European Journal of Control 3:92–103

European Journal of Control

On the Role of Passivity and Output Injection in the Output Feedback Stabilisation Problem: Application to Robot Control*

S. Battilotti[1], L. Lanari[1] and R. Ortega[2]

[1]Dipartimento di Informatica e Sistemistica, Roma, Italy; [2]Laboratoire des Signaux et Systèmes CNRS–SUPELEC, Gif sur Yvette, France

Existing dynamic output feedback stabilisation techniques of non-linear systems are applicable mainly when the unmeasurable part of the state enters linearly. Our objective in this paper is to identify a class of systems, non-linear with respect to the unmeasured variables, which are globally asymptotically stabilisable via dynamic output feedback. To achieve this characterisation we explore two routes: systems passifiability, and the combination of state feedback plus output injection. While the first line of research is motivated by the recent developments in robot control, the second one is an attempt to extend the work of the first author to the non-linear dependence case. Our contributions are twofold: first, we present a series of general theorems that identify the class mentioned above. Second, we apply these theorems to derive in a systematic manner, and using a unified framework, some recent schemes on global set point control of rigid and elastic joint robots using only position measurements.

Keywords: Global set point control; Output feedback stabilisation; Non-linear systems; Rigid and elastic joint robots

1. Problem Formulation

We are interested in this paper in the problem of global asymptotic stabilisation of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $f(0) = 0$, $h(0) = 0$, via dynamic output feedback[1], that is

$$u = u(y, \sigma) \tag{1.2}$$

$$\dot{\sigma} = \eta(y, \sigma) \tag{1.3}$$

with $\sigma \in \mathbb{R}^s$. In the case where the unmeasurable part of the state enters linearly, systematic techniques for stabilisation are now available, see, for example, [5,20] and references therein. Our interest here is to identify some structural properties of the system that allow us to stabilise it even in the non-linear case. To this end we explore two mechanisms for stabilisation: passifiability *à la* [9], and the possibility of combining the solutions of output injection and state feedback stabilisation problems as done for the linear case in [5].

Before presenting our contributions we first briefly recall some preliminary results instrumental for our further developments.

2. Theoretical Background

2.1. Passivity and Internal Stability

Following [26] we say the system (1.1) with $m = p$ defines a passive map $u \to y$ if there exists a C^0 storage function $V(x) \geq 0$, $V(0) = 0$ such that for all u, x, y, with x solution of (1.1) we have

$$\int_0^t u(\tau)^{\mathrm{T}} y(\tau) \mathrm{d}\tau \geq V[x(t)] - V[x(0)] \qquad t \geq 0$$

Correspondence and offprint requests to: S. Battilotti, Dipartimento di Informatica e Sistemistica, Via Eudossiana 18-00184, Rome, Italy. E-mail: stefbatt@riscdis.ing.uniroma1.ital.

* This research was supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica and by Agenzia Spaziale Italiana under contract *RS*17, 1994, and by the Commission of European Communities under contract ERB CHRX CT 93-0380.

[1]To avoid further technical discussions we will assume throughout the paper that all functions are "sufficiently" smooth.

Received 22 January 1996; Accepted in revised form 6 January 1997
Recommended by C. Samson and O. H. Bosgra

It is well known [12] that if a passive system has a C^1 storage function then

$$L_{f(x)}V(x) \leq 0$$
$$L_{g(x)}V(x) = h^{\mathrm{T}}(x) \tag{2.1}$$

where $L_{\tau_1}\tau_2$ denotes the Lie derivative of a function τ_2 along a vector field τ_1. With abuse of notation τ_1 can be a set of vector fields.

It is shown in [9] (see also [19]) that passive, zero-state detectable systems with proper positive definite storage functions are globally asymptotically stabilisable by static output feedbacks of the form

$$u = -\phi(y)$$

where ϕ is a smooth function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for each $y \neq 0$. It is also shown in [9] that, under mild regularity conditions, the system is passifiable (via static state feedback) iff it has relative degree $\{1, \cdots, 1\}$ and is weakly minimum phase.

2.2. State Feedback Plus Output Injection

It is well known that in linear systems the notion of detectability plays a fundamental role in the output feedback stabilisation problem. A lot of research has recently been devoted to define an equivalent notion for non-linear systems (e.g. [10]). The importance of this property has been established in [3–5] where it is shown that for systems linear with respect to the unmeasured variables

$$\dot{x} = A(x_y)x + B(x_y)u$$
$$y = x_y \tag{2.2}$$

with x_y the first p components of the state, the *global stabilisation problem via output feedback* of $x = 0$ can be split into two independent subproblems. First, a state-feedback stabilisation problem, which consists of globally stabilising (2.2) in $x = 0$ through a state-feedback control law

$$u = u_{\mathrm{SF}}(x), \qquad u_{\mathrm{SF}}(0) = 0 \tag{2.3}$$

Second, an output injection stabilisation problem which consists of globally stabilising in $x = 0$

$$\dot{x} = A(x_y)x + v, \qquad v \in \mathbb{R}^n$$
$$y = x_y \tag{2.4}$$

through an output feedback

$$v = v_{\mathrm{OI}}(y), \qquad v_{\mathrm{OI}}(0) = 0 \tag{2.5}$$

From the solution of these separate subproblems, one derives two Lyapunov functions V_{ST} and V_{OI}, which, suitably combined, give a candidate Lyapunov function for the closed-loop system. The design is completed proposing the actual dynamic output feedback control, which follows in a straightforward manner, from the solution of the state-feedback and output injection subproblems.

While the state-feedback stabilisation problem has a clear interpretation, the output injection problem has no direct physical meaning in the sense that one tries to force the term $v_{OI}(y)$ not through the input of the system but directly into the state equations. In order to clarify its theoretical significance, let us consider for a moment the case of *linear* systems

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{2.6}$$

It is well known that in this case output feedback stabilisability is equivalent to stabilisability and detectability, detectability being in turn equivalent to solvability of the output injection problem. That is, (C, A) is detectable if and only if there exists a matrix K such that

$$\mathrm{spectrum}(A - KC) \in \mathbf{C}^- \tag{2.7}$$

which holds if and only if there exists a symmetric positive definite matrix P_{OI} such that

$$x^{\mathrm{T}}\left[P_{\mathrm{OI}}(A - KC) + (A - KC)^{\mathrm{T}}P_{\mathrm{OI}}\right]x < 0$$
$$\forall x \neq 0 \tag{2.8}$$

In particular (2.8) implies the following detectability condition

$$y = 0, \;\; x \neq 0 \Rightarrow x^{\mathrm{T}}[P_{\mathrm{OI}}A + A^{\mathrm{T}}P_{\mathrm{OI}}]x < 0 \tag{2.9}$$

Going back to non-linear systems (2.2), since there does not exist a non-linear equivalent of condition (2.7), the output injection problem can be stated in terms of the existence of a proper Lyapunov function $V_{\mathrm{OI}}(x)$ and an at least C^0 function $v_{\mathrm{OI}}(y)$, vanishing at the origin, such that, along the trajectories of (2.4) with $v = v_{\mathrm{OI}}(y)$,

$$\dot{V}_{\mathrm{OI}}(x) = \frac{\partial V_{\mathrm{OI}}(x)}{\partial x}\left[A(x_y)x + v_{\mathrm{OI}}(x_y)\right] < 0$$
$$\forall x \neq 0 \tag{2.10}$$

which is the non-linear counterpart of (2.8).

