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A note on reduced order stabilizing output feedback controllers[1]

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Abstract

In this paper we show that if a certain class of nonlinear systems is globally asymptotically stabilizable through an n-dimensional output feedback controller then it can be always stabilized through an $(n-p)$-dimensional output feedback controller, where p is the number of outputs and n is the dimension of the state space. This result gives an alternative construction of reduced order controllers for linear systems, and recovers in a more general framework the concept of *dirty derivative*, used in the framework of rigid and elastic joint robots, and gives an alternative procedure for designing reduced-order controllers for nonlinear systems considered in the existing literature. © 1997 Elsevier Science B.V.

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1. Introduction

Let us consider nonlinear systems of the form

$$\dot{x} = A(x_1)x + B(x_1)u, \qquad y = x_1 \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, x_1 being the first p components of the state vector, $A(\cdot)$ and $B(\cdot)$ are smooth functions. Moreover, for our purposes we will consider the class of smooth output feedback controllers

$$u = k(y, \sigma), \qquad \dot{\sigma} = \eta(y, \sigma), \quad \sigma \in \mathbb{R}^q \tag{2}$$

with $k(0,0) = 0$, $\eta(0,0) = 0$. In this paper, we will study the following control problem.

Global stabilization problem (GS). Given $A(x_1)$ and $B(x_1)$, find, if possible, a smooth positive definite and proper function $V(x, \sigma)$ and a control law (2) such that along the trajectories of the closed-loop system (1), (2) $\dot{V}(x(t), \sigma(t))$ is negative definite.

The class of systems (1) has been studied by several researchers [1–8]. Recently, in [7] it has been shown that, under certain assumptions, the global stabilization problem for systems (1) can be split up into two subproblems: a *state-feedback* problem and an *output injection* problem. As is well known, the first

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subproblem represents the possibility of globally stabilizing (1) via *state*-feedback. In order to clarify the significance of the second subproblem, let us consider for a moment the case of *linear* systems

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{3}$$

and define the system

$$\dot{x} = Ax + v, \quad v \in \mathbb{R}^n, \qquad y = Cx \tag{4}$$

as the *associated* system of (3). In this case, the vector v is conceptually different from the control vector u and has dimension n. The problem of globally stabilizing (4) via *output* feedback is commonly known as stabilization via *output injection* of (3) and is *dual* to the stabilization problem via *state*-feedback of (3). A necessary and sufficient condition for (3) to be stabilizable via output injection is that the pair (C, A) is *detectable*. This condition implies, in particular, that the *unforced* system (3) satisfies a suitable *detectability* property.

More generally, for nonlinear systems the stabilization problem of (1) via state-feedback can be characterized through the existence of a Lyapunov function $V(x)$ which decreases along the trajectories of the closed-loop system. Similarly, the stabilization problem of (1) via output injection can be characterized through the existence of a Lyapunov function $\bar{V}(x)$ which decreases along the trajectories of the closed-loop associated system. In [7] it is shown that a nonlinear "separation" principle holds for systems (1) as long as the Lyapunov functions $V(x)$ and $\bar{V}(x)$ have the form

$$\tfrac{1}{2}z^{\mathrm{T}}Pz + z^{\mathrm{T}}\zeta(x_1) + \xi(x_1), \tag{5}$$

where z is the vector of the last $n - p$ components of x. It is easy to see for many existing results on output feedback stabilization of (1) (see [1–6], for example) that the Lyapunov function $V(x)$ does not exhibit the form (5). However, in [9] it has been shown that in these cases, by *dynamically* extending the system (1), the state-feedback problem can be always solved with a Lyapunov functions of the form

$$\tfrac{1}{2}z^{\mathrm{T}}Pz + z^{\mathrm{T}}\zeta(x_1, \sigma) + \xi(x_1, \sigma)$$

together with a *dynamic-state* feedback $u = k(x, \sigma)$, $\dot{\sigma} = \eta(x, \sigma)$. Thus, the separation principle holds also in these cases (see Remark 3).

In this paper, we show that, under the same conditions for which a system (1) is globally asymptotically stabilizable through an n-dimensional output feedback controller, one can always construct $(n - p)$-dimensional output feedback controller (Theorem 2). The procedure proposed gives an alternative construction of reduced order controllers for linear systems, recovers in a more general framework the concept of *dirty derivative*, used in [10] and in [11] for the set point control of a rigid robot and an elastic joint robot, respectively (see Remark 1) and gives an alternative procedure for designing reduced-order controllers for the nonlinear systems studied in [1–6]. Finally, we show that, under the same conditions for which an "uncertain" system (1) is "robustly" globally asymptotically stabilizable through an n-dimensional output feedback controller, one can always construct a "robust" $(n - p)$-dimensional output feedback controller (Theorem 4).

