On the Role of Passivity and Output Injection in the Output Feedback Stabilisation Problem: Application to Robot Control

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Existing dynamic output feedback stabilisation techniques of non-linear systems are applicable mainly when the unmeasurable part of the state enters linearly. Our objective in this paper is to identify a class of systems, non-linear with respect to the unmeasured variables, which are globally asymptotically stabilisable via dynamic output feedback. To achieve this characterisation we explore two routes: systems passifiability, and the combination of state feedback plus output injection. While the first line of research is motivated by the recent developments in robot control, the second one is an attempt to extend the work of the first author to the non-linear dependence case. Our contributions are twofold: first, we present a series of general theorems that identify the class mentioned above. Second, we apply these theorems to derive in a systematic manner, and using a unified framework, some recent schemes on global set point control of rigid and elastic joint robots using only position measurements.

Keywords: Global set point control; Output feedback stabilisation; Non-linear systems; Rigid and elastic joint robots

\[ \begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*} \tag{1.1} \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \), and \( f(0) = 0 \), \( h(0) = 0 \), via dynamic output feedback¹, that is

\[ \begin{align*}
u &= u(y, \sigma) \\
\dot{\sigma} &= \eta(y, \sigma)
\end{align*} \tag{1.2} \tag{1.3} \]

with \( \sigma \in \mathbb{R}^q \). In the case where the unmeasurable part of the state enters linearly, systematic techniques for stabilisation are now available, see, for example, [5,20] and references therein. Our interest here is to identify some structural properties of the system that allow us to stabilise it even in the non-linear case. To this end we explore two mechanisms for stabilisation: passifiability \( \text{à la} \) [9], and the possibility of combining the solutions of output injection and state feedback stabilisation problems as done for the linear case in [5].

Before presenting our contributions we first briefly recall some preliminary results instrumental for our further developments.

2. Theoretical Background

2.1. Passivity and Internal Stability

Following [26] we say the system (1.1) with \( m = p \) defines a passive map \( u \to y \) if there exists a \( C^0 \) storage function \( V(x) \geq 0 \), \( V(0) = 0 \) such that for all \( u, x, y \), with \( x \) solution of (1.1) we have

\[ \int_0^t u(\tau) y(\tau) d\tau \geq V[x(t)] - V[x(0)] \quad t \geq 0. \]

¹To avoid further technical discussions we will assume throughout the paper that all functions are "sufficiently" smooth.

Received 22 January 1996; Accepted in revised form 6 January 1997

Recommended by C. Samson and O. H. Bosgra

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* This research was supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica and by Agenzia Spaziale Italiana under contract R517, 1994, and by the Commission of European Communities under contract ERB CHRX CT 93-0380.
It is well known [12] that if a passive system has a C^1 storage function then
\[ L_T[x]V(x) \leq 0 \]
\[ L_T[x]V(x) = h^T(x) \]  \hspace{1cm} (2.1)
where \( L_T[\tau_j] \) denotes the Lie derivative of a function \( \tau_j \) along a vector field \( \tau_j \). With abuse of notation \( \tau_j \) can be a set of vector fields.

It is shown in [9] (see also [19]) that passive, zero-state detectable systems with proper positive definite storage functions are globally asymptotically stabilizable by static output feedbacks of the form
\[ u = -\phi(y) \]
where \( \phi \) is a smooth function such that \( \phi(0) = 0 \) and \( y^T\phi(y) > 0 \) for each \( y \neq 0 \). It is also shown in [9] that, under mild regularity conditions, the system is passifiable (via static state feedback) iff it has relative degree \( \{1, \ldots, 1\} \) and is weakly minimum phase.

### 2.2. State Feedback Plus Output Injection

It is well known that in linear systems the notion of detectability plays a fundamental role in the output feedback stabilisation problem. A lot of research has recently been devoted to define an equivalent notion for non-linear systems (e.g. [10]). The importance of this property has been established in [3–5] where it is shown that for systems linear with respect to the unmeasured variables
\[ \dot{x} = A(x) x + B(x) u \\
y = x 
\]  \hspace{1cm} (2.2)
with \( x \) the first \( p \) components of the state, the global stabilisation problem via output feedback of \( x = 0 \) can be split into two independent subproblems. First, a state-feedback stabilisation problem, which consists of globally stabilising (2.2) in \( x = 0 \) through a state-feedback control law
\[ u = u_{SF}(x), \quad u_{SF}(0) = 0 \]  \hspace{1cm} (2.3)
Second, an output injection stabilisation problem which consists of globally stabilising in \( x = 0 \)
\[ \dot{x} = A(x) x + v, \quad v \in \mathbb{R}^p \\
y = x \]
\[ \dot{v} = v_{OF}(y), \quad v_{OF}(0) = 0 \]  \hspace{1cm} (2.4)
through an output feedback
\[ \dot{v} = v_{OF}(y), \quad v_{OF}(0) = 0 \]  \hspace{1cm} (2.5)
From the solution of these separate subproblems, one derives two Lyapunov functions \( V_{SF} \) and \( V_{OF} \), which, suitably combined, give a candidate Lyapunov function for the closed-loop system. The design is completed proposing the actual dynamic output feedback control, which follows in a straightforward manner, from the solution of the state-feedback and output injection subproblems.

While the state-feedback stabilisation problem has a clear interpretation, the output injection problem has no direct physical meaning in the sense that one tries to force the term \( v_{OF}(y) \) not through the input of the system but directly into the state equations. In order to clarify its theoretical significance, let us consider for a moment the case of linear systems
\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]  \hspace{1cm} (2.6)
It is well known that in this case output feedback stabilisability is equivalent to stablisability and detectability, detectability being in turn equivalent to solvability of the output injection problem. That is, \((C, A)\) is detectable if and only if there exists a matrix \( K \) such that
\[ \text{spectrum}(A - KC) \in \mathbb{C}^+ \]  \hspace{1cm} (2.7)
which holds if and only if there exists a symmetric positive definite matrix \( P_{OF} \) such that
\[ x^T [P_{OF}(A - KC) + (A - KC)^T P_{OF}] x < 0 \]
\[ \forall x \neq 0 \]  \hspace{1cm} (2.8)
In particular (2.8) implies the following detectability condition
\[ y = 0, \quad x \neq 0 \Rightarrow x^T [P_{OF} A + A^T P_{OF}] x < 0 \]  \hspace{1cm} (2.9)
Going back to non-linear systems (2.2), since there does not exist a non-linear equivalent of condition (2.7), the output injection problem can be stated in terms of the existence of a proper Lyapunov function \( V_{OF}(x) \) and an at least \( C^1 \) function \( v_{OF}(y) \), vanishing at the origin, such that, along the trajectories of (2.4) with \( v = v_{OF}(y) \),
\[ \dot{V}_{OF}(x) = \frac{\partial V_{OF}(x)}{\partial x} [A(x) x + v_{OF}(x)] < 0 \]
\[ \forall x \neq 0 \]  \hspace{1cm} (2.10)
which is the non-linear counterpart of (2.8).