Similarly, the state-feedback problem for (2.2) can be stated in terms of the existence of a smooth proper Lyapunov function $V_{\mathrm{SF}}(x)$ and an at least continuous function $u_{\mathrm{SF}}(x)$, vanishing at the origin, such that, along the trajectories of (2.2) with $u = u_{\mathrm{SF}}(x)$ and $\forall x \neq 0$

$$\dot{V}_{\mathrm{SF}}(x) = \frac{\partial V_{\mathrm{SF}}(x)}{\partial x}\left[A(x_y)x + B(x_y)u_{\mathrm{SF}}(x)\right] < 0 \tag{2.11}$$

In [3-5] it has been shown that, if P_{SF} and P_{OI} are symmetric positive definite matrices and x_z denotes the vector of state variables not available for feedback, the existence of two proper smooth Lyapunov functions $V_{SF}(x)$ and $V_{OI}(x)$ of the form

$$\begin{aligned} V_{SF}(x) &= \tfrac{1}{2}x_z^T P_{SF} x_z + x_z^T \zeta_{SF}(x_y) + \xi_{SF}(x_y) \\ V_{OI}(x) &= \tfrac{1}{2}x_z^T P_{OI} x_z + x_z^T \zeta_{OI}(x_y) + \xi_{OI}(x_y) \end{aligned} \tag{2.12}$$

satisfying respectively (2.11) and (2.10) implies global output feedback stabilisability of (2.2). This result is a non-linear version of the separation principle for systems (2.2).

The construction of the stabilising output feedback controller is based on the definition of a suitable Lyapunov function $V^e(x,\sigma)$, for the closed-loop system

$$\begin{aligned} \dot{x} &= A(x_y)x + B(x_y)u(y,\sigma) \\ \dot{\sigma} &= \eta(y,\sigma) \\ y &= x_y \end{aligned} \tag{2.13}$$

with $\sigma \in \mathbb{R}^n$, $u(y,\sigma)$ and $\eta(y,\sigma)$ to be defined. Let

$$\sigma = \begin{pmatrix} \sigma_y \\ \sigma_z \end{pmatrix}$$

denote the partition of σ corresponding to the partition

$$x = \begin{pmatrix} x_y \\ x_z \end{pmatrix}$$

The proposed Lyapunov function candidate $V^e(x,\sigma)$ for (2.13) is defined as

$$\begin{aligned} V^e(x,\sigma) &= \lambda V_{SF}(x) + \tfrac{1}{2}(x_z - \hat{\sigma}_z)^T P_m (x_z - \hat{\sigma}_z) \\ &\quad + \xi_m(y,\sigma_y) \\ P_m &= P_{OI} - \lambda P_{SF} \\ \hat{\sigma}_z &= \sigma_z - P_m^{-1}\zeta_m(y,\sigma_y) \\ \zeta_m(y,\sigma_y) &= -\lambda\zeta_{SF}(y) + \zeta_{OI}(y) + \lambda\zeta_{SF}(\sigma_y) \\ &\quad - \zeta_{OI}(\sigma_y) \\ \xi_m(y,\sigma_y) &= \tfrac{1}{2}\|y - \sigma_y\|^2 \end{aligned} \tag{2.14}$$

with $\lambda \in \mathbb{R}^+$ such that P_m is positive definite. It has been proved in [5] that choosing $u(y,\sigma)$ equal to the smooth feedback $u_{SF}(x)$, evaluated for $x_z = \sigma_z$, i.e.

$$u(y,\sigma) = u_{SF}(y,\sigma_z)$$

$\eta(y,\sigma)$ can be designed in such a way that the derivative of $V^e(x,\sigma)$ along the trajectories of (2.13) is negative definite. The same result can be obtained by choosing $u(y,\sigma) = u_{SF}(y,\hat{\sigma}_z)$ (see [6]).

3. Main Results

Our objective in this section is to push forward the results of Section 2 by combining passivity, output injection and LaSalle's principle. The non-linear systems (1.1) considered for this task are based on the interconnection of a 'good' system with a 'bad' system. A 'good' system is one for which the non-linear separation principle of Section 2 works. A 'bad' system is a highly non-linear one for which neither passivity nor any other analysis tools are available to date.

Throughout the paper we will use $(\cdot)_y$, $(\cdot)_z$ to denote the measurable and unmeasurable components of the state, respectively, as well as the corresponding terms of any given vector. Similarly, by $[\cdot]_y$ and $[\cdot]_z$ we will denote the rows of any matrix corresponding to measurable and unmeasurable components of the state. We will consider systems of the form

$$\begin{aligned} \dot{\delta} &= f_1(x) + g_1(x)u \\ \dot{\vartheta} &= f_{21}(y)\vartheta_z + f_{22}(y) + g_2(y)u \\ y &= h(x) = \begin{pmatrix} \delta_y \\ \vartheta_y \end{pmatrix} \end{aligned} \tag{3.1}$$

with $x = (\delta^T \quad \vartheta^T)$.

A similar 'good+bad' structure will be imposed on the Lyapunov functions. A 'good' Lyapunov function has the structure (2.12), while a 'bad' Lyapunov function does not necessarily have this form.

Our strategy in achieving output feedback stabilisation will be the following. First, by combining output injection and passivity, we guarantee the existence of a proper, positive definite and smooth $V^e(x,\sigma)$ and smooth dynamic output feedback controller of the form

$$\begin{aligned} u &= u_{SF}(y,\sigma_z) + \ell(y,\sigma)(y - \sigma_y) \\ \dot{\sigma} &= \eta(y,\sigma) \end{aligned} \tag{3.2}$$

such that, along the trajectories of (3.1) and (3.2), one has

$$\dot{V}^e \leq 0 \tag{3.3}$$

i.e., the closed-loop system is stable. Our Lyapunov function candidate for the closed-loop system has the form

$$V^e(x,\sigma) = \lambda V_{SF}(x) + V_m(x,\sigma) \tag{3.4}$$

where $\lambda > 0$ and $V_m(x,\sigma)$ is a positive semidefinite function such that $V_m(x,x) = 0$.

Finally, suitable detectability assumptions guarantee *global asymptotic stability* via LaSalle's principle.

Let

$$f(x) = \begin{pmatrix} f_1(x) \\ f_{21}(y)\vartheta_z + f_{22}(y) \end{pmatrix}$$
$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(y) \end{pmatrix}$$

with $f(0) = 0$. As usual, $L_\varphi v$ denotes the Lie derivative of a function φ along a vector field v. With some abuse of notation, we have

$$\begin{aligned} L_\varphi v_y &= [L_\varphi v]_y \\ L_\varphi v_z &= [L_\varphi v]_z \end{aligned} \tag{3.5}$$

The first result of the section is a combination of *passivity* and *output injection* tools.

Theorem 1. *Assume that*
(H1) (State-feedback plus passivity) there exists a smooth, proper and positive definite function $V_{\mathrm{SF}}(x)$

$$\begin{aligned} V_{\mathrm{SF}}(x) &= V^b_{\mathrm{SF}}(x) + V^g_{\mathrm{SF}}(\vartheta) \\ V^g_{\mathrm{SF}}(\vartheta) &= \frac{1}{2}\vartheta_z^{\mathrm{T}} P_{\mathrm{SF}}\vartheta_z + \vartheta_z^{\mathrm{T}}\zeta_{\mathrm{SF}}(\vartheta_y) + \xi_{\mathrm{SF}}(\vartheta_y) \end{aligned}$$

and a smooth feedback $u_{\mathrm{SF}}(y, \vartheta_z)$, vanishing at the origin, such that, along the trajectories of the system (3.1) with $u = u_{\mathrm{SF}}(y, \vartheta_z)$, one has

$$\begin{aligned} L_{g(x)} V^b_{\mathrm{SF}}(x) &= \delta_y^{\mathrm{T}} \\ \dot V_{\mathrm{SF}}(x) &= -\alpha_{\mathrm{SF}}(x) \end{aligned} \tag{3.6}$$

with positive semidefinite α_{SF}
(H2) (Output injection) there exists a smooth, proper and positive definite function

$$V^g_{\mathrm{OI}}(\vartheta) = \frac{1}{2}\vartheta_z^{\mathrm{T}} P_{\mathrm{OI}}\vartheta_z + \vartheta_z^{\mathrm{T}}\zeta_{\mathrm{OI}}(\vartheta_y) + \xi_{\mathrm{OI}}(\vartheta_y)$$

and a smooth function $v_{\mathrm{OI}}(y)$, vanishing at the origin, such that

$$\frac{\partial V^g_{\mathrm{OI}}(\vartheta)}{\partial\vartheta}[f_{21}(y)\vartheta_z + f_{22}(y) + v_{\mathrm{OI}}(y)] = -\alpha_{\mathrm{OI}}(\vartheta) \tag{3.7}$$