Although based on a procedure conceptually similar to the one adopted for reduced observers, we obtain stabilizing controllers which, in general, do not exhibit an observer-like structure. The meaning of this statement is clarified by the following example. Let us consider

$$\dot{y} = z + u, \qquad \dot{z} = y^2 - z^3.$$

A stabilizing smooth dynamic output feedback is [7]

$$u = k(y, \sigma_2) = -y - \sigma_2 - y\sigma_2,$$

$$\dot{\sigma}_1 = \sigma_2 + k(y, \sigma_2) + y - \sigma_1, \qquad \dot{\sigma}_2 = -\sigma_2^3 + y^2 + y(1 + y)\frac{\lambda}{1 - \lambda} + y - \sigma_1$$

for any $0 < \lambda < 1$. The structure of the above controller differs from that of a classical identity observer in the term $y(1 + y)\lambda/(1 - \lambda)$ (in the $\dot{\sigma}_2$ equation). Note that, if $\lambda \to 0$, this extra term tends to zero. A similar phenomenon happens in the case of reduced order stabilizing controllers.

2. Reduced order controllers

One of the main results of [7] states that, for the class of systems (1), if it is possible to achieve global stabilization via *state* feedback and via *output injection*, separately, then it is possible to achieve it by dynamic output feedback. The dimension of the dynamic controller is n. More precisely, one has the following result. Let z be the vector of the last $n - p$ components of x.

Theorem 1. *Assume that*

(State feedback) *there exist a smooth, proper and positive definite function* $V(x) = \frac{1}{2}z^{\mathrm{T}}Pz + z^{\mathrm{T}}\zeta(x_1) + \xi(x_1)$ *and a smooth function* $k(x)$, *vanishing at the origin, such that*

$$\frac{\partial V}{\partial x}(x)[A(x_1)x + B(x_1)k(x)]$$

is negative definite;

(Output injection) *there exist a smooth, proper and positive definite function* $\bar{V}(x) = \frac{1}{2}z^{\mathrm{T}}\bar{P}z + z^{\mathrm{T}}\bar{\zeta}(x_1) + \bar{\xi}(x_1)$ *and an (at least)* C^0 *function* $l(y)$, *vanishing at the origin, such that*

$$\frac{\partial \bar{V}}{\partial x}(x)[A(x_1)x + l(y)]$$

is negative definite.

Under the above assumptions, it is possible to find an n-dimensional control law (2) *which solves the global stabilization problem.*

Theorem 1 gives a nonlinear analogue of the fact that if a linear system (1) is both stabilizable via *state*-feedback and via *output injection* it can be asymptotically stabilized via *dynamic output*-feedback. The procedure proposed in [7] gives an explicit expression of a stabilizing output feedback controller, once the functions $V(x)$, $u(x)$ and $\bar{V}(x)$ are known.

In this note, we want to show that under the same assumptions of Theorem 1, it is always possible to construct an $(n - p)$-dimensional stabilizing output feedback controller. This is stated in the following theorem.

Theorem 2. *Under the same assumptions as for Theorem* 1, *it is possible to find an* $(n - p)$*-dimensional controller* (2) *which solves the global stabilization problem.*

Proof. Let

$$\mathscr{H}_S(x) = \frac{\partial V}{\partial x}[A(x_1)x + B(x_1)k(x)], \qquad \mathscr{H}_I(x) = \frac{\partial \bar{V}}{\partial x}[A(x_1) + l(y)].$$

By our assumptions, $\mathscr{H}_S(x)$ and $\mathscr{H}_I(x)$ are both negative definite. Moreover, $\mathscr{H}_I(x)$ exhibits the structure

$$z^{\mathrm{T}}D(x_1)z + z^{\mathrm{T}}E(x_1) + F(x_1)$$

with $F(0) = 0$ and $E(0) = 0$. Since $\mathscr{H}_I(x)$ is negative definite, the Hessian of $\mathscr{H}_I(x)$ with respect to z, i.e. $D(x_1)$, is negative definite for $x_1 = 0$ and negative semidefinite for all $x_1 \neq 0$.

Let $\sigma \in \mathbb{R}^{n-p}$. Define the following Lyapunov function candidate

$$V^e(x,\sigma) = \tfrac{1}{2}(z - \sigma + P_m^{-1}\zeta_m(x_1))^T P_m (z - \sigma + P_m^{-1}\zeta_m(x_1)) + \lambda V(x) \tag{6}$$

with

$$P_m = \bar{P} - \lambda P, \qquad \zeta_m(x_1) = -\lambda\zeta(x_1) + \bar{\zeta}(x_1)$$

and $\lambda > 0$ such that

$$\bar{P} - \lambda P > 0.$$

It is easy to see that the function (6) is smooth, positive definite and proper. Moreover, let

$$u = k(x_1, \sigma - P_m^{-1}\zeta_m(x_1)),$$

where $k(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ is the function $k(x)$ with $z = \sigma - P_m^{-1}\zeta_m(x_1)$.