Similarly, the state-feedback problem for (2.2) can be stated in terms of the existence of a smooth proper Lyapunov function \( V_{SF}(x) \) and an at least continuous function \( u_{SF}(x) \), vanishing at the origin, such that, along the trajectories of (2.2) with \( u = u_{SF}(x) \) and \( \forall x \neq 0 \),
\[ \dot{V}_{SF}(x) = \frac{\partial V_{SF}(x)}{\partial x} [A(x) x + B(x) u_{SF}(x)] < 0 \]  \hspace{1cm} (2.11)
In [3-5] it has been shown that, if $P_{\text{SF}}$ and $P_{\text{OI}}$ are symmetric positive definite matrices and $x_t$ denotes the vector of state variables not available for feedback, the existence of two proper smooth Lyapunov functions $V_{\text{SF}}(x)$ and $V_{\text{OI}}(x)$ of the form

$$V_{\text{SF}}(x) = \frac{1}{2}x^TP_{\text{SF}}x + x^T \xi_{\text{SF}}(x_t) + \xi_{\text{SF}}(x_t)$$

$$V_{\text{OI}}(x) = \frac{1}{2}x^TP_{\text{OI}}x + x^T \xi_{\text{OI}}(x_t) + \xi_{\text{OI}}(x_t)$$

(2.12)

satisfying respectively (2.11) and (2.10) implies global output feedback stabilisability of (2.2). This result is a non-linear version of the separation principle for systems (2.2).

The construction of the stabilising output feedback controller is based on the definition of a suitable Lyapunov function $V^\sigma(x, \sigma)$, for the closed-loop system

$$\dot{x} = A(x, \sigma)x + B(x, \sigma)u(y, \sigma)$$

$$\dot{\sigma} = \eta(y, \sigma)$$

$$y = x_y$$

(2.13)

with $\sigma \in \mathbb{R}^r$, $u(y, \sigma)$ and $\eta(y, \sigma)$ to be defined. Let

$$\sigma = \begin{pmatrix} \sigma_x \\ \sigma_z \end{pmatrix}$$

denote the partition of $\sigma$ corresponding to the partition

$$x = \begin{pmatrix} x_z \\ x_y \end{pmatrix}$$

The proposed Lyapunov function candidate $V^\sigma(x, \sigma)$ for (2.13) is defined as

$$V^\sigma(x, \sigma) = \lambda V_{\text{SF}}(x) + \frac{1}{2}(x_z - \tilde{\sigma}_z)^TP_m(x_z - \tilde{\sigma}_z)$$

$$+ \xi_m(y, \sigma_x) + \xi_m(y, \sigma_z)$$

$$P_m = P_{\text{OI}} - \lambda P_{\text{SF}}$$

$$\tilde{\sigma}_z = \sigma_z - P_m^{-1}\xi_m(y, \sigma_z)$$

$$\xi_m(y, \sigma_x) = -\lambda \xi_{\text{SF}}(y) + \xi_{\text{OI}}(y) + \lambda \xi_m(y, \sigma_z)$$

$$\xi_m(y, \sigma_z) = \frac{1}{2}||y - \sigma_y||^2$$

(2.14)

with $\lambda \in \mathbb{R}^+$ such that $P_m$ is positive definite. It has been proved in [5] that choosing $u(y, \sigma)$ equal to the smooth feedback $u_{\text{SF}}(x)$, evaluated for $x_z = \sigma_z$, i.e.

$$u(y, \sigma) = u_{\text{SF}}(y, \sigma_z)$$

$\eta(y, \sigma)$ can be designed in such a way that the derivative of $V^\sigma(x, \sigma)$ along the trajectories of (2.13) is negative definite. The same result can be obtained by choosing $u(y, \sigma) = u_{\text{OI}}(y, \sigma_z)$ (see [6]).

3. Main Results

Our objective in this section is to push forward the results of Section 2 by combining passivity, output injection and LaSalle’s principle. The non-linear systems (1.1) considered for this task are based on the interconnection of a ‘good’ system with a ‘bad’ system. A ‘good’ system is one for which the non-linear separation principle of Section 2 works. A ‘bad’ system is a highly non-linear one for which neither passivity nor any other analysis tools are available to date.

Throughout the paper we will use $(\cdot)_x$, $(\cdot)_y$ to denote the measurable and unmeasurable components of the state, respectively, as well as the corresponding terms of any given vector. Similarly, by $[\cdot]$ and $[\cdot]_y$ we will denote the rows of any matrix corresponding to measurable and unmeasurable components of the state. We will consider systems of the form

$$\dot{\delta} = f_1(x) + g_1(x)u$$

$$\dot{\theta} = f_2(y)\theta_x + f_2(y) + g_2(y)u$$

$$y = h(x) = \begin{pmatrix} \xi_x \\ \theta_y \end{pmatrix}$$

(3.1)

with $x = (\delta^T \theta^T)$.

A similar ‘good-bad’ structure will be imposed on the Lyapunov functions. A ‘good’ Lyapunov function has the structure (2.12), while a ‘bad’ Lyapunov function does not necessarily have this form.

Our strategy in achieving output feedback stabilisation will be the following. First, by combining output injection and passivity, we guarantee the existence of a proper, positive definite and smooth $V^\sigma(x, \sigma)$ and smooth dynamic output feedback controller of the form

$$u = u_{\text{SF}}(y, \sigma_y) + \xi(y, \sigma)(y - \sigma_y)$$

$$\dot{\sigma} = \eta(y, \sigma)$$

(3.2)

such that, along the trajectories of (3.1) and (3.2), one has

$$\dot{V} \leq 0$$

(3.3)

i.e., the closed-loop system is stable. Our Lyapunov function candidate for the closed-loop system has the form

$$V^\sigma(x, \sigma) = \lambda V_{\text{SF}}(x) + V_m(x, \sigma)$$

(3.4)

where $\lambda > 0$ and $V_m(x, \sigma)$ is a positive semidefinite function such that $V_m(x, x) = 0$.