with positive definite α_{OI}
(H3) (Structure) $L_{f(x)}V^b_{\mathrm{SF}}(x) = \varrho_1(y, \delta_z)$ and $L_{g(x)}V^b_{\mathrm{SF}}(x) = \varrho_2(y, \delta_z)$ for some functions ϱ_1 and ϱ_2.
(H4) (Detectability) $\alpha_{\mathrm{SF}}(x) = 0 \Rightarrow \vartheta = 0$; moreover, along the trajectories of (3.1) with $u = u_{\mathrm{SF}}(y, \vartheta_z)$

$$\dot\delta_y = 0, \quad \vartheta = 0 \Rightarrow \delta_z \to 0$$

Under the above conditions, the dynamic output feedback controller

$$\begin{aligned} u &= u_{\mathrm{SF}}(y, \widehat\sigma_z) + \ell(y,\sigma)(y - \sigma_y) \\ \eta(y,\sigma) &= \begin{pmatrix} \eta_y(y,\sigma) \\ P_m^{-1}(\lambda\frac{\partial\zeta_{\mathrm{SF}}}{\partial\sigma_{\vartheta_y}}(y) - \frac{\partial\zeta_{\mathrm{OI}}}{\partial\sigma_{\vartheta_y}}(y))\eta_y(y,\sigma) + a(y,\sigma) \end{pmatrix} \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} \eta_y(y,\sigma) &= \begin{pmatrix} 0 \\ L_{f_{21}(y)\widehat\sigma_z + f_{22}(y) + g_2(y)u(y,\widehat\sigma_z)}\vartheta_y \end{pmatrix} \\ &\quad + k(y - \sigma_y) + b(y,\sigma) \\ \ell(y,\sigma) &= -\frac{1}{\lambda}(1 \quad 0) \\ \widehat\sigma_z &= \sigma_{\vartheta_z} - P_m^{-1}\zeta_m(\vartheta_y, \sigma_{\vartheta_y}) \\ P_m &= P_{\mathrm{OI}} - \lambda P_{\mathrm{SF}} \\ \zeta_m(\vartheta_y, \sigma_{\vartheta_y}) &= -\lambda\zeta_{\mathrm{SF}}(\vartheta_y) + \zeta_{\mathrm{OI}}(\vartheta_y) + \lambda\zeta_{\mathrm{SF}}(\sigma_{\vartheta_y}) \\ &\quad - \zeta_{\mathrm{OI}}(\sigma_{\vartheta_y}) \\ a(y,\sigma) &= P_m^{-1}\{[\frac{\partial\zeta_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)L_{g_2(y)}\vartheta_y \\ &\quad + P_{\mathrm{OI}}L_{g_2(y)}\vartheta_z]\ell(y,\sigma)(y - \sigma_y) \\ &\quad + [f_{21}]^{\mathrm{T}}_{\vartheta_y}(y)(\vartheta_y - \sigma_{\vartheta_y} \\ &\quad + \lambda(\widehat\sigma_z^{\mathrm{T}}\frac{\partial\zeta_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y) \\ &\quad + \frac{\partial\xi_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)))^{\mathrm{T}} \\ &\quad + \lambda[f_{21}]^{\mathrm{T}}_{\vartheta_y}(y)(\widehat\sigma_z^{\mathrm{T}}P_{\mathrm{SF}} + \zeta_{\mathrm{SF}}(\vartheta_y))^{\mathrm{T}} \\ &\quad + [\frac{\partial\zeta_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)L_{f_{21}(y)\widehat\sigma_z + f_{22}(y) + g_2(y)u}\vartheta_y \\ &\quad + P_{\mathrm{OI}}L_{f_{21}(y)\widehat\sigma_z + f_{22}(y) + g_2(y)u}\vartheta_z]\} \\ b(y,\sigma) &= \lambda\ell^{\mathrm{T}}(y,\sigma)[L^{\mathrm{T}}_{g_2(y)}\vartheta_y(\frac{\partial\zeta^{\mathrm{T}}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)\widehat\sigma_z \\ &\quad + \frac{\partial\xi^{\mathrm{T}}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)) + L^{\mathrm{T}}_{g_2(y)}\vartheta_z(P_{\mathrm{SF}}(\widehat\sigma_z + \zeta^{\mathrm{T}}_{\mathrm{SF}}(\vartheta_y))] \end{aligned} \tag{3.9}$$

for any $\lambda > 0$ such that $P_m > 0$ and any $k > 0$, globally asymptotically stabilises the trivial equilibrium.
Proof. Let

$$\sigma = \begin{pmatrix} \sigma_y \\ \sigma_{\vartheta_z} \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} \sigma_{\delta_y} \\ \sigma_{\vartheta_y} \end{pmatrix}$$

Consider the Lyapunov function candidate

$$\begin{aligned} V^\sigma(x,\sigma) = &\lambda V_{\mathrm{SF}}(x) + \frac{1}{2}(\vartheta_z - \widehat\sigma_z)^T P_m(\vartheta_z - \widehat\sigma_z) \\ &+ \frac{1}{2}\|y - \sigma_y\|^2 \end{aligned}$$

and let

$$
\begin{aligned}
u &= u_{\mathrm{SF}}(y,\widehat{\sigma}_z) + \ell(y,\sigma)(y-\sigma_y)\\
\eta(y,\sigma) &= \begin{pmatrix} \eta_y(y,\sigma)\\ \eta_{\vartheta_z}(y,\sigma)\end{pmatrix},\quad \eta_y(y,\sigma) = \begin{pmatrix} \eta_{\delta_y}(y,\sigma)\\ \eta_{\vartheta_y}(y,\sigma)\end{pmatrix}
\end{aligned}
\tag{3.10}
$$

with $\ell(y,\sigma)$ and $\eta(y,\sigma)$ to be determined. Moreover, define $\widehat{e}_z = \vartheta_z - \widehat{\sigma}_z$.

Using (H1)–(H3) and (3.10), by direct calculations one obtains

$$
\begin{aligned}
\dot{V}^{e}(x,\sigma) =\ &\lambda(\varrho_1(y,\delta_z) + \varrho_2(y,\delta_z)u_{\mathrm{SF}}(y,\widehat{\sigma}_z))\\
&+ \lambda\delta_y^{\mathrm{T}}\ell(y,\sigma)(y-\sigma_y)\\
&+ \lambda[(\widehat{\sigma}_z^{\mathrm{T}}\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)\\
&+ \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y))L_{f_{21}(y)\vartheta_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_y\\
&+ (\widehat{\sigma}_z^{\mathrm{T}}P_{\mathrm{SF}} + \zeta_{\mathrm{SF}}(\vartheta_y))L_{f_{21}(y)\vartheta_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_z]\\
&+ \widehat{e}_z^{\mathrm{T}}[\frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)(L_{f_{21}(y)\vartheta_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_y)\\
&+ P_{\mathrm{OI}}L_{f_{21}(y)\vartheta_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_z]\\
&- \widehat{e}_z^{\mathrm{T}}P_m[\eta_{\vartheta_z}(y,\sigma)\\
&+ (-\lambda\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\sigma_{\vartheta_y}}(\sigma_{\vartheta_y}) + \frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\sigma_{\vartheta_y}}(\sigma_{\vartheta_y}))\eta_{\vartheta_y}(y,\sigma)]\\
&+ (\delta_y - \sigma_{\delta_y})^{\mathrm{T}}(\dot{\delta}_y - \eta_{\delta_y})\\
&+ (\vartheta_y - \sigma_{\vartheta_y})^{\mathrm{T}}[L_{f_{21}(y)\vartheta_z+f_{22}(y)+g_2(y)u}\vartheta_y - \eta_{\vartheta_y}]\\
&+ (y-\sigma_y)^{\mathrm{T}}\lambda\ell^{\mathrm{T}}(y,\sigma)[L^{\mathrm{T}}_{g_2(y)}\vartheta_y(\frac{\partial \mathcal{K}^{\mathrm{T}}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)\widehat{\sigma}_z\\
&+ \frac{\partial \mathcal{K}^{\mathrm{T}}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)) + L^{\mathrm{T}}_{g_2(y)}\vartheta_z(P_{\mathrm{SF}}(\widehat{\sigma}_z + \zeta^{\mathrm{T}}_{\mathrm{SF}}(\vartheta_y))]\\
&+ \widehat{e}_z^{T}[\frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)L_{g_2(y)}\vartheta_y\\
&+ P_{\mathrm{OI}}L_{g_2(y)}\vartheta_z]\ell(y,\sigma)(y-\sigma_y)
\end{aligned}
\tag{3.11}
$$