One has

$$\frac{\partial V^e}{\partial x_1, z}(x,\sigma) = \lambda \frac{\partial V}{\partial x_1, z}(x_1, \sigma - P_m^{-1}\zeta_m(x_1)) + (z - \sigma + P_m^{-1}\zeta_m(x_1))^T \left(\frac{\partial \bar{\zeta}}{\partial x_1}(x_1), \bar{P}\right),$$

$$\frac{\partial V^e}{\partial \sigma}(x,\sigma) = -(z - \sigma + P_m^{-1}\zeta_m(x_1))^T P_m,$$

where $(\partial V/\partial x_1, z)(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ is the function $(\partial V/\partial x_1, z)(x_1, z)$ with $z = \sigma - P_m^{-1}\zeta_m(x_1)$.

After straightforward computations, along the trajectories of

$$\dot{x} = A(x_1)x + B(x_1)u, \qquad \dot{\sigma} = \eta(x_1, \sigma)$$

with $u = k(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ and $\eta(x_1,\sigma)$ to be determined, one obtains for some smooth $\beta(x_1,\sigma)$

$$\begin{aligned}\dot{V}^e(x,\sigma) = {} & [z - \sigma + P_m^{-1}\zeta_m(x_1)]^T D(x_1)[z - \sigma + P_m^{-1}\zeta_m(x_1)] \\ & + [z - \sigma + P_m^{-1}\zeta_m(x_1)]^T [\beta(x_1,\sigma) - P_m\eta(x_1,\sigma)] + \lambda \mathscr{H}_S(x_1, \sigma - P_m^{-1}\zeta_m(x_1)),\end{aligned}$$

where $\mathscr{H}_S(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ is the function $\mathscr{H}_S(x)$ with $z = P_m^{-1}\zeta_m(x_1)$.

Choose

$$\eta(x_1,\sigma) = P_m^{-1}\beta(x_1,\sigma). \tag{7}$$

Since $D(x_1) \leqslant 0$ for all x_1 and $\mathscr{H}_S(x_1, \sigma - P_m^{-1}\zeta_m(x_1)) \leqslant 0$, it follows that (7) is semidefinite negative. On the other hand, since $D(0) < 0$, $\dot{V}^e$ vanishes only when $z = \sigma - P_m^{-1}\zeta_m(x_1)$, $x_1 = 0$ and $\sigma = P_m^{-1}\zeta_m(x_1)$ or, equivalently, since $\zeta_m(0) = 0$, when $z = \sigma = 0$ and $x_1 = 0$. This, by standard arguments on Lyapunov functions, proves our theorem. □

Remark 1. The procedure proposed in the proof of Theorem 2 recovers in a more general framework the concept of *dirty derivative*, used in [9] and in [10] for the set point control of a rigid robot and an elastic joint robot, respectively. Indeed, let $u(x_1, w)$ be a smooth function, vanishing at the origin, and C, D and E be matrices such that the system

$$\dot{x} = A(x) + B(x)u(x_1, w), \qquad \dot{w} = Cw + D\dot{x}_1 + Ex_1, \quad w \in \mathbb{R}^{n-p} \tag{8}$$

is globally asymptotically stable at the origin. Note that the *linear* filter $\dot{w} = Cw + D\dot{x}_1 + Ex_1$ requires the knowledge of $\dot{x}_1$. However, one can implement the same filter without knowing $\dot{x}_1$ as follows. Let $\sigma = w - Dx_1$. In the new coordinates the system (8) rewrites as

$$\dot{x} = A(x) + B(x)k(x_1, \sigma + Dx_1), \qquad \dot{\sigma} = C\sigma + (CD + E)x_1$$

so that $u = k(x_1, \sigma + Dx_1)$, $\dot{\sigma} = C\sigma + (CD + E)x_1$ is a stabilizing dynamic output feedback [10, 11]. It is possible to see that the dirty derivative is intrinsically related to reduced order stabilizing controllers. Let $k(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$, $\zeta_m(x_1)$, P_m and $\eta(x_1, \sigma)$ as in the proof of Theorem 2. It is easy to see that $\eta(x_1, \sigma)$, for each x_1, is a *linear* function of σ; thus, one can write

$$\dot{\sigma} = \tilde{C}(x_1)\sigma + \tilde{D}(x_1) \tag{9}$$

for some $\tilde{C}(x_1)$ and $\tilde{D}(x_1)$. Moreover, define $w = \sigma - P_m^{-1}\zeta_m(x_1)$. In the new coordinates the system (1), with $u = k(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ and $\eta(x_1, \sigma)$ as in the proof of Theorem 2, is given by

$$\dot{x} = A(x_1)x + B(x_1)k(x_1, w),$$
$$\dot{w} = \tilde{C}(x_1)w + P_m^{-1}\,\partial\zeta_m/\partial x_1(x_1)\dot{x}_1 + \tilde{C}(x_1)P_m^{-1}\zeta_m(x_1) + \tilde{E}(x_1).$$

In this case we obtain a *nonlinear filter* of the form $\dot{w} = C(x_1)w + D(x_1)\dot{x}_1 + E(x_1)$. This in some sense generalizes to a nonlinear setting the concept of dirty derivative.