Finally, suitable detectability assumptions guarantee global asymptotic stability via LaSalle’s principle.
Let
\[
    f(x) = \begin{pmatrix} f_1(x) \\ f_{21}(y)\delta_z + f_{22}(y) \end{pmatrix}, \\
    g(x) = \begin{pmatrix} g_1(x) \\ g_2(y) \end{pmatrix}
\]
with \( f(0) = 0 \). As usual, \( L_\psi v \) denotes the Lie derivative of a function \( \psi \) along a vector field \( v \). With some abuse of notation, we have
\[
    L_\psi v_x = [L_\psi v]_x, \\
    L_\psi v_z = [L_\psi v]_z
\]
(3.5)
The first result of the section is a combination of passivity and output injection tools.

**Theorem 1.** Assume that

(H1) (State-feedback plus passivity) there exists a smooth, proper and positive definite function \( V_{\text{SF}}(x) \)
\[
    V_{\text{SF}}(x) = V_{\text{SF}}^p(x) + V_{\text{SF}}^q(\theta)
\]
and a smooth feedback \( u_{\text{SF}}(y, \delta_z) \), vanishing at the origin, such that, along the trajectories of the system (3.1) with \( u = u_{\text{SF}}(y, \delta_z) \), one has
\[
    L_{\psi}(x) V_{\text{SF}}(x) = \delta_y^T \\
    \dot{V}_{\text{SF}}(x) = -\alpha_{\text{SF}}(x)
\]
with positive semidefinite \( \alpha_{\text{SF}} \)

(H2) (Output injection) there exists a smooth, proper and positive definite function
\[
    V_{\text{OI}}(\theta) = \frac{1}{2} \vartheta^T P_{\text{OI}} \vartheta + \zeta_{\text{OI}}(\theta) + \xi_{\text{OI}}(\theta)
\]
and a smooth function \( v_{\text{OI}}(y) \), vanishing at the origin, such that
\[
    \frac{\partial V_{\text{OI}}(\theta)}{\partial \vartheta} [f_{21}(y)\vartheta_z + f_{22}(y) + v_{\text{OI}}(y)] = -\alpha_{\text{OI}}(\vartheta)
\]
with positive definite \( \alpha_{\text{OI}} \)

(H3) (Structure) \( L_{\psi}(x) V_{\text{SF}}^p(x) = \vartheta_1(y, \delta_z) \) and \( L_{\psi}(x) V_{\text{SF}}^q(x) \)
\[
    = \vartheta_2(y, \delta_z) \quad \text{for some functions } \vartheta_1 \text{ and } \vartheta_2.
\]

(H4) (Detectability) \( \alpha_{\text{SF}}(x) = 0 \Rightarrow \vartheta = 0 \); moreover, along the trajectories of (3.1) with \( u = u_{\text{SF}}(y, \delta_z) \)
\[
    \delta_y = 0, \quad \vartheta = 0 \Rightarrow \delta_z \to 0
\]
Under the above conditions, the dynamic output feedback controller
\[
    u = u_{\text{SF}}(y, \delta_z) + \ell(y, \sigma)(y - \sigma)
\]
\[
    \eta(y, \sigma) = P_m^{-1} \left( \lambda \frac{\partial \xi_{\text{SF}}}{\partial \vartheta} (y) - \frac{\partial \zeta_{\text{SF}}}{\partial \vartheta} (y) \eta(y, \sigma) \right) + a(y, \sigma)
\]
(3.8)
with
\[
    \eta_{\gamma}(y, \sigma) = \begin{pmatrix} 0 \\ f(y, \sigma) \end{pmatrix}, \\
    f(y, \sigma) = -\frac{1}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
\[
    \sigma_{\theta} = \sigma_{\vartheta} - P_m^{-1} \zeta_{\text{SF}}(\theta, \sigma_{\theta}), \\
    P_m = P_{\text{OI}} - \lambda P_{\text{SF}}
\]
\[
    \zeta_{\text{SF}}(\theta, \sigma_{\theta}) = -\lambda \xi_{\text{SF}}(\theta) + \zeta_{\text{OI}}(\theta) + \lambda \xi_{\text{OI}}(\theta)
\]
\[
    \xi_{\text{OI}}(\theta) = \begin{pmatrix} 0 \\ \vartheta \end{pmatrix}
\]
(3.9)
for any \( \lambda > 0 \) such that \( P_m > 0 \) and any \( k > 0 \), globally asymptotically stabilizes the trivial equilibrium.

**Proof.** Let
\[
    \sigma = \begin{pmatrix} \sigma_{\gamma} \\ \sigma_{\vartheta} \end{pmatrix}, \quad \sigma_{\theta} = \begin{pmatrix} \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix}
\]
Consider the Lyapunov function candidate
\[
    V(x, \sigma) = \lambda V_{\text{SF}}(x) + \frac{1}{2} \vartheta_z^T P_m \vartheta_z + \frac{1}{2} \vartheta^T P_m \vartheta
\]
and let
\[ u = u_{SF}(y, \tilde{\sigma}_y) + \ell(y, \sigma)(y - \sigma_y) \]
\[ \eta(y, \sigma) = \begin{pmatrix} \eta_0(y, \sigma) \\ \eta_1(y, \sigma) \end{pmatrix}, \quad \eta_0(y, \sigma) = \begin{pmatrix} \eta_{0_0}(y, \sigma) \\ \eta_{0_1}(y, \sigma) \end{pmatrix} \]  
(3.10) 

with \( \ell(y, \sigma) \) and \( \eta(y, \sigma) \) to be determined. Moreover, define \( \tilde{\sigma}_y = \sigma_z - \sigma_y \).