Collecting common terms in (3.11)

$$
\begin{aligned}
\dot{V}^{e}(x,\sigma) =\ &\lambda(\varrho_1(y,\delta_z) + \varrho_2(y,\delta_z)u_{\mathrm{SF}}(y,\widehat{\sigma}_z))\\
&+ \lambda[(\widehat{\sigma}_z^{\mathrm{T}}\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y)\\
&+ \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\vartheta_y}(\vartheta_y))L_{f_{21}(y)\widehat{\sigma}_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_y\\
&+ (\widehat{\sigma}_z^{\mathrm{T}}P_{\mathrm{SF}} + \zeta_{\mathrm{SF}}(\vartheta_y))L_{f_{21}(y)\widehat{\sigma}_z+f_{22}(y)+g_2(y)u_{\mathrm{SF}}(y,\widehat{\sigma}_z)}\vartheta_z]\\
&+ \lambda\delta_y^{\mathrm{T}}[\ell(y,\sigma)(y-\sigma_y) + \frac{1}{\lambda}(\delta_y - \sigma_{\delta_y})]\\
&+ \widehat{e}_z^{\mathrm{T}}(\frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)[f_{21}]_{\vartheta_y}(y) + P_{\mathrm{OI}}[f_{21}]_{\vartheta_z}(y))\widehat{e}_z\\
&- \widehat{e}_z^{\mathrm{T}}\{[P_m\eta_{\vartheta_z}(y,\sigma)\\
&+ (-\lambda\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial\sigma_{\vartheta_y}}(\sigma_{\vartheta_y})\\
&+ \frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\sigma_{\vartheta_y}}(\sigma_{\vartheta_y}))\eta_{\vartheta_y}(y,\sigma) - a(y,\sigma)]\\
&- \left((\delta_y - \sigma_{\delta_y})^{\mathrm{T}}(\vartheta_y - \sigma_{\vartheta_y})^{\mathrm{T}}\right)\\
&\times\Bigg[\begin{pmatrix} \eta_{\delta_y}\\ -L_{f_{21}(y)\widehat{\sigma}_z+f_{22}(y)+g_2(y)u}\vartheta_y + \eta_{\vartheta_y}\end{pmatrix}\\
&- b(y,\sigma)\Bigg]
\end{aligned}
\tag{3.12}
$$

From (3.12), choosing $\ell(y,\sigma)$ and $\eta(y,\sigma)$ as in (3.8), one obtains

$$
\begin{aligned}
\dot{V}^{e}(x,\sigma) \le -\ &\lambda\alpha_{\mathrm{SF}}(\delta,\vartheta_y,\widehat{\sigma}_z)\\
&+ \widehat{e}_z^{T}(\frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)[f_{21}]_{\vartheta_y}(y)\\
&+ P_{\mathrm{OI}}[f_{21}]_{\vartheta_z}(y))\widehat{e}_z\\
&- k\|y-\sigma_y\|^2
\end{aligned}
\tag{3.13}
$$

where $\alpha_{\mathrm{SF}}(\delta,\vartheta_y,\widehat{\sigma}_z)$ is the function $\alpha_{\mathrm{SF}}(x)$ evaluated for $\vartheta_z = \widehat{\sigma}_z$.

Since

$$
\frac{\partial \mathcal{K}_{\mathrm{OI}}}{\partial\vartheta_y}(\vartheta_y)[f_{21}]_{\vartheta_y}(y) + P_{\mathrm{OI}}[f_{21}]_{\vartheta_z}(y)
\tag{3.14}
$$

is the Hessian of (3.7) with respect to ϑ_z, by (H2) it is easy to see that (3.14); is semidefinite negative for all y and negative definite for $\vartheta_y = 0$. This, together with (3.13), implies that

$$
\dot{V}^{e} \le 0
$$

i.e., the closed-loop system is Lyapunov stable.

Combining LaSalle's invariance principle with (H3) and (3.13), we conclude global asymptotic stability. Indeed, $\dot{V}^{e} = 0$ only if $\sigma_{\delta_y} = \delta_y$, $\vartheta = 0$, $\sigma_{\vartheta_y} = 0$ and $\widehat{\sigma}_z = 0$. This, by definition of $\widehat{\sigma}_z$ and since $\zeta_m(\vartheta_y,\vartheta_y) = 0$, implies $\sigma_{\vartheta_z} = 0$. It follows from the definition of η_{δ_y} that $b(y,\sigma) = 0$ and $\dot{\delta}_y = 0$. This, by (H4), implies $x \to 0$. □

Remark 1. Potential applications of theorem 1 rely on the output feedback stabilisation of non-linear systems driven by non-linear ones, as for example robot with motor dynamics. A more general version of theorem 1 is obtained by replacing $\dot{\delta}_y$ by

$S(y)(\dot{\delta}_y - \varrho_3(y)u)$, where $S(y)$ is any matrix, invertible for all y, and $L_{g_1(x)}\delta_y = \varrho_3(y)$. Notice that when $\varrho_3 = 0$ and $S = I$, we recover theorem 1 in the above form. The case $S \neq I$ is of interest in some other applications (e.g., output feedback stabilisation of satellites [16]).

If $\vartheta = \{\emptyset\}$, structure assumption (H3) is automatically satisfied. On the other hand, (3.6) becomes the passivity condition (2.1) w.r.t. the output $\dot{y}$ or, equivalently, the 'bad' system, with $u = u_{SF}(y)$, is passive w.r.t. $\dot{y}$. Thus, theorem 1 reads as follows.

Theorem 2. Assume $\vartheta = \{\emptyset\}$ and that
(H1) (State-feedback plus passivity) there exists a smooth, proper and positive definite function $V_{SF}(x)$ and a smooth feedback $u_{SF}(y)$, vanishing at the origin, such that, along the trajectories of the system (3.1) with $u = u_{SF}(y)$, one has

$$\begin{aligned} L_{g(x)}V_{SF}(x) &= \dot{y}^{\mathrm{T}} \\ \dot{V}_{SF}(x) &= -\alpha_{SF}(x) \end{aligned} \tag{3.15}$$

with positive semidefinite α_{SF}
(H2) (Detectability) along the trajectories of (3.1) with $u = u_{SF}(y)$

$$\dot{y} = 0 \Rightarrow x \to 0$$

Under the above conditions, the dynamic output feedback controller

$$\begin{aligned} u(y,\sigma) &= u_{SF}(y) + \ell(y,\sigma)(y - \sigma_y) \\ \eta(y,\sigma) &= k(y - \sigma_y) \\ \ell(y,\sigma) &= -\frac{1}{\lambda} \end{aligned} \tag{3.16}$$

for any $\lambda > 0$ and $k > 0$, globally asymptotically stabilises the trivial equilibrium.

In the next theorem, only state–feedback plus output injection is considered. A 'generalised' separation principle is used to prove *stability* of the closed-loop system and additional detectability assumptions are invoked to infer *asymptotic* stability via LaSalle's principle.