A converse procedure can be outlined. Indeed, if

$$\dot{x} = A(x) + B(x)k(x_1, w), \qquad \dot{w} = C(x_1)w + D(x_1)\dot{x}_1 + E(x_1)$$

is globally asymptotically stable at the origin and if one can find a function $\tilde{\zeta}(x_1)$ such that

$$\frac{\partial\tilde{\zeta}}{\partial x_1}(x_1) = D(x_1)$$

then $u = k(x_1, \sigma - \tilde{\zeta}(x_1))$, $\dot{\sigma} = C(x_1)\sigma - C(x_1)\tilde{\zeta}(x_1) + E(x_1)$ is a globally stabilizing output feedback controller of the form (9).

Remark 2. The procedure proposed in the proof of Theorem 2 is similar, in principle, to the well-known one for designing reduced observers. Indeed, for constructing reduced observers one considers as a new "estimate" of z the quantity $\sigma - \tilde{L}x_1$ and choose $\tilde{L}$ in such a way that, under the assumption that the system is observable, a certain matrix has all its eigenvalues in the left-half open complex plane. This guarantees, among other things, that $z \to \sigma - \tilde{L}x_1$ as $t \to \infty$. In our procedure, one may consider as an "estimate" of z the quantity $\sigma - P_m^{-1}\zeta_m(x_1)$.

3. Robust reduced order controllers

Let us consider the *uncertain* nonlinear system

$$\dot{x} = A(x_1)x + B(x_1)u + \Delta A(x, t)x + \Delta B(x, t)u, \qquad y = x_1 \tag{10}$$

with $\Delta A(x, t)$ and $\Delta B(x, t)$ *unknown* functions, x_1 being the first p components of x and z being the last $n - p$ ones. We will focus our attention on the following control problem, under the same class of controllers (2). Assume that $\Delta A(x, t)$ and $\Delta B(x, t)$ are continuous functions of x and t (but weaker assumptions can be stated) and that

$$\|\Delta A(x, t)x + \Delta B(x, t)u\|^2 \leqslant \|\Phi(x_1)x + \Psi(x_1)u\|^2$$

for all x, u and t and for some *known* smooth functions $\Phi(x_1)$ and $\Psi(x_1)$.

Global robust stabilization problem (GRS). Given $A(x_1)$, $B(x_1)$, $\Phi(x_1)$ and $\Psi(x_1)$, find, if possible, a smooth positive definite and proper function $V(x, \sigma)$, a C^0 positive definite function ϱ and a control law (2) such that, along the trajectories of the closed-loop system (10)–(2), $\dot{V}(x(t), \sigma(t), t) \leqslant -\varrho(x(t), \sigma(t))$.

In [7, 8], for the class of systems (10), it has been shown that, if it is possible to achieve robust global stabilization via *state* feedback and via *output injection*, separately, and if a coupling condition is satisfied between the two Lyapunov functions $V(x)$ and $\bar{V}(x)$, then it is possible to achieve robust stabilization via dynamic output feedback. The dimension of the dynamic controller is n. More precisely, one has the following result.

Theorem 3. *Assume that*

(Robust state feedback) *there exist a smooth, proper and positive definite function* $V(x) = \frac{1}{2}z^{\mathrm{T}}Pz + z^{\mathrm{T}}\zeta(x_1) + \xi(x_1)$ *and a smooth function* $k(x)$, *vanishing at the origin, such that*

$$\frac{\partial V}{\partial x}(x)[A(x_1)x + B(x_1)k(x)] + \frac{1}{2}\frac{\partial V}{\partial x}(x)\frac{\partial V^{\mathrm{T}}}{\partial x}(x) + \tfrac{1}{2}\|\Phi(x_1)x + \Psi(x_1)k(x)\|^2$$

is negative definite;

(Robust output injection) *there exist a smooth, proper and positive definite function* $\bar{V}(x) = \frac{1}{2}z^{\mathrm{T}}\bar{P}z + z^{\mathrm{T}}\bar{\zeta}(x_1) + \bar{\xi}(x_1)$ *and an (at least)* C^0 *function* $l(x_1)$, *vanishing at the origin, such that*

$$\frac{\partial \bar{V}}{\partial x}(x)[A(x_1)x + l(x_1)] + \frac{1}{2}\frac{\partial \bar{V}}{\partial x}(x)\frac{\partial \bar{V}^{\mathrm{T}}}{\partial x}(x) + \tfrac{1}{2}\|\Phi(x_1)x\|^2$$

is negative definite;

(Coupling) $\bar{P} - P > 0$.

Under the above assumptions, it is possible to find an n-dimensional control law (2) *which solves the global robust stabilization problem.*

Similar arguments to those used in the proof of Theorem 2 lead to the following result.