Using (H1)-(H3) and (3.10), by direct calculations one obtains

\[
\dot{V}^e(x, \sigma) = \lambda(\vartheta_1(y, \tilde{\sigma}_y) + \vartheta_2(y, \tilde{\sigma}_y)u_{SF}(y, \tilde{\sigma}_y)) + \lambda \delta^T \ell(y, \sigma)(y - \sigma_y) + \lambda |(\tilde{\sigma}_y^2 P_{SF} + \zeta_{SF}(\vartheta_2))(\eta_0(y, \sigma))|
\]
\[
+ \delta^T \zeta_{SF}(\vartheta_2)(L_{f_1(y, \tilde{\sigma}_y), f_2(y, \tilde{\sigma}_y), \eta_0(y, \sigma), \eta_0(y, \sigma)) \dot{\vartheta}_z
\]
\[
+ \delta^T \zeta_{SF}(\vartheta_2)(L_{f_1(y, \tilde{\sigma}_y), f_2(y, \tilde{\sigma}_y), \eta_0(y, \sigma), \eta_0(y, \sigma)) \dot{\vartheta}_z
\]
\[
+ \zeta^T \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[
+ \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[
+ \lambda \zeta_{SF}(\vartheta_2)(\eta_0(y, \sigma), \eta_0(y, \sigma)) - \lambda \delta \vartheta_2 \| \eta_0(y, \sigma) \|^2
\]
\[
\leq \lambda \vartheta_1(y, \tilde{\sigma}_y) + \vartheta_2(y, \tilde{\sigma}_y)u_{SF}(y, \tilde{\sigma}_y)
\]
\[ + \delta^T \zeta_{SF}(\vartheta_2)(L_{f_1(y, \tilde{\sigma}_y), f_2(y, \tilde{\sigma}_y), \eta_0(y, \sigma), \eta_0(y, \sigma)) \dot{\vartheta}_z
\]
\[ + \zeta^T \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[ + \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[ + \lambda \zeta_{SF}(\vartheta_2)(\eta_0(y, \sigma), \eta_0(y, \sigma)) - \lambda \delta \vartheta_2 \| \eta_0(y, \sigma) \|^2
\]
(3.12)

From (3.12), choosing \( \ell(y, \sigma) \) and \( \eta(y, \sigma) \) as in (3.8), one obtains

\[
\dot{V}^e(x, \sigma) \leq -\lambda \vartheta_1(y, \tilde{\sigma}_y, \tilde{\sigma}_z)
\]
\[ + \delta^T \zeta_{SF}(\vartheta_2)(L_{f_1(y, \tilde{\sigma}_y), f_2(y, \tilde{\sigma}_y), \eta_0(y, \sigma), \eta_0(y, \sigma)) \dot{\vartheta}_z
\]
\[ + \zeta^T \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[ + \frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[ + \lambda \zeta_{SF}(\vartheta_2)(\eta_0(y, \sigma), \eta_0(y, \sigma)) - \lambda \delta \vartheta_2 \| \eta_0(y, \sigma) \|^2
\]
(3.13)

where \( \alpha_{SF}(\vartheta_1, \tilde{\sigma}_y, \tilde{\sigma}_z) \) is the function \( \alpha_{SF}(x) \) evaluated for \( \vartheta_1 = \tilde{\sigma}_y \).

Since

\[
\frac{\partial \zeta_{SF}(\vartheta_2)}{\partial \vartheta_2}(\vartheta_2)(f_2(y, \tilde{\sigma}_y)) \eta_0(y, \sigma)
\]
\[ + \lambda \zeta_{SF}(\vartheta_2)(\eta_0(y, \sigma), \eta_0(y, \sigma)) - \lambda \delta \vartheta_2 \| \eta_0(y, \sigma) \|^2
\]
\[ + \delta \vartheta_2 \| \eta_0(y, \sigma) \|^2
\]
(3.14)

is the Hessian of (3.7) with respect to \( \vartheta_2 \). By (H2) it is easy to see that (3.14); is semidefinite negative for all \( y \) and negative definite for \( \vartheta_2 = 0 \). This, together with (3.13), implies that

\[ \dot{V}^e \leq 0 \]

i.e., the closed-loop system is Lyapunov stable.

Combining LaSalle’s invariance principle with (H3) and (3.13), we conclude global asymptotic stability. Indeed, \( \dot{V}^e = 0 \) only if \( \vartheta_2 = \tilde{\sigma}_z = 0, \vartheta_1 = 0 \) and \( \tilde{\sigma}_y = 0 \). This, by definition of \( \tilde{\sigma}_y \) and since \( \zeta_{SF}(\vartheta_2, \vartheta_2) = 0 \), implies \( \vartheta_2 = 0 \). It follows from the definition of \( \eta_0 \) that \( b(y, \sigma) = 0 \) and \( \tilde{\sigma}_y = 0 \). This, by (H4), implies \( x \to 0 \).

\[ \square \]

**Remark 1.** Potential applications of theorem 1 rely on the output feedback stabilisation of non-linear systems driven by non-linear ones, as for example robot with motor dynamics. A more general version of theorem 1 is obtained by replacing \( \tilde{\sigma}_y \) by...
\[ S(y)(\delta_y - \vartheta_1(y)u) \], where \( S(y) \) is any matrix, invertible for all \( y \), and \( L_y(x) \delta_y = \vartheta_1(y) \). Notice that when \( \vartheta_1 = 0 \) and \( S = I \), we recover theorem 1 in the above form. The case \( S \neq I \) is of interest in some other applications (e.g., output feedback stabilisation of satellites [16]).

If \( \vartheta = \{0\} \), structure assumption (H3) is automatically satisfied. On the other hand, (3.6) becomes the passivity condition (2.1) w.r.t. the output \( \dot{y} \) or, equivalently, the ‘bad’ system, with \( u = u_{SF}(y) \), is passive w.r.t. \( \dot{y} \). Thus, theorem 1 reads as follows.

**Theorem 2.** Assume \( \vartheta = \{0\} \) and that (H1) (State-feedback plus passivity) there exists a smooth, proper and positive definite function \( V_{SF}(x) \) and a smooth feedback \( u_{SF}(y) \), vanishing at the origin, such that, along the trajectories of the system (3.1) with \( u = u_{SF}(y) \), one has
\[
L_{x}(x)V_{SF}(x) = \dot{y}^T \\
\dot{V}_{SF}(x) = -\alpha_{SF}(x) 
\] (3.15)
with positive semidefinite \( \alpha_{SF} \)

(H2) (Detectability) along the trajectories of (3.1) with \( u = u_{SF}(y) \)
\[ \dot{y} = 0 \Rightarrow x \rightarrow 0 \]
Under the above conditions, the dynamic output feedback controller
\[
\eta(y, \sigma) = k(y) + \ell(y, \sigma)(y - \sigma) \\
\ell(y, \sigma) = -\frac{1}{\lambda} 
\] (3.16)
for any \( \lambda > 0 \) and \( k > 0 \), globally asymptotically stabilises the trivial equilibrium.

In the next theorem, only state-feedback plus output injection is considered. A ‘generalised’ separation principle is used to prove stability of the closed-loop system and additional detectability assumptions are invoked to infer asymptotic stability via LaSalle’s principle.