Theorem 3. Assume that
(H1) (State-feedback) there exists smooth, proper and positive definite function $V_{SF}(x)$ and smooth function $u_{SF}(y,\vartheta_z)$, vanishing at the origin, such that

$$\begin{aligned} V_{SF}(x) &= V_{SF}^{b}(x) + V_{SF}^{g}(y,\vartheta_z) \\ V_{SF}^{g}(y,\vartheta_z) &= \frac{1}{2}\vartheta_z^{\mathrm{T}}P_{SF}\vartheta_z + \vartheta_z^{\mathrm{T}}\zeta_{SF}(y) + \xi_{SF}(y) \\ u_{SF}(y,\vartheta_z) &= q_1(y)\vartheta_z + q_2(y), \qquad q_2(0) = 0 \\ \dot{V}_{SF}(x) &= -\alpha_{SF}(y,\vartheta_z) \\ \dot{V}_{SF}^{b}(x) &= \Phi_1(y,\delta_z)\vartheta_z + \Phi_2(y,\delta_z), \\ \Phi_j(0,0) &= 0, j = 1,2 \end{aligned}$$

with positive semidefinite α_{SF}
(H2) (Output injection) there exists smooth, proper and positive definite functions $V_{OI}(x)$ and smooth function $v_{OI}(y)$, vanishing at the origin, such that

$$\begin{aligned} V_{OI}(x) &= V_{OI}^{b}(x) + V_{OI}^{g}(y,\vartheta_z) \\ V_{OI}^{g}(y,\vartheta_z) &= \frac{1}{2}\vartheta_z^{\mathrm{T}}P_{OI}\vartheta_z + \vartheta_z^{\mathrm{T}}\zeta_{OI}(y) + \xi_{OI}(y) \\ \frac{\partial V_{OI}(x)}{\partial x}[f(x) + v_{OI}(y)] &= -\alpha_{OI}(y,\vartheta_z) \leq 0 \\ \frac{\partial V_{OI}^{b}(x)}{\partial x}[f(x) + v_{OI}(y)] &= \Psi_1(y,\delta_z)\vartheta_z + \Psi_2(y,\delta_z) \\ \Psi_j(0,0) &= 0, j = 1,2 \end{aligned}$$

(H3) (Structure) $g_1(x) = 0$, $L_{g(x)}V_{SF}^{b}(x) = 0$ and $L_{f(x)}h(x) = \varrho_1(y,\delta_z)\vartheta_z + \varrho_2(y,\delta_z)$, with $\varrho_1(y,0) = 0$ for all y and $\varrho_2(0,0) = 0$
Under the above assumptions, the system (3.1) together with the dynamic output controller

$$\begin{aligned} u &= u_{SF}(y,\sigma) \\ \eta(y,\sigma) &= L_{f_{21}(y)\sigma + f_{22}(y) + g_2(y)u_{SF}(y,\sigma)}\vartheta_z + \lambda P_m^{-1}[k(y) \\ &\quad + \left(L_{f_{21}(y)}^{\mathrm{T}}\vartheta_z\right)P_{SF}\sigma \\ &\quad + P_{SF}L_{f_{21}(y)\sigma + g_2(y)u_{SF}(y,\sigma)}\vartheta_z \\ &\quad + \frac{\partial\zeta_{SF}}{\partial\vartheta_y}(y))\left(L_{g_2(y)}\vartheta_z\right)u_{SF}(y,\sigma)] \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} k(y) = &P_{SF}L_{f_{22}(y)}\vartheta_z + \frac{\partial\zeta_{SF}}{\partial y}(y)\varrho_2(y,0) \\ &+ \Phi_1^{\mathrm{T}}(y,0) + \left(L_{f_{21}(y)}^{\mathrm{T}}\vartheta_z\right)\zeta_{SF}(y) \end{aligned}$$

is Lyapunov stable. If, in addition,
(H4) (Detectability) along the trajectories of (3.1)–(3.17),

$$\begin{aligned} &\alpha_{SF}(y,\sigma) = 0, \\ &\alpha_{OI}(y,\vartheta_z - \sigma) = 0 \Rightarrow \sigma = 0,\ y = 0 \end{aligned}$$

and, along the trajectories of (3.1), with $u = 0$,

$$y = 0 \Rightarrow x \to 0$$

the dynamic output controller (3.17) globally asymptotically stabilises the trivial equilibrium.

Proof. Consider the Lyapunov function candidate

$$V^{e}(x,\sigma) = \lambda V_{SF}(x) + \frac{1}{2}(\vartheta_z - \sigma)^{\mathrm{T}}P_m(\vartheta_z - \sigma)$$

Moreover, let

$$u = u_{SF}(y,\sigma) \tag{3.18}$$

and $e_z = \vartheta_z - \sigma$.

Under assumptions (H1) and (H3), along the trajectories of (3.1), with $u = 0$, one has

$$
\begin{aligned}
\dot V_{\mathrm{SF}}(x) &= \dot V^{h}_{\mathrm{SF}}(x) + \dot V^{g}_{\mathrm{SF}}(x)\\
&= L_{f(x)}V^{h}_{\mathrm{SF}}(x) + [\vartheta_z^T \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial y}(y)\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial y}(y)]L_{f(x)}h(x)\\
&\quad + (\vartheta_z^{\mathrm{T}} P_{\mathrm{SF}} + \zeta^{\mathrm{T}}_{\mathrm{SF}}(y))L_{f_{21}(y)\vartheta_z + f_{22}(y)}\vartheta_z\\
&= \vartheta_z^{\mathrm{T}}\left[P_{\mathrm{SF}}L_{f_{21}(y)}\vartheta_z + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial y}(y)\varrho_1(y,\delta_z)\right]\vartheta_z\\
&\quad + \vartheta_z^{\mathrm{T}}\Bigg[P_{\mathrm{SF}}L_{f_{22}(y)}\vartheta_z + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial y}(y)\varrho_2(y,\delta_z)\\
&\quad + \varrho_1^{\mathrm{T}}(y,\delta_z)\frac{\partial \mathcal{K}^{\mathrm{T}}_{\mathrm{SF}}}{\partial y}(y)\\
&\quad + \Phi_1^{\mathrm{T}}(y,\delta_z) + \left(L^{\mathrm{T}}_{f_{21}(y)}\vartheta_z\right)\zeta_{\mathrm{SF}}(y)\Bigg]\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial y}(y)\varrho_2(y,\delta_z) + \zeta^{\mathrm{T}}_{\mathrm{SF}}(y)L_{f_{22}(y)}\vartheta_z\\
&\quad + \Phi_2(y,\delta_z)\\
&= \vartheta_z^{\mathrm{T}}k_1(y,\delta_z)\vartheta_z + \vartheta_z^{\mathrm{T}}k_2(y,\delta_z) + k_3(y,\delta_z)
\end{aligned}
\tag{3.19}
$$

Since $u_{\mathrm{SF}}(y,\vartheta_z) = q_1(y)\vartheta_z + q_2(y)$, $q_2(0) = 0$ and $L_{g(x)}V^{h}_{\mathrm{SF}}(x) = 0$, from (3.19) it follows that along the trajectories of (3.1), with $u = u_{\mathrm{SF}}(y,\vartheta_z)$, one has

$$
\begin{aligned}
\dot V_{\mathrm{SF}}(x) &= \dot V^{h}_{\mathrm{SF}}(x) + \dot V^{g}_{\mathrm{SF}}(x)\\
&= \vartheta_z^{\mathrm{T}}[k_1(y,\delta_z) + P_{\mathrm{SF}}(L_{g_2(y)}\vartheta_z)q_1(y)\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)q_1(y)]\vartheta_z\\
&\quad + \vartheta_z^{\mathrm{T}}\Big[k_2(y,\delta_z) + q_1^{\mathrm{T}}(y)\left(L^{\mathrm{T}}_{g_2(y)}\vartheta_z\right)\zeta_{\mathrm{SF}}(y)\\
&\quad + P_{\mathrm{SF}}(L_{g_2(y)}\vartheta_z)q_2(y)\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)q_2(y)\\
&\quad + (\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)q_1(y))^{\mathrm{T}}]\\
&\quad + k_3(y,\delta_z) + \zeta^{\mathrm{T}}_{\mathrm{SF}}(y)(L_{g_2(y)}\vartheta_z)q_2(y)\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)q_2(y)\\
&= \vartheta_z^{\mathrm{T}}r_1(y,\delta_z)\vartheta_z + \vartheta_z^{\mathrm{T}}r_2(y,\delta_z) + r_3(y,\delta_z)
\end{aligned}
\tag{3.20}
$$