Theorem 4. *Under the same assumptions of Theorem* 3, *it is possible to find an* $(n-p)$*-dimensional control law* (2) *which solves the global robust stabilization problem.*

Proof. Let $\sigma \in \mathbb{R}^{n-p}$. Define the following Lyapunov function candidate

$$V^e(x,\sigma) = \tfrac{1}{2}(z - \sigma + P_m^{-1}\zeta_m(x_1))^{\mathrm{T}}P_m(z - \sigma + P_m^{-1}\zeta_m(x_1)) + V(x) \tag{11}$$

with

$$P_m = \bar{P} - P, \qquad \zeta_m(x_1) = -\zeta(x_1) + \bar{\zeta}(x_1).$$

Moreover, let

$$u = k(x_1, \sigma - P_m^{-1}\zeta_m(x_1)),$$

where $k(x_1, \sigma - P_m^{-1}\zeta_m(x_1))$ is the function $k(x)$ with $z = \sigma - P_m^{-1}\zeta_m(x_1)$ and

$$\mathscr{H}_S(x) = \frac{\partial V}{\partial x}(x)[A(x_1)x + B(x_1)k(x)] + \frac{1}{2}\frac{\partial V}{\partial x}(x)\frac{\partial V^{\mathrm{T}}}{\partial x}(x) + \tfrac{1}{2}\|\Phi(x_1)x + \Psi(x_1)k(x)\|^2,$$

$$\mathscr{H}_I(x) = \frac{\partial \bar{V}}{\partial x}(x)[A(x_1)x + l(x_1)] + \frac{1}{2}\frac{\partial \bar{V}}{\partial x}(x)\frac{\partial \bar{V}^{\mathrm{T}}}{\partial x}(x) + \tfrac{1}{2}\|\Phi(x_1)x\|^2.$$

Since

$$\begin{aligned}&\frac{\partial V}{\partial x}(x)[A(x_1)x + B(x_1)k(x) + \Delta A(x,t)x + \Delta B(x,t)k(x)]\\ &\quad\leqslant \frac{\partial V}{\partial x}(x)[A(x_1)x + B(x_1)k(x)] + \frac{1}{2}\frac{\partial V}{\partial x}(x)\frac{\partial V^{\mathrm{T}}}{\partial x}(x) + \tfrac{1}{2}\|\Phi(x_1)x + \Psi(x_1)k(x)\|^2\end{aligned}$$

from this point onwards, one proceeds exactly as in the proof of Theorem 2. □

As already pointed out, for many existing results on output feedback stabilization (see [1–6], for example) the Lyapunov function $V(x)$ does not exhibit the form (5). In [9] it has been shown that in these cases, by *dynamically* extending (1), the state-feedback problem can be always solved with a Lyapunov function of the form

$$\tfrac{1}{2}z^{\mathrm{T}}Pz + z^{\mathrm{T}}\zeta(x_1,\sigma) + \xi(x_1,\sigma) \tag{12}$$

together with a state-feedback $u = k(x,\sigma)$, $\dot{\sigma} = \eta(x,\sigma)$. Thus, the separation principle can be applied to the dynamically extended system.

For lack of space, we will shortly outline the construction of the function (12) in the case of the systems considered in [2], with no zero dynamics, i.e.

$$\dot{x}_1 = x_2 + \Delta f_1(x), \ \ldots, \ \dot{x}_r = u + \Delta f_r(x), \qquad y = x_1 \tag{13}$$

with $x = (x_1, \ldots, x_r)$ and *unknown* functions Δf_j satisfying $\sum_{j=1}^{r}\|\Delta f_j(x)\|^2 \leqslant \bar{\delta}^2(|x_1|)$ for all x and for some $\bar{\delta} \in \mathscr{K}_\infty$, (bounded by) a linear function near the origin (see [9] for the discussion in the case of nontrivial zero dynamics). We will denote by $\tilde{X}_k$ the vector $(x_1, \tilde{x}_2, \ldots, \tilde{x}_k)$, by $\tilde{Z}_k$ the vector $(\tilde{x}_2, \ldots, \tilde{x}_k)$ and by $\tilde{\Sigma}_k$ the vector $(\tilde{\sigma}_2, \ldots, \tilde{\sigma}_k)$ ($\tilde{\sigma}_j$ and $\tilde{x}_j$ will be defined later).

The constructive procedure is based on the combination of a step-by-step procedure with $\mathscr{H}_\infty$ arguments. This procedure gives an alternative algorithm for designing reduced-order output feedback stabilizing controllers for classes of nonlinear systems considered in the literature (see [2] for comparisons).

First of all, we perform a global coordinate change on (1), which will remarkably simplify the step-by-step procedure, For, it is always possible to find a positive definite function $\bar{V}(x) = \frac{1}{2}x^{\mathrm{T}}\bar{W}x$, with $\bar{W}$ symmetric and positive definite, and a smooth function $\bar{q}(y)$, $\bar{q}(0) = 0$, such that

$$\frac{\partial \bar{V}}{\partial x}(x)\left[\begin{pmatrix} x_2 \\ \vdots \\ x_r \\ 0 \end{pmatrix} + \bar{q}(y)\right] + \frac{1}{2}\frac{\partial \bar{V}}{\partial x}(x)\frac{\partial \bar{V}^{\mathrm{T}}}{\partial x}(x) + \bar{\delta}^2(|x_1|) \leqslant -\bar{\varepsilon}\bar{V}(x) \tag{14}$$

for some $\bar{\varepsilon} \in \mathbb{R}^+$.