**Theorem 3.** Assume that (H1) (State-feedback) there exists smooth, proper and positive definite function \( V_{SF}(x) \) and smooth function \( u_{SF}(y, \vartheta) \), vanishing at the origin, such that
\[
V_{SF}(x) = V_{SF}^l(x) + V_{SF}^e(y, \vartheta_2) \\
V_{SF}^l(y, \vartheta_2) = \frac{1}{2} \vartheta_2^T P_{SF} \vartheta_2 + \vartheta_2^T \zeta_{SF}(y) + \xi_{SF}(y) \\
u_{SF}(y, \vartheta_2) = \vartheta_1(y) \dot{y} + \vartheta_2(y) \\
\vartheta_2(0) = 0 \\
V_{SF}^e(y, \vartheta_2) = -\alpha_{SF}(y, \vartheta_2) \\
\dot{V}_{SF}(x) = \dot{\vartheta}_1(y, \vartheta_2)(y) + \dot{\vartheta}_2(y, \vartheta_2), \\
\Phi_0(0, 0) = 0, j = 1, 2
\]
with positive semidefinite \( \alpha_{SF} \)

(H2) (Output injection) there exists smooth, proper and positive definite functions \( V_{OL}(x) \) and smooth function \( v_{OL}(y) \), vanishing at the origin, such that
\[
V_{OL}(x) = V_{OL}^l(x) + V_{OL}^e(y, \vartheta_2) \\
V_{OL}^l(y, \vartheta_2) = \frac{1}{2} \vartheta_2^T P_{OL} \vartheta_2 + \vartheta_2^T \zeta_{OL}(y) + \xi_{OL}(y) \\
\frac{\partial V_{OL}(x)}{\partial x}[f(x) + v_{OL}(y)] = -\alpha_{OL}(y, \vartheta_2) \leq 0 \\
\frac{\partial V_{OL}^e(y, \vartheta_2)}{\partial y}[f(x) + v_{OL}(y)] = \Phi_1(y, \vartheta_2) \vartheta_2 + \Phi_2(y, \vartheta_2) \\
\Phi_0(0, 0) = 0, j = 1, 2
\]

(H3) (Structure) \( g_1(x) = 0 \), \( L_{x}(x)V_{SF}^l(x) = 0 \) and \( L_{x}(x)h(x) = \vartheta_1(y, \vartheta_2) \vartheta_2 + \vartheta_2(y, \vartheta_2), \) with \( \vartheta_1(y, 0) = 0 \) for all \( y \) and \( \vartheta_2(0, 0) = 0 \)
Under the above assumptions, the system (3.1) together with the dynamic output controller
\[
u(y, \sigma) = L_{y}(y) + \vartheta_2(y) \eta_{SF}(y, \sigma) \vartheta_2 + \lambda P_m^{-1}[k(y) \\
+ (L_{y}(y) \vartheta_2) P_{SF} \sigma \\
+ P_{SF} L_{y}(y) + \vartheta_2(y) \eta_{SF}(y, \sigma) \vartheta_2 \\
+ \frac{\partial \Phi_2}{\partial y}(y)](L_{y}(y) \vartheta_2) u_{SF}(y, \sigma) 
\] (3.17)
where
\[
k(y) = P_{SF} L_{y}(y) \vartheta_2 + \frac{\partial \Phi_2}{\partial y}(y) \vartheta_2 + \Phi_0(y, 0) + (L_{y}(y) \vartheta_2) \zeta_{SF}(y)
\]
is Lyapunov stable. If, in addition, (H4) (Detectability) along the trajectories of (3.1)–(3.17),
\[
\alpha_{SF}(y, \sigma) = 0, \\
\alpha_{OL}(y, \vartheta_2 - \sigma) = 0 \Rightarrow \sigma = 0, \ y = 0
\]
and, along the trajectories of (3.1), with \( u = 0, \ y = 0 \Rightarrow x \rightarrow 0 \)
the dynamic output controller (3.17) globally asymptotically stabilises the trivial equilibrium.

**Proof.** Consider the Lyapunov function candidate
\[
V^e(x, \sigma) = \lambda V_{SF}(x) + \frac{1}{2} (\vartheta_2 - \sigma)^T P_{m}(\vartheta_2 - \sigma)
\]
Moreover, let
\[
u(y, \sigma) 
\]
and \( e_2 = \vartheta_2 - \sigma \).
Under assumptions (H1) and (H3), along the trajectories of (3.1), with \( u = 0 \), one has
\[
\dot{V}_{SF}(x) = \dot{V}_{SF}^+(x) + \dot{V}_{SF}^-(x) \\
= L_{f_5}(x) V_{SF}^+(x) + \left[ a^2 \frac{\partial V_{SF}^-(x)}{\partial y}(y) \right] L_{f_5}(x) \dot{\delta}(x) \\
+ \left[ \theta_1^T P_{SF} + \frac{\partial V_{SF}^-(x)}{\partial y}(y) L_{f_5}(x) \dot{\delta}(x) \right] \theta_3 \\
+ a_1 \left[ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \dot{\delta}(x) \right] \theta_3 \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \dot{\delta}(x) \theta_3 \\
+ \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \dot{\delta}(x) \right] \theta_3 \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \dot{\delta}(x) \theta_3 \\
= \theta_3^T k_1(y, \dot{\delta}) \theta_3 + \theta_3^T k_2(y, \dot{\delta}) + \theta_3^T k_3(y, \dot{\delta}) \\
\tag{3.19}
\]

Since \( u_{SF}(y, \delta) = q_1(y) \delta_t + q_2(y), \quad q_2(0) = 0 \) and \( L_{f_5}(x) V_{SF}^-(x) = 0 \), from (3.19) it follows that along the trajectories of (3.1), with \( u = u_{SF}(y, \delta) \), one has

\[
\dot{V}_{SF}(x) = V_{SF}^+(x) + V_{SF}^-(x) \\
= \theta_3^T k_1(y, \dot{\delta}) \theta_3 + \theta_3^T k_2(y, \dot{\delta}) + \theta_3^T k_3(y, \dot{\delta}) \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] q_1(y) \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] q_2(y) \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] q_3(y) \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] q_4(y) \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] q_5(y) \\
= \theta_3^T r_1(y, \dot{\delta}) \theta_3 + \theta_3^T r_2(y, \dot{\delta}) + r_3(y, \dot{\delta}) \\
\tag{3.20}
\]