From (H1) and (3.20) it follows that $r_3(y,\delta_z)$, $r_2(y,\delta_z)$ and $r_1(y,\delta_z)$ are independent from δ_z. This, in turn, implies that $k_3(y,\delta_z)$, $k_2(y,\delta_z)$ and $k_1(y,\delta_z)$ depend only on y and that, along the trajectories of (3.1), with $u = u_{\mathrm{SF}}(y,\vartheta_z)$, one has

$$
\begin{aligned}
-\alpha_{\mathrm{SF}}(y,\vartheta_z) = \dot V_{\mathrm{SF}}(x) &= \vartheta_z^{\mathrm{T}}r_1(y,0)\vartheta_z\\
&\quad + \vartheta_z^{\mathrm{T}}r_2(y,0) + r_3(y,0)
\end{aligned}
\tag{3.21}
$$

Since by (H3) $\varrho_1(y,0) = 0$ for all y,

$$
\begin{aligned}
r_1(y,0) &= P_{\mathrm{SF}}[f_{21}(y)\\
&\quad + g_2(y)q_1(y)]_{\vartheta_z}\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)q_1(y)
\end{aligned}
$$

In a similar way, we conclude for the output injection problem that, as a consequence of (H2), one has

$$
\begin{aligned}
-\alpha_{\mathrm{OI}}(y,\vartheta_z) &= \vartheta_z^{\mathrm{T}}m_1(y,\delta_z)\vartheta_z + \vartheta_z^{\mathrm{T}}m_2(y,\delta_z)\\
&\quad + m_3(y,\delta_z)\\
&= \vartheta_z^{\mathrm{T}}m_1(y,0)\vartheta_z\\
&\quad + \vartheta_z^{\mathrm{T}}m_2(y,0) + m_3(y,0)
\end{aligned}
\tag{3.22}
$$

for some functions m_1, m_2 and m_3. Note that, since $\varrho_1(y,0) = 0$ for all y, $m_1(y,0) = P_{\mathrm{OI}}L_{f_{21}(y)}\vartheta_z$. Moreover, since α_{OI} is semidefinite positive, $m_1(y,0) \le 0$ for all y.

Along the trajectories of the closed-loop system (3.1)–(3.17), with $u = u_{\mathrm{SF}}(y,\sigma)$, one has

$$
\begin{aligned}
\dot V^{e}(x,\sigma) &= \lambda[\dot V^{h}_{\mathrm{SF}}(x) + \dot V^{g}_{\mathrm{SF}}(x)]\\
&\quad + (\vartheta_z - \sigma)^{\mathrm{T}}P_m(\dot\vartheta_z - \eta(y,\sigma))\\
&= \lambda\big[\vartheta_z^{\mathrm{T}}k_1(y,\delta_z)\vartheta_z + \vartheta_z^{\mathrm{T}}k_2(y,\delta_z) + k_3(y,\delta_z)\\
&\quad + (\vartheta_z^{\mathrm{T}}P_{\mathrm{SF}} + \zeta^{\mathrm{T}}_{\mathrm{SF}}(y))(L_{g_2(y)}\vartheta_z)u_{\mathrm{SF}}(y,\sigma).\\
&\quad + (\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y) + \vartheta_z^{\mathrm{T}}\frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y))(L_{g_2(y)}\vartheta_y)u_{\mathrm{SF}}(y,\sigma)\Big]\\
&\quad + (\vartheta_z - \sigma)^{\mathrm{T}}P_m(L_{f_{21}(y)}\vartheta_z)(\vartheta_z - \sigma)\\
&\quad + (\vartheta_z - \sigma)^{\mathrm{T}}P_m[(L_{g_2(y)}\vartheta_z)u_{\mathrm{SF}}(y,\sigma)\\
&\quad + (L_{f_{21}(y)}\vartheta_z)\sigma + L_{f_{22}(y)}\vartheta_z - \eta(y,\sigma)]\\
&= \lambda[\sigma^{\mathrm{T}}r_1(y,0)\sigma + \sigma^{\mathrm{T}}r_2(y,0) + r_3(y,0)]\\
&\quad + (\vartheta_z - \sigma)^{\mathrm{T}}m_1(y,0)(\vartheta_z - \sigma)\\
&\quad + (\vartheta_z - \sigma)^{\mathrm{T}}P_m\Big\{L_{f_{21}(y)\sigma + f_{22}(y) + g_2(y)u_{\mathrm{SF}}(y,\sigma)}\vartheta_z\\
&\quad - \eta(y,\sigma)\\
&\quad + \lambda P_m^{-1}[k(y) + \left(L^{\mathrm{T}}_{f_{21}(y)}\vartheta_z\right)P_{\mathrm{SF}}\sigma\\
&\quad + P_{\mathrm{SF}}L_{f_{21}(y)\sigma + g_2(y)u_{\mathrm{SF}}(y,\sigma)}\vartheta_z\\
&\quad + \frac{\partial \mathcal{K}_{\mathrm{SF}}}{\partial \vartheta_y}(y)(L_{g_2(y)}\vartheta_y)u_{\mathrm{SF}}(y,\sigma)]\Big\}
\end{aligned}
\tag{3.23}
$$

Finally, choosing $\eta(y,\sigma)$ as in (3.17), from (3.23) one obtains

$$
\dot V^{e}(x,\sigma) = -\lambda\alpha_{\mathrm{SF}}(y,\sigma) - \alpha_{\mathrm{OI}}(y,\vartheta_z - \sigma) \le 0
$$

This proves stability of the closed-loop system. Global asymptotic stability follows as in the proof of theorem 1 from detectability assumptions (H3) and LaSalle's principle. □

Remark 2. Note that the detectability assumption (H4) is stated on the closed-loop system (3.1)–(3.17). A deeper analysis shows that asymptotic stability of (3.1)–(3.17) can be shown as well assuming some more complicated, but directly on the open-loop system, detectability conditions.

4. Application to the Robot Stabilisation Problem

Motivated by practical considerations the control problem of robots using only position measurement has attracted the attention of several researchers. The global set point control problem for rigid robots, assuming that only joint position measurement is available, has been recently solved, simultaneously and independently, in [1,13,11,7]. The controller in all those cases consists of a gravitation compensation and a *linear* dynamic output feedback. Also, it is worth noting that the approach pursued there is not based on constructing an observer that tries to estimate the velocity signal. For elastic joint robots the global set point control problem was first solved in [1], (see also [14]), when only position measurement on the motor side is available, and in [2], when link position measurement is available. Again, the approach followed in those papers is not observer-based. Semiglobal tracking is addressed in [17] with link position measurements.

In this section, we use theorems 1–3 to put into a unified framework the results, for both the rigid and elastic joint robots, contained in [14,2,7]. This framework is strongly based on the combination of passivity, output injection and LaSalle's principle. Even though we will not present in this paper any new controller for the robotic problem, our contribution is to put in perspective various existing solutions. Other comparative studies for robot controllers have been reported in [8,18] where the schemes are compared using passivity and energy shaping properties, respectively.

4.1. Rigid Robots

The equations describing the dynamics of an n-degree-of-freedom rigid robot manipulator are given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g(q) = \tau$$

with q the $(n \times 1)$ measurable vector of joint angular positions (we assume rotational joints), τ the $(n \times 1)$ vector of external torques, $D(q)$ is the positive definite inertia matrix, $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centrifugal torques and $\tau_g(q)$ is the vector of gravitational torques. It is well known that the robot equations define a passive map $u \to \dot{q}$. Furthermore, there exists a suitable choice of the matrix $C(q,\dot{q})$ such that the matrix

$$\dot{D}(q) - 2C(q,\dot{q})$$

is skew-symmetric.

Choosing as state vector $\begin{pmatrix} z_1^T & z_2^T \end{pmatrix} = \begin{pmatrix} q^T & \dot{q}^T \end{pmatrix}$, the second order model can be rewritten as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -D^{-1}(z_1)\left[C(z_1,z_2)z_2 + \tau_g(z_1) - \tau\right] \end{aligned}$$

Define as the desired configuration $\begin{pmatrix} q_d \\ 0 \end{pmatrix}$. In the error coordinates $x_1 = z_1 - q_d$, $x_2 = z_2$ the system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -D^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 \\ &\quad + \quad \tau_g(x_1 + q_d) - u] \end{aligned} \tag{4.1}$$

which is of the form (3.1) with $y = x_1$, $\vartheta = \{\emptyset\}$ and

$$\begin{aligned} f(x) &= \begin{pmatrix} x_2 \\ -D^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + \tau_g(x_1 + q_d) - \tau_g(q_d)] \end{pmatrix}, \\ g(x) &= \begin{pmatrix} 0 \\ D^{-1}(x_1 + q_d) \end{pmatrix} \\ u &= \tau + \tau_g(q_d) \end{aligned}$$

We will check assumptions (H1) and (H2) of theorem 2.