Let $\tilde{x} = Tx$ with

$$T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & a_r & 1 \end{pmatrix}.$$

It is easy to see that one can choose the numbers $a_j \in \mathbb{R}$ in such a way that $\tilde{W} = T^{\mathrm{T}}\bar{W}T$, with $\tilde{W} > 0$ having the form

$$\tilde{W} = \begin{pmatrix} \tilde{w}_{11} & 0 & \cdots & 0 \\ 0 & \tilde{w}_{22} & \cdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \tilde{w}_{rr} \end{pmatrix}.$$

The considered system of coordinates (13) has the form

$$\begin{aligned} \dot{x}_1 &= \tilde{x}_2 + \tilde{a}_{11}x_1 + \Delta\tilde{f}_1(x), \\ \dot{\tilde{x}}_2 &= \tilde{x}_3 + \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 + \Delta\tilde{f}_2(x), \\ &\vdots \\ \dot{\tilde{x}}_r &= \tilde{a}_{rr}x + \tilde{a}_{r,r-1}\tilde{x}_{r-1} + \cdots + \tilde{a}_{r2}\tilde{x}_2 + \tilde{a}_{r1}x_1 + u + \Delta\tilde{f}_r(x), \\ y &= x_1, \end{aligned} \tag{15}$$

where $\Delta\tilde{f}_j(x)$, $j = 1, \ldots, r$, satisfies $\sum_{j=1}^{r}\|\Delta\tilde{f}_j(x)\|^2 \leqslant \tilde{\delta}^2(|x_1|)$ for all x and $\tilde{\delta} = H\bar{\delta}$ for some $H > 0$.

We are left to prove that there exist smooth functions $u(y, \Sigma)$ and $F(y, \Sigma)$, vanishing at the origin, and a smooth, positive definite and proper function $V(x, \Sigma)$ such that

$$\frac{\partial V}{\partial x, \Sigma}(x, \Sigma)\begin{pmatrix} \tilde{x}_2 + \tilde{a}_{11}x_1 \\ \vdots \\ \tilde{x}_r + \tilde{a}_{r-1,r-1}\tilde{x}_{r-1} + \tilde{a}_{r-1,r-2}\tilde{x}_{r-2} + \cdots + \tilde{a}_{r-1,2}\tilde{x}_2 + \tilde{a}_{r-1,1}x_1 \\ \tilde{a}_{rr}x_r + \tilde{a}_{r,r-1}\tilde{x}_{r-1} + \cdots + \tilde{a}_{r2}\tilde{x}_2 + \tilde{a}_{r1}x_1 + u(y, \Sigma) \\ F(x_1, \Sigma) \end{pmatrix}$$

$$+ \frac{1}{2}\frac{\partial V}{\partial x}(x, \Sigma)TT^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial x}(x, \Sigma) + \frac{\|T^{-1}\|}{2}\tilde{\delta}^2(|x_1|) \leqslant -\alpha V(x, \Sigma) \tag{16}$$

for some $\alpha \in \mathbb{R}^+$. Since, along the trajectories of (15) with $u = u(y, \Sigma)$ and $\dot{\sigma} = F(y, \Sigma)$, $\dot{V}$ is less than or equal to the left-hand part of (16), and (16) would clearly imply global asymptotic stability of the closed-loop system (15).

For satisfying (11), we will proceed by steps. We will prove only the first three steps of the procedure, since any other step can be performed in the same way as the third one. Moreover, we can assume that $\tilde{\delta}^2$ is a smooth function of x_1 and define $k = \frac{1}{2}\|T^{-1}\|$.

Robust output feedback problem #1. If, for any smooth function γ, linear near the origin and vanishing at the origin, $V_1(x_1) = \gamma^2(x_1)$ and $u_1(x_1) = x_1L_1(x_1) - \tilde{a}_{11}x_1$ then

$$\frac{\partial V_1}{\partial x_1}(x_1)u_1(x_1) + \frac{1}{2}\frac{\partial V_1^2}{\partial x_1}(x_1) + k\tilde{\delta}^2(x_1) \leqslant -\alpha_1 V_1(x_1)$$

for some $\alpha_1 \in \mathbb{R}^+$ and for some smooth $L_1(x_1) < 0$.

Let

$$T_2 = \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}.$$

Robust state feedback problem #2. If

$$V_{S2}(\tilde{X}_2) = V_1(x_1) + \frac{1}{2l_2}(\tilde{x}_2 - u_1(x_1))^2$$

and

$$u_2(\tilde{X}_2) = (\tilde{x}_2 - u_1(x_1))L_2(x_1) - \tilde{a}_{22}\tilde{x}_2 - \tilde{a}_{21}x_1$$

with $l_2 > 1$, then

$$\frac{\partial V_{S2}}{\partial \tilde{X}_2}(\tilde{X}_2)\begin{pmatrix} \tilde{x}_2 \\ \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 + u_2(x_1) \end{pmatrix} + \frac{1}{2}\frac{\partial V_{S2}}{\partial \tilde{X}_2}(\tilde{X}_2)T_2T_2^{\mathrm{T}}\frac{\partial V_{S2}^{\mathrm{T}}}{\partial \tilde{X}_2}(\tilde{X}_2) + k\tilde{\delta}^2(x_1) \leqslant -\alpha_{S2}V_2(\tilde{X}_2)$$

for some $\alpha_{S2} \in \mathbb{R}^+$ and smooth $L_2(x_1) < 0$.