From (H1) and (3.20) it follows that \( r_1(y, \dot{\delta}), r_2(y, \dot{\delta}) \) and \( r_3(y, \dot{\delta}) \) are independent from \( \delta \). This, in turn, implies that \( k_1(y, \dot{\delta}), k_2(y, \dot{\delta}) \) and \( k_3(y, \dot{\delta}) \) depend only on \( y \) and that, along the trajectories of (3.1), with \( u = u_{SF}(y, \delta) \), one has

\[
-\alpha_{SF}(y, \delta) = \dot{V}_{SF}(x) = \theta_3^T r_1(y, 0) \theta_3 \\
+ \theta_3^T r_2(y, 0) + r_3(y, 0) \\
\tag{3.21}
\]

Since by (H3) \( q_1(y, 0) = 0 \) for all \( y \),

\[
r_1(y, 0) = P_{SF} f_{21}(y) \\
+ g_2(y) q_1(y) \delta_t \\
+ \frac{\partial V_{SF}^-(x)}{\partial y}(y) \left( L_{f_5}(x) \theta_3 \right) q_1(y) \\
\]

In a similar way, we conclude for the output injection problem that, as a consequence of (H2), one has

\[
-\alpha_{Qi}(y, \delta) = \theta_3^T m_1(y, \delta) \theta_3 + \theta_3^T m_2(y, \delta) \\
+ m_3(y, \delta) \\
= \theta_3^T m_1(y, 0) \theta_3 + \theta_3^T m_2(y, 0) + m_3(y, 0) \\
\tag{3.22}
\]

for some functions \( m_1, m_2 \) and \( m_3 \). Note that, since \( q_1(y, 0) = 0 \) for all \( y \), \( m_1(y, 0) = P_{Qi} L_{f_5}(y) \theta_3 \). Moreover, since \( \alpha_{Qi} \) is semidefinite positive, \( m_1(y, 0) \leq 0 \) for all \( y \).

Along the trajectories of the closed-loop system (3.1)–(3.17), with \( u = u_{SF}(y, \sigma) \), one has

\[
\dot{V}(x, \sigma) = \lambda \left[ V_{SF}^+(x) + V_{SF}^+(x) \right] \\
+ (\sigma_t - \sigma)^T P_{m} (\sigma_t - \eta(y, \sigma)) \\
= \lambda \left[ \theta_3^T k_1(y, \dot{\sigma}) \theta_3 + \theta_3^T k_2(y, \dot{\sigma}) + k_3(y, \dot{\sigma}) \right] \\
+ (\sigma_t - \sigma)^T \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \right] u_{SF}(y, \sigma) \\
+ \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \right] u_{SF}(y, \sigma) \\
+ (\sigma_t - \sigma)^T \frac{\partial V_{SF}^+(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] u_{SF}(y, \sigma) \\
+ (\sigma_t - \sigma)^T \frac{\partial V_{SF}^+(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] u_{SF}(y, \sigma) \\
+ (\sigma_t - \sigma)^T \frac{\partial V_{SF}^+(x)}{\partial y}(y) \left[ L_{f_5}(x) \theta_3 \right] u_{SF}(y, \sigma) \\
+ \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \right] u_{SF}(y, \sigma) \\
= \lambda \left[ \theta_3^T r_1(y, 0) \sigma_t + \theta_3^T r_2(y, 0) + r_3(y, 0) \right] \\
+ (\sigma_t - \sigma)^T m_1(y, 0) (\sigma_t - \sigma) \\
+ (\sigma_t - \sigma)^T m_2 \left[ L_{f_5}(x) \sigma_t + \eta(y, \sigma) \right] \theta_3 \\
+ (\sigma_t - \sigma)^T \left[ L_{f_5}(x) \sigma_t + \eta(y, \sigma) \right] \theta_3 \\
+ \theta_3^T \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \right] u_{SF}(y, \sigma) \\
+ \left[ \frac{\partial V_{SF}^+(x)}{\partial y}(y) \right] u_{SF}(y, \sigma) \\
\tag{3.23}
\]

Finally, choosing \( \eta(y, \sigma) \) as in (3.17), from (3.23) one obtains

\[
\dot{V}(x, \sigma) = -\lambda \alpha_{SF}(y, \sigma) - \alpha_{Qi}(y, \dot{\sigma} - \sigma) \leq 0
\]

This proves stability of the closed-loop system. Global asymptotic stability follows as in the proof of theorem 1 from detectability assumptions (H3) and LaSalle's principle. \( \square \)
Remark 2. Note that the detectability assumption (H4) is stated on the closed-loop system (3.1)–(3.17). A deeper analysis shows that asymptotic stability of (3.1)–(3.17) can be shown as well assuming some more complicated, but directly on the open-loop system, detectability conditions.

4. Application to the Robot Stabilisation Problem

Motivated by practical considerations the control problem of robots using only position measurement has attracted the attention of several researchers. The global set point control problem for rigid robots, assuming that only joint position measurement is available, has been recently solved, simultaneously and independently, in [1,13,11,7]. The controller in all those cases consists of a gravitation compensation and a linear dynamic output feedback. Also, it is worth noting that the approach pursued there is not based on constructing an observer that tries to estimate the velocity signal. For elastic joint robots the global set point control problem was first solved in [1], (see also [14]), when only position measurement on the motor side is available, and in [2], when link position measurement is available. Again, the approach followed in those papers is not observer-based. Semiglobal tracking is addressed in [17] with link position measurements.

In this section, we use theorems 1–3 to put into a unified framework the results, for both the rigid and elastic joint robots, contained in [14,2,7]. This framework is strongly based on the combination of passivity, output injection and LaSalle’s principle. Even though we will not present in this paper any new controller for the robotic problem, our contribution is to put in perspective various existing solutions. Other comparative studies for robot controllers have been reported in [8,18] where the schemes are compared using passivity and energy shaping properties, respectively.

4.1. Rigid Robots

The equations describing the dynamics of an n-degree-of-freedom rigid robot manipulator are given by

\[ D(q) \ddot{q} + C(q, \dot{q}) \dot{q} + \tau_e(q) = \tau \]

with \( q \) the \((n \times 1)\) measurable vector of joint angular positions (we assume rotational joints), \( \tau \) the \((n \times 1)\) vector of external torques, \( D(q) \) is the positive definite inertia matrix, \( C(q, \dot{q}) \dot{q} \) is the vector of Coriolis and centrifugal torques and \( \tau_e(q) \) is the vector of gravitational torques. It is well known that the robot equations define a passive map \( u \rightarrow \dot{q} \). Furthermore, there exists a suitable choice of the matrix \( C(q, \dot{q}) \) such that the matrix

\[ \dot{D}(q) = 2C(q, \dot{q}) \]

is skew-symmetric.