- *State-feedback plus passivity*. Choose

$$\begin{aligned} V_{SF}(x) = \frac{1}{2}x_2^T D(x_1 + q_d)x_2 + \frac{1}{2}x_1^T K_p x_1 \\ + G(x_1 + q_d) - G(q_d) - x_1^T \tau_g(q_d) \end{aligned}$$

with K_p symmetric positive definite and $G(\cdot)$ denoting the potential energy associated with the conservative force $\tau_g(\cdot)$, that is, $G(x_1 + q_d)$ is such that

$$\frac{\partial G(x_1 + q_d)}{\partial x_1} = \tau_g(x_1 + q_d)$$

It has been shown [24] that $V_{SF}(x)$ is positive definite with a global minimum in $x = 0$ if K_p is sufficiently large, i.e.,

$$\lambda_{min}(K_p) > \gamma$$

with γ defined as

$$\|\tau_g(q_1) - \tau_g(q_2)\| \le \gamma\|q_1 - q_2\|, \quad \forall q_1, q_2 \in \mathbb{R}^n$$

Along the trajectories of (4.1), one has

$$\dot{V}_{SF}(x) = x_2^T\left[K_p x_1 + u\right]$$

so that, choosing

$$u = u_{SF}(y) = -K_p x_1$$

one has

$$\dot{V}_{SF}(x) = -\alpha_{SF}(y) = 0$$

One has

$$L_{g(x)} V_{SF}(x) = \dot{y}^{\mathrm{T}} = x_2^{\mathrm{T}}$$

- *Detectability*. Along the trajectories of (4.1) with $u = u_{SF}(y)$, if $x_2 = 0$ then $x = 0$.

Therefore assumptions (H1)–(H2) of theorem 2 are satisfied and a dynamic output controller which achieves global asymptotic stabilisation of the trivial equilibrium has the form (3.16).

Remark 3. Theorem 2 still holds with $y - \sigma_y$ replaced by $Ny - M\sigma_y$ with N and M invertible matrices thus recovering the same structure proposed in [7,13].

4.2. Elastic Joint Robots

The simplified dynamic model of an elastic joint robot is given by [22]

$$D_1(q_l)\ddot{q}_l + C(q_l, \dot{q}_l)\dot{q}_l + K(q_l - q_m) + \tau_g(q_l) = 0$$
$$D_2\ddot{q}_m - K(q_l - q_m) = \tau$$

with q_l and q_m being respectively the $(n \times 1)$ vectors of the links and rotors relative displacements, $K = \mathrm{diag}\{k_i\} > 0$ with k_i the elastic constant of the ith joint and $\tau_g(q_l)$ representing the gravity forces acting on the links. We assume that no damping is present. The positive definite inertia matrix is

$$D(q_l) = \begin{pmatrix} D_1(q_l) & 0 \\ 0 & D_2 \end{pmatrix}$$

while $C(q_l, \dot{q}_l)\dot{q}_l$ represent the centrifugal and Coriolis forces.

Choosing as state variables $\begin{pmatrix} q_l^{\mathrm{T}} & \dot{q}_l^{\mathrm{T}} & q_m^{\mathrm{T}} & \dot{q}_m^{\mathrm{T}} \end{pmatrix}$ and defining the desired state

$$\begin{pmatrix} q_d \\ 0 \\ q_d + K^{-1}\tau_g(q_d) \\ 0 \end{pmatrix}$$

we have in the error coordinates

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -D_1^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + K(x_1 - x_3) + \tau_g(x_1 + q_d) - \tau_g(q_d)]$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = D_2^{-1}[K(x_1 - x_3) - \tau_g(q_d) + u]$$

which is of the form (3.1) with

$$f(x) = \begin{pmatrix} x_2 \\ -D_1^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + K(x_1 - x_3) + \tau_g(x_1 + q_d) - \tau_g(q_d)] \\ x_4 \\ D_2^{-1}K(x_1 - x_3) \end{pmatrix}$$

$$g(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_2^{-1} \end{pmatrix}$$

$$u = \tau - \tau_g(q_d)$$

For flexible joint robots we have two possible measured variables, motor or link angular position. We will solve each one of them separately.

4.3. Motor Position Measurement

In this case since $y = x_3$ one has $\vartheta = \{\emptyset\}$. We will check assumptions (H1) and (H2) of theorem 2.

- *State-feedback plus passivity*. Choose

$$V_{SF}(x) = \frac{1}{2}\begin{pmatrix} x_2^{\mathrm{T}} & x_4^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1^{\mathrm{T}} & x_3^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + G(x_1 + q_d) - G(q_d) - x_1^{\mathrm{T}}\tau_g(q_d)$$

which is positive definite provided the linear spring at the joint is sufficiently stiff and K_p is sufficiently large [2]

$$K + \left.\frac{\partial \tau_g(q)}{\partial q}\right|_{q_d} > 0, \quad \forall q_d$$

$$K_p + K - K\left(K + \left.\frac{\partial \tau_g(q)}{\partial q}\right|_{q_d}\right)^{-1} K > 0, \quad \forall q_d$$

Along the trajectories of the system, one has

$$\dot{V}_{SF}(x) = x_4^{\mathrm{T}}[K_p x_3 + u]$$

so that, if

$$u = u_{SF}(y) = -K_p x_3$$

then

$$\dot{V}_{SF}(x) = -\alpha_{SF}(y) = 0$$

One has

$$L_{g(x)} V_{SF}(x) = \dot{y}^{\mathrm{T}} = x_4^{\mathrm{T}}$$

- *Detectability.* With $u = u_{SF}(y)$, if $x_4 = 0$ then $x = 0$. Indeed, with $u = u_{SF}(y)$, if $x_4 = 0$ then

$$x_3 = \text{constant} \Rightarrow x_1 = \text{constant} \Rightarrow x_2 = 0$$

and the following equations must hold

$$K(x_1 - x_3) + \tau_g(x_1 + q_d) - \tau_g(q_d) = 0$$
$$K(x_3 - x_1) + K_p x_3 = 0 \qquad (4.2)$$

It has been shown (see [2] for example) that, under the conditions on K_p above, we have that (4.2) is true only for $(x_1, x_3) = (0, 0)$. Thus, $x = 0$.

Therefore assumptions (H1) and (H2) of theorem 2 are satisfied and a dynamic output controller which achieves global asymptotic stabilisation of the trivial equilibrium has the form (3.16).

Remark 4. Note that replacing $y - \sigma_y$ by $Ny - M\sigma_y$, with N and M invertible matrices, one recovers the same structure proposed in [14].

4.4. Link Position Measurement

We have $y = x_1$. In this case, we will check assumptions (H1)–(H3) of theorem 3.