Robust output injection problem #2. It is always possible to find a positive definite function $V_{I2}(\tilde{X}_2) = \frac{1}{2}[\tilde{w}_{11}x_1^2 + \tilde{w}_{22}\tilde{x}_2^2]$ and a smooth function $\tilde{q}_2(y)$, $\tilde{q}_2(0) = 0$, such that

$$\frac{\partial V_{I2}}{\partial \tilde{X}_2}(\tilde{X}_2)\left[\begin{pmatrix} \tilde{Z}_2 \\ \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 \end{pmatrix} + \tilde{q}_2(y)\right] + \frac{1}{2}\frac{\partial V_{I2}}{\partial \tilde{X}_2}(\tilde{X}_2)T_2T_2^{\mathrm{T}}\frac{\partial V_{I2}^{\mathrm{T}}}{\partial \tilde{X}_2}(\tilde{X}_2) + k\tilde{\delta}^2(x_1) \leqslant -\varepsilon_2 V_{I2}(\tilde{X}_2)$$

for some $\varepsilon_2 > 0$.

Robust output feedback problem #2. Let P_{I2}, P_{S2}, $\zeta_{I2}(x_1)$, $\zeta_{S2}(x_1)$, $\xi_{I2}(x_1)$ and $\xi_{S2}(x_1)$ be such that

$$V_{I2}(\tilde{X}_2) = \tfrac{1}{2}\tilde{Z}_2^2 P_{I2} + \tilde{Z}_2\zeta_{I2}(x_1) + \xi_{I2}(x_1)$$

and

$$V_{S2}(\tilde{X}_2) = \tfrac{1}{2}\tilde{Z}_2^2 P_{S2} + \tilde{Z}_2\zeta_{S2}(x_1) + \xi_{S2}(x_1).$$

Note that $\zeta_{I2}(x_1) = 0$.

Pick $l_2 > 1$ large enough so that

$$P_{m2} = P_{I2} - P_{S2} > 0. \tag{17}$$

From (12), it follows that the function $V_2(\tilde{X}_2, \tilde{\Sigma}_2) = V_{S2}(\tilde{X}_2) + V_{m2}(\tilde{X}_2, \tilde{\Sigma}_2)$ with

$$V_{m2}(\tilde{X}_2, \tilde{\Sigma}_2) = \tfrac{1}{2}\tilde{e}_2^2(\tilde{X}_2, \tilde{\Sigma}_2)P_{m2}, \qquad \tilde{e}_2(\tilde{X}_2, \tilde{\Sigma}_2) = \tilde{x}_2 - \tilde{\sigma}_2,$$

$$\tilde{\sigma}_2 = \sigma_2 - P_{m2}^{-1}\zeta_{m2}(x_1), \qquad \zeta_{m2}(x_1, \sigma_1) = \zeta_{I2}(x_1) - \zeta_{S2}(x_1)$$

is smooth, proper and positive definite.

Denote by $u_2(y, \tilde{\Sigma}_2)$ the function $u_1(\tilde{X}_2)$ with $\tilde{x}_2 = \tilde{\sigma}_2$. By direct calculations, we obtain

$$\frac{\partial V_2}{\partial \tilde{X}_2, \tilde{\Sigma}_2}(\tilde{X}_2, \tilde{\Sigma}_2)\begin{pmatrix} \tilde{x}_2 + \tilde{a}_{11}x_1 \\ \tilde{a}_{21}\tilde{x}_2 + \tilde{a}_{21}x_1 + u_2(y, \tilde{\Sigma}_2) \\ F_{21}(y, \tilde{\Sigma}_2) \end{pmatrix} + \frac{1}{2}\frac{\partial V_2}{\partial \tilde{X}_2}(\tilde{X}_2, \tilde{\Sigma}_2)T_2T_2^{\mathrm{T}}\frac{\partial V_2^{\mathrm{T}}}{\partial \tilde{X}_2}(\tilde{X}_2, \tilde{\Sigma}_2) + k\bar{\delta}^2(|x_1|)$$
$$\leqslant -\alpha_2 V_2(\tilde{X}_2, \tilde{\Sigma}_2)$$

with $\alpha_2 = \min\{\alpha_{S2}, \varepsilon_2\}$ and for some smooth $F_{21}(y, \tilde{\Sigma}_2)$. Moreover, by construction $V_2(\tilde{X}_2, \tilde{\Sigma}_2) \geqslant \gamma^2(x_1)$.