Choosing as state vector \( (z_1^T \quad z_2^T) = (q^T \quad \dot{q}^T) \), the second order model can be rewritten as

\[ \begin{align*}
    \dot{z}_1 &= z_2 \\
    \dot{z}_2 &= -D^{-1}(z_1)[C(z_1, z_2)z_2 + \tau_e(z_1) - \tau]
\end{align*} \]

Define as the desired configuration \( (q_d \ 0) \). In the error coordinates \( x_1 = z_1 - q_d \), \( x_2 = z_2 \) the system becomes

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -D^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + \tau_e(x_1 + q_d) - \tau]
\end{align*} \]

which is of the form (3.1) with \( y = x_1, \theta = \{0\} \)

\[ f(x) = \begin{pmatrix}
    x_1 \\
    0
\end{pmatrix} \\
\begin{pmatrix}
    x_2 \\
    \theta
\end{pmatrix} \]

\[ g(x) = \begin{pmatrix}
    0 \\
    \theta
\end{pmatrix} \]

We will check assumptions (H1) and (H2) of theorem 2.

- **State-feedback plus passivity.** Choose

\[ V_{SF}(x) = \frac{1}{2} x_1^T D(x_1 + q_d) x_2 + \frac{1}{2} x_1^T K_p x_1 \\
  + G(x_1 + q_d) - G(q_d) - x_1^T \tau_e(q_d) \]

with \( K_p \) symmetric positive definite and \( G(\cdot) \) denoting the potential energy associated with the conservative force \( \tau_e(\cdot) \), that is, \( G(x) \) is such that

\[ \frac{\partial G(x_1 + q_d)}{\partial x_1} = \tau_e(x_1 + q_d) \]

It has been shown [24] that \( V_{SF}(x) \) is positive definite with a global minimum in \( x = 0 \) if \( K_p \) is sufficiently large, i.e.,

\[ \lambda_{\min}(K_p) > \gamma \]

with \( \gamma \) defined as

\[ \|\tau_e(q_1) - \tau_e(q_2)\| \leq \gamma \|q_1 - q_2\|, \quad \forall q_1, q_2 \in \mathbb{R}^n \]

Along the trajectories of (4.1), one has

\[ V_{SF}(x) = x_2^T [K_p x_1 + u] \]
so that, choosing
\[ u = u_{SF}(y) = -K_p x_1 \]
one has
\[ \dot{V}_{SF}(x) = -\alpha_{SF}(y) = 0 \]

One has
\[ L_{\pi(x)} V_{SF}(x) = \dot{y}^T = x_2^T \]

- Detectability. Along the trajectories of (4.1) with
  \[ u = u_{SF}(y), \text{ if } x_2 = 0 \text{ then } x = 0. \]

Therefore assumptions (H1)–(H2) of theorem 2 are satisfied and a dynamic output stabiliser which achieves global asymptotic stabilisation of the trivial equilibrium has the form (3.16).

**Remark 3.** Theorem 2 still holds with \( y - \sigma_y \) replaced by \( N y - M \sigma_y \), with \( N \) and \( M \) invertible matrices thus recovering the same structure proposed in [7,13].

### 4.2. Elastic Joint Robots

The simplified dynamic model of an elastic joint robot is given by [22]
\[
D_1(q_i) \ddot{q}_i + C(q_i, \dot{q}_i) \dot{q}_i + K(q_i - q_m) + \tau_g(q_i) = 0
\]
\[
D_2 \ddot{q}_m - K(q_i - q_m) = \tau
\]
with \( q_i \) and \( q_m \) being respectively the \((n \times 1)\) vectors of the links and rotors relative displacements, \( K = \text{diag}(k_i) > 0 \) with \( k_i \) the elastic constant of the \( i \)-th joint and \( \tau_g(q_i) \) representing the gravity forces acting on the links. We assume that no damping is present. The positive definite inertia matrix is
\[
D(q_i) = \begin{pmatrix} D_1(q_i) & 0 \\ 0 & D_2 \end{pmatrix}
\]
while \( C(q_i, \dot{q}_i) \dot{q}_i \) represent the centrifugal and Coriolis forces.

Choosing as state variables \((q_i^T \ q_i^T \ q_m^T \ \ddot{q}_m^T)\) and defining the desired state
\[
\begin{pmatrix} q_d \\ 0 \\ q_d + K^{-1} \tau_g(q_d) \\ 0 \end{pmatrix}
\]
we have in the error coordinates
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -D_1^{-1}(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + K(x_1 - x_3) + \tau_g(x_1 + q_d) - \tau_g(q_d)] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= D_2^{-1}[K(x_1 - x_3) - \tau_g(q_d) + u]
\end{align*}
\]
which is of the form (3.1) with
\[
f(x) = \begin{pmatrix} x_1 \\ x_4 \\ \tau + \tau_g(q_d) \\ 0 \\ 0 \\ 0 \\ 0 \\ D_2^{-1}K(x_1 - x_3) \end{pmatrix}
\]
\[ g(x) - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau - \tau_g(q_d) \end{pmatrix} \]
For flexible joint robots we have two possible measured variables, motor or link angular position. We will solve each one of them separately.

### 4.3. Motor Position Measurement

In this case since \( y = x_1 \) one has \( \theta = \{\emptyset\} \). We will check assumptions (H1) and (H2) of theorem 2.

- **State-feedback plus passivity.** Choose
\[
V_{SF}(x) = \frac{1}{2} \begin{pmatrix} x_1^T & x_3^T \end{pmatrix} \begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1^T & x_3^T \end{pmatrix} \begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + G(x_1 + q_d) - G(q_d) - x_1^T \tau_g(q_d)
\]
which is positive definite provided the linear spring at the joint is sufficiently stiff and \( K_p \) is sufficiently large [2]
\[
K + \frac{\partial \tau_g(q)}{\partial q} \bigg|_{q_d} > 0, \ \forall q_d
\]
\[
K_p + K - K_p + K \left( K + \frac{\partial \tau_g(q)}{\partial q} \bigg|_{q_d} \right)^{-1} > 0, \ \forall q_d
\]
Along the trajectories of the system, one has
\[
\dot{V}_{SF}(x) = x_4^T K_p x_3 + u
\]
so that, if
\[
u = u_{SF}(y) = -K_p x_3
\]
then
\[
\dot{V}_{SF}(x) = -\alpha_{SF}(y) = 0
\]
One has
\[
L_{\pi(x)} V_{SF}(x) = \dot{y}^T = x_4^T
\]
Detectability. With $u = u_{\text{SF}}(y)$, if $x_4 = 0$ then $x = 0$. Indeed, with $u = u_{\text{SF}}(y)$, if $x_4 = 0$ then $x_3 = \text{constant} \Rightarrow x_1 = \text{constant} \Rightarrow x_2 = 0$

and the following equations must hold

$$K(x_1 - x_3) + \tau_p(x_1 + q_d) - \tau_e(q_d) = 0$$
$$K(x_3 - x_1) + K_p x_3 = 0 \quad (4.2)$$

It has been shown (see [2] for example) that, under the conditions on $K_p$ above, we have that (4.2) is true only for $(x_1, x_3) = (0, 0)$. Thus, $x = 0$.

