + k(\mathbf{x},\mathbf{z},\mathbf{u})\mathbf{u}, \qquad (45)\end{aligned}$$

with $y \in \mathbb{R}^d$, $u \in \mathbb{R}$. Moreover, $s(0) = 0$, $f(0) = 0$, $a(0) = 0$, $h|_{z=0} = 0$ for all x, $\frac{\partial h}{\partial z}|_{(x,z)=(0,0)} := H$, $q|_{(x,z)=(0,0)} = 0$ for all y, $\frac{\partial q}{\partial x}|_{(y,x,z)=(0,0,0)} = L_1$, $\frac{\partial q}{\partial z}|_{(y,x,z)=(0,0,0)} = L_2$, $\frac{\partial s}{\partial y}|_{y=0} = S$, $\frac{\partial f}{\partial x}|_{x=0} = F$, $\frac{\partial a}{\partial z}|_{z=0} = A$, $r|_{(x,z)=(0,0)} = r_0 \in \mathbb{R}^d$ for all y and u, $g|_{z=0} = g_0 \in \mathbb{R}^n$ and $k|_{z=0} = k_0 \in \mathbb{R}^m$ for all x and u. Moreover, $g \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n)$, $k \in \mathbf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \mathbb{R}^m)$, $r \in \mathbf{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \mathbb{R}^d)$ and $q \in \mathbf{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^d)$. A stabilizing feedback law for (4) can be designed by iterating the procedure illustrated below for (44)-(45). The system (45) has the form (3). Theorem V.1 establishes under assumptions (**A2**)-(**A5**) the existence of a stabilizing feedback law $u = \tilde v(x,z,p)$ for (45). Therefore, (44)-(45) has once again the form (3) with x, z, v replaced respectively by y, (x,z), $u - \tilde v(x,z,p)$. Therefore, in the context of stabilization of (45) it is natural to retain assumptions (**A2**)-(**A5**) for (45) which boil down to

(A1') *there exist a symmetric positi definite* $W_0 \in \mathbb{R}^{n\times n}$ *and* $V_0 \in \mathbb{R}^{m\times m}$ *such that* $W_0F + F^TW_0 \leqslant 0$ *and* $V_0A + A^TV_0 < 0$ *and a partition of* $x = (x_1^T, x_2^T)^T$ *such that the uncontrolled system (45) is globally asymptotically and locally exponentially stable with respect to* x_2,

(A2') *there exist* $\gamma_2 \in \mathscr{K}_{\mathscr{O}(1)}$ *and* $\gamma_0, \gamma_1, \gamma_3 \in \mathscr{K}_{\mathscr{O}(2)}$ *such that*

$$\begin{aligned}&\|f(x) - Fx\| \leqslant \gamma_0(\|x_2\|),\\ &\|h(x,z) - Hz\| \leqslant \gamma_1(\|z\|) + \gamma_2(\|z\|)\|x\|,\\ &\|a(z) - Az\| \leqslant \gamma_3(\|z\|) \qquad (46)\end{aligned}$$

for all x, z,

and assume the following additional ones for (44):

(B1) *there exist* $\rho_2, \xi_2 \in \mathscr{K}_{\mathscr{O}(1)}$ *and* $\xi_0, \rho_1, \xi_1 \in \mathscr{K}_{\mathscr{O}(2)}$ *such that for all* y, x, z, u

$$\begin{aligned}&\|s(y) - Sy\| \leqslant \xi_0(\|y_2\|),\\ &\|q(y,x,z) - L_1x - L_2z\|\\ &\leqslant \xi_1(\|(x,z)\|) + \xi_2(\|(x,z)\|)\|y\|,\\ &\|r(y,x,z,u) - r_0\| \leqslant \rho_1(\|(x,z,u)\|) + \rho_2(\|(x,z,u)\|)\|y\|,\end{aligned}$$

(B2) *there exists a symmetric positive definite* $U_0 \in \mathbb{R}^{d\times d}$ *such that* $U_0S + S^TU_0 \leqslant 0$ *and a partition of*

$y = (y_1^T, y_2^T)^T$ *such that the uncontrolled system (45) is globally asymptotically and locally exponentially stable with respect to* y_2 *and* x_2,
(B3) *the pair*

$$\begin{pmatrix} S & L_1 & L_2 \\ 0 & F & H \\ 0 & 0 & A \end{pmatrix}, \begin{pmatrix} r_0 \\ g_0 \\ k_0 \end{pmatrix} \tag{47}$$

is stabilizable.

Using the procedure of sections IV and V, we can establish the following result which is the basic step for iterating the above procedure and finding a stabilizing feedback law for (4). Let $\tilde{v}$ be as in (35).

Theorem VII.1 *Under assumptions (**A1'**)-(**A3'**), (**B1**)-(**B3**) there exist* $\overline{p}^\circ \in \mathscr{P}$ *and* $u^\circ \in \mathbf{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}^\circ), \mathbb{R})$, *with* $\mathscr{P}^\circ := (0, \overline{p}^\circ)$, *such that (44)-(45) with* $u = u^\circ(y, x, z, p) + \tilde{v}(x, z, p)$ *is globally asymptotically and locally exponentially stable with respect to* (y, x, z). *Moreover, there exists a smooth strict strongly convex FLP* $U^\circ \in \mathbf{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{clos}(\mathscr{P}^\circ), \mathbb{R}_+)$ *for the closed-loop system (44)-(45) with locally quadratic at zero stability margin and such that* $\frac{\partial U^\circ}{\partial p}(x, z, p) \geqslant 0$ *for all* x, z *and* $p \in \mathscr{P}^\circ$.

VIII. CONCLUSIONS

In this paper we have introduced the notion of filtered Lyapunov functions (FLP) and studied their properties. These functions have the same properties as classical Lyapunov function but are much easier to construct. FLP's can be used in cascaded design to obtain composite FLP's for feedback stabilization using passification. Iterative design with FLP's has also been studied for upper block-triangular systems. Further study will be devoted to enlarge the class of systems for which filtered Lyapunov functions can be easily designed.

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