- *State-feedback plus output injection.* We have

$$\delta_y = x_1, \qquad \delta_z = x_2, \qquad \vartheta_y = \{\emptyset\},$$
$$\vartheta_z = \vartheta = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Choose

$$V_{SF}(x) = \frac{1}{2}\begin{pmatrix} x_2^T & x_4^T \end{pmatrix}\begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1^T & x_3^T \end{pmatrix}\begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + G(x_1 + q_d) - G(q_d) - x_1^T \tau_g(q_d)$$

$$V_{SF}^{\phi}(x) = \frac{1}{2} x_2^T D_1(x_1 + q_d) x_2$$

Define the feedback law

$$u_{SF}(y, \vartheta_z) = -K_p x_3 - K_d x_4$$

with K_p and K_d positive definite symmetric matrices, that is

$$q_1(y) = -\begin{pmatrix} K_p & K_d \end{pmatrix}$$
$$q_2(y) = 0$$

Along the trajectories of the system with $u = u_{SF}(y, \vartheta_z)$, one has

$$\dot V_{SF}(x) = -\alpha_{SF}(x_1, x_3, x_4) = -x_4^T K_d x_4$$

$$\dot V_{SF}^{\phi}(x) = -x_2^T[K(x_1 - x_3) + \tau_g(x_1 + q_d) - \tau_g(q_d)]$$

with

$$\Phi_1(y, x_2) = \begin{pmatrix} x_2^T K & 0 \end{pmatrix}$$
$$\Phi_2(y, x_2) = -x_2^T[K x_1 + \tau_g(x_1 + q_d) - \tau_g(q_d)]$$

Choose

$$V_{OI}(x) = \frac{1}{2}\begin{pmatrix} x_2^T & x_4^T \end{pmatrix}\begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1^T & x_3^T \end{pmatrix}\begin{pmatrix} K + K_p & -K \\ -K & K \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + G(x_1 + q_d) - G(q_d) - x_1^T \tau_g(q_d)$$

and

$$V_{OI}^{\phi}(x) = V_{SF}^{\phi}(x)$$

Again, with the assumption on K_p above we have definiteness of V_{OI}. Let the output injection be

$$v_{OI}(x_1) = \begin{pmatrix} 0 \\ -D_1(x_1 + q_d)^{-1} K_p x_1 \\ 0 \\ 0 \end{pmatrix}$$

obtaining

$$\frac{\partial V_{OI}(x)}{\partial x}[f(x) + v_{OI}(y)] = -\alpha_{OI}(x_3, x_4) = 0$$

$$\frac{\partial V_{OI}^{\phi}(x)}{\partial x}[f(x) + v_{OI}(y)] = -x_2^T[K(x_1 - x_3) + K_p x_1 + \tau_g(x_1 + q_d) - \tau_g(q_d)]$$

with

$$\Psi_1(y, x_2) = \begin{pmatrix} x_2^T K & 0 \end{pmatrix}$$
$$\Psi_2(y, x_2) = -x_2^T[(K + K_p)x_1 + \tau_g(x_1 + q_d) - \tau_g(q_d)]$$

- *Structure.*

$$g_1(x) = 0, \qquad L_{g(x)} V_{SF}^{\phi}(x) = 0$$

$$\varrho_1(y, x_2) = 0, \qquad \varrho_2(y, x_2) = x_2$$

- *Detectability.* Following [3], also (H3) can be shown to hold, so that theorem 3 applies and global asymptotic stability is guaranteed by the dynamic output feedback controller (3.17), with

$$\sigma = \begin{pmatrix} \sigma_{x_1} \\ \sigma_{x_4} \end{pmatrix}$$

$$\eta(y,\sigma) = \lambda P_m^{-1} \begin{pmatrix} K_p \\ -K_d \end{pmatrix} \sigma_{x_4} + \begin{pmatrix} \sigma_{x_4} \\ D_2^{-1}[K(x_1 - \sigma_{x_1}) + u_{SF}(y,\sigma)] \end{pmatrix}$$

5. Concluding Remarks

We have presented in this paper some results on output feedback stabilisation of systems which are nonlinear in the unmeasurable part of the state. The mechanisms for stabilisation we have explored are passifiability and the solvability of output injection and state feedback stabilisation problems. In the second approach the design is decomposed into a state feedback and an output injection subproblem, and then combining the two solutions in a certainty equivalent–like strategy. This procedure, advocated in [5], leads to a systematic design procedure, and sheds light on the structural obstacles for output feedback stabilisation that stem from a lack of detectability.

We have shown how the state feedback plus output injection route allows us to design output feedback regulators for rigid and flexible joint manipulators. Even though we have not presented in this paper any new controller for the robotic problem, our contribution is to put in perspective various existing solutions. This, in principle, allows for an easy comparison of their benefits and shortcomings. Other comparative studies for robot controllers have been reported in [8,18] where the schemes are compared using passivity and energy shaping properties, respectively. Potential applications of theorems 1–3 can range from rigid satellites to induction motors.

References

1. Ailon A. and Ortega R. An observer–based set–point controller for robot manipulators with flexible joints, Systems & Control Letters 1993; 21: 329–335
2. Battilotti S, Lanari L. Global set point control via link position measurement for flexible joint robots. Systems & Control Letters 1995; 25: 21–29
3. Battilotti S. Output feedback stabilisation of nonlinear systems: tools and examples. First IFAC Workshop on New Trends in Design of Control Systems, Smolenice, SL, 1994 pp 11–14
4. Battilotti S. Sufficient conditions for global robust stabilisation via measurement feedback for some classesw > of nonlinear systems. Proc 33rd Conf on Dec and Control, Lake Buena Vista, FL, 1994, pp 808–813
5. Battilotti S. Global output regulation and disturbance attenuation with global stability via measurement feedback for a class of nonlinear systems, IEEE Trans on Automatic Control 1996; 41: 315–327
6. Battilotti S. A note on reduced order dynamic output feedback stabilising controllers. Syst Contr Lett (in press)
7. Berghuis H, Nijmeijer H. Global regulation of robots using only position measurements. Systems & Control Letters. 1993; 21: 289–293
8. Brogliato B, Ortega R and Lozano R. Globally stable nonlinear controllers for flexible joint manipulators: A comparative study, Automatica 1995; 31: 6
9. Byrnes C, Isidori CJ, Willems JC. Passivity, feedback equivalence and the global stabilization of minimum phase nonlinear systems. IEEE Trans on Automatic Control 1991; 36, 11: 1228–1240
10. Byrnes C, Martin C. An integral–invariance principle for nonlinear systems. IEEE Trans on Automatic Control 1995; 40, 6: 983–994
11. Burkov I. Asymptotic stabilisation of nonlinear Lagrangian systems without measuring velocities,. Int Symp on Active Control in Mech Engg, Lyon, France, July 1993
12. Hill D, Moylan P. The stability of nonlinear dissipative systems. IEEE Trans on Automatic Control 1976; 21: 708–711
13. Kelly R. A simple robot set point controller using only position measurements. Proc 12th IFAC World Cong Sydney, Australia 1993
14. Kelly R, Ortega R, Ailon A and Loria A. Global regulation of flexible joint robots using approximate differentiation IEEE Transactions on Automatic Control 1995; 39, 6: 1222–1224
15. Lanari L, Wen JT. Asymptotically stable set point control laws for flexible robots, Systems & Control Letters 1992; 19: 119–129
16. Lizarralde F, Wen JT. Attitude control without angular velocity measurement: a passivity approach. IEEE Trans on Automatic Control 1996; 41, 3: 468–472
17. Nicosia S, Tome P. A tracking controller for flexible joint robots using only link position feedback. IEEE Trans on Automatic Control 1995; 40, 5: 885–890
18. Ortega R, Kelly R, Loria A. A class of output feedback globally stabilizing controllers for flexible joint robots, IFAC SyROCO'94, Capri, Italy, Sept. 19-21, 1994. See also IEEE Trans on Robotics and Automation 1995; 11, 5: 766–769
19. Ortega R. Passivity properties for stabilisation of cascaded nonlinear systems. Automatica 1991; 27, 2: 423–424.
20. Pomet JB, Hirschorn RM, Cebuhar WA. Dynamic output feedback regulation for a class of nonlinear systems. Math of Cont Sign and Syst 1993; 6: 106–124
21. Sontag E. State-space and I/O stability for nonlinear systems. Feedback control, nonlinear systems, and complexity. (Francis B, Tannenbaum A. (ed) LNCIS, Springer-Verlag, Berlin 1995.

22. Spong MW, Vidyasagar M. Robot dynamics and control. John Wiley (New York) 1989
23. Tomei P. A simple PD controller for robots with elastic joints. IEEE Trans Autom Control 1991; 36: 1208–1213
24. Tomei P.Adaptive PD controller for robot manipulators. IEEE Transactions on Robotics and Automation 1991; 7, 4: 565–570
25. Tsinias J, Kalouptsidis N. Output feedback stabilization. IEEE Trans on Automatic Control 1990; 35: 951–954
26. Willems JC. The Analysis of feedback systems. MIT Press, Cambridge, MA,1971
27. Willems JC. Dissipative dynamical systems, Part I: general theory. Arch Rational Mechanics Analysis 1972; 45: 321–351