Therefore assumptions (H1) and (H2) of theorem 2 are satisfied and a dynamic output controller which achieves global asymptotic stabilisation of the trivial equilibrium has the form (3.16).

Remark 4. Note that replacing $y - \sigma_j$ by $Ny - M\sigma_j$, with $N$ and $M$ invertible matrices, one recovers the same structure proposed in [14].

4.4. Link Position Measurement

We have $y = x_1$. In this case, we will check assumptions (H1)–(H3) of theorem 3.

- State-feedback plus output injection. We have $\delta_j = x_1$, $b_2 = x_2$, $\theta_j = \{0\}$, $\theta_i = \emptyset$.

Choose

$$V_{\text{SF}}(x)$$
$$= \frac{1}{2} \begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$
$$+ \frac{1}{2} \begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{pmatrix} K & -K \\ -K & K + K_p \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$
$$+ G(x_1 + q_d) - G(q_d) - x_1^T \tau_e(q_d)$$

$$V_{\text{SF}}(x) = \frac{1}{2} x_1^T D_1(x_1 + q_d) x_2$$

Define the feedback law

$$u_{\text{SF}}(y, \theta_j) = -K_p x_3 - K_d x_4$$

with $K_p$ and $K_d$ positive definite symmetric matrices, that is

$$q_1(y) = -(K_p \quad K_d)$$
$$q_2(y) = 0$$

Along the trajectories of the system with $u = u_{\text{SF}}(y, \theta_j)$, one has

$$\dot{V}_{\text{SF}}(x) = -\alpha_{\text{SF}}(x_1, x_3, x_4) = -x_2^T K_d x_4$$

$$\dot{V}_{\text{SF}}(x) = -x_2^T [K(x_1 - x_3)$$
$$+ \tau_e(x_1 + q_d) - \tau_e(q_d)]$$

with

$$\Phi_1(x_2, y) = (x_2^T K \quad 0)$$
$$\Phi_2(x_2, y) = -x_2^T [K(x_1 + \tau_e(x_1 + q_d) - \tau_e(q_d)]$$

Choose

$$V_{\text{OF}}(x)$$
$$= \frac{1}{2} \begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{pmatrix} D_1(x_1 + q_d) & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$
$$+ \frac{1}{2} \begin{pmatrix} x_1^T & x_2^T \end{pmatrix} \begin{pmatrix} K + K_p & -K \\ -K & K \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$
$$+ G(x_1 + q_d) - G(q_d) - x_1^T \tau_e(q_d)$$

and

$$V_{\text{OF}}(x) = V_{\text{SF}}(x)$$

Again, with the assumption on $K_p$ above we have definiteness of $V_{\text{OF}}$. Let the output injection be

$$\nu_{\text{OF}}(x_1) = \begin{pmatrix} 0 \\ -D_1(x_1 + q_d)^{-1} K_p x_1 \\ 0 \\ 0 \end{pmatrix}$$

obtaining

$$\frac{\partial V_{\text{OF}}(x)}{\partial x} [f(x) + \nu_{\text{OF}}(y)] = -\alpha_{\text{OF}}(x_3, x_4) = 0$$

$$\frac{\partial V_{\text{OF}}(x)}{\partial x} [f(x) + \nu_{\text{OF}}(y)] =$$

$$-x_2^T [K(x_1 - x_3) + K_p x_1 + \tau_e(x_1 + q_d) - \tau_e(q_d)]$$

with

$$\Psi_1(x_2, y) = (x_2^T K \quad 0)$$
$$\Psi_2(x_2, y) = -x_2^T [(K + K_p) x_1$$
$$+ \tau_e(x_1 + q_d) - \tau_e(q_d)]$$

- Structure.

$$g_1(x) = 0, \quad L_2(x) V_{\text{SF}}(x) = 0$$
$$g_1(y, x_2) = 0, \quad g_2(y, x_2) = x_2$$

- Detectability. Following [3], also (H3) can be shown to hold, so that theorem 3 applies and global asymptotic stability is guaranteed by the dynamic output feedback controller (3.17), with
\[
\sigma = \begin{pmatrix} \sigma_{s1} \\ \sigma_{s4} \end{pmatrix} \\
\eta(y, \sigma) = \lambda P_m^{-\frac{1}{2}} \begin{pmatrix} K_p & -K_d \end{pmatrix} \sigma_{s4} + \begin{pmatrix} \sigma_{s4} \\ D_2^{-\frac{1}{2}}[K(x_1 - \sigma_{s1}) + w_{SP}(y, \sigma)] \end{pmatrix}
\]

5. Concluding Remarks

We have presented in this paper some results on output feedback stabilisation of systems which are nonlinear in the unmeasurable part of the state. The mechanisms for stabilisation we have explored are passivity and the solvability of output injection and state feedback stabilisation problems. In the second approach the design is decomposed into a state feedback and an output injection subproblem, and then combining the two solutions in a certain equivalence-like strategy. This procedure, advocated in [5], leads to a systematic design procedure, and sheds light on the structural obstacles for output feedback stabilisation that stem from a lack of detectability.

We have shown how the state feedback plus output injection route allows us to design output feedback regulators for rigid and flexible joint manipulators. Even though we have not presented in this paper any new controller for the robotic problem, our contribution is to put in perspective various existing solutions. This, in principle, allows for an easy comparison of their benefits and shortcomings.

Other comparative studies for robot controllers have been reported in [8,18] where the schemes are compared using passivity and energy shaping properties, respectively. Potential applications of theorems 1–3 can range from rigid satellites to induction motors.

References