Let

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ 0 & a_3 & 1 \end{pmatrix}.$$

Robust state feedback problem #3. As in step #2, if

$$V_{S3}(\tilde{X}_3, \tilde{\Sigma}_2) = V_2(\tilde{X}_2, \tilde{\Sigma}_2) + \frac{1}{2l_3}(\tilde{x}_3 - u_2(x_1, \tilde{\Sigma}_2))^2$$

and

$$u_3(\tilde{X}_3, \tilde{\Sigma}_2) = (\tilde{x}_3 - u_2(x_1, \tilde{\Sigma}_2))L_3(x_1, \tilde{\Sigma}_2) - \tilde{a}_{33}\tilde{x}_3 - \tilde{a}_{32}x_2 - \tilde{a}_{31}x_1$$

with $l_3 > 1$, then

$$\frac{\partial V_{S3}}{\partial \tilde{X}_3, \tilde{\Sigma}_2}(\tilde{X}_3, \tilde{\Sigma}_2)\begin{pmatrix} \tilde{x}_2 + \tilde{a}_{11}x_1 \\ \tilde{x}_3 + \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 \\ \tilde{a}_{33}\tilde{x}_3 + \tilde{a}_{32}\tilde{x}_2 + \tilde{a}_{31}x_1 + u_3(\tilde{X}_3, \Sigma_2) \\ F_{21}(y, \tilde{\Sigma}_2) + F_{22}(y, \Sigma_3) \end{pmatrix}$$
$$+\frac{1}{2}\frac{\partial V_{S3}}{\partial \tilde{X}_3}(\tilde{X}_3, \tilde{\Sigma}_2)T_3T_3^{\mathrm{T}}\frac{\partial V_{S3}^{\mathrm{T}}}{\partial \tilde{X}_3}(\tilde{X}_3, \tilde{\Sigma}_2) + k\bar{\delta}^2(|x_1|) \leqslant -\alpha_{S3}V_{S3}(\tilde{X}_3, \tilde{\Sigma}_2) \tag{18}$$

for some $\alpha_{S3} \in \mathbb{R}^+$ and smooth $L_3(x_1, \tilde{\Sigma}_2) < 0$ and for any smooth $F_{22}(y, \Sigma_3)$.

Robust output injection problem #3. As in step #2, it is always possible to find a positive definite function $V_{I3}(\tilde{X}_3) = \frac{1}{2}[\tilde{w}_{11}x_1^2 + \tilde{w}_{22}\tilde{x}_2^2 + \tilde{w}_{33}\tilde{x}_3^2]$ and a smooth function $\tilde{q}_3(y)$, $\tilde{q}_3(0) = 0$, such that

$$\frac{\partial V_{I3}}{\partial \tilde{X}_3}(\tilde{X}_3)\left[\begin{pmatrix} \tilde{x}_2 + \tilde{a}_{11}x_1 \\ \tilde{x}_3 + \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 \\ \tilde{a}_{33}\tilde{x}_3 + \tilde{a}_{32}\tilde{x}_2 + \tilde{a}_{31}x_1 \end{pmatrix} + \tilde{q}_3(y)\right]\frac{1}{2}\frac{\partial V_{I3}}{\partial \tilde{X}_3}(\tilde{X}_3)T_3T_3^{\mathrm{T}}\frac{\partial V_{I3}^{\mathrm{T}}}{\partial \tilde{X}_3}(\tilde{X}_3) + k\tilde{\delta}^2(x_1)$$

$$\leqslant -\varepsilon_3 V_{I3}(\tilde{X}_3) \tag{19}$$

for some $\varepsilon_3 > 0$.

Robust output feedback problem #3. Pick $l_3 > 1$ large enough so that

$$P_{m3} = \tilde{w}_{33} - \frac{1}{l_3} > 0. \tag{20}$$

From (20), it follows that the function $V_3(\tilde{X}_3, \tilde{\Sigma}_3) = V_{S3}(\tilde{X}_3, \tilde{\Sigma}_2) + V_{m3}(\tilde{X}_3, \tilde{\Sigma}_3)$, with

$$V_{m3}(\tilde{X}_3, \tilde{\Sigma}_3) = \tfrac{1}{2}\tilde{e}_3^2(\tilde{X}_3, \tilde{\Sigma}_3)P_{m3}, \qquad \tilde{e}_3(\tilde{X}_3, \tilde{\Sigma}_3) = \tilde{x}_3 - \tilde{\sigma}_3,$$

$$\tilde{\sigma}_3 = \sigma_3 - P_{m3}^{-1}\zeta_{m3}(x_1, \tilde{\Sigma}_2), \qquad \zeta_{m3}(x_1, \tilde{\Sigma}_2) = -\frac{1}{l_3}u_2(x_1, \tilde{\Sigma}_2)$$

is smooth, proper and positive definite.

Moreover, denote by $u_3(x_1, \tilde{\Sigma}_3)$ the function $u_3(\tilde{X}_3, \tilde{\Sigma}_2)$ with $\tilde{Z}_3 = \tilde{\Sigma}_3$. By tedious but straightforward calculations, one obtains

$$\frac{\partial V_3}{\partial \tilde{X}_3, \tilde{\Sigma}_3}(\tilde{X}_3, \tilde{\Sigma}_3)\begin{pmatrix} \tilde{x}_2 + \tilde{a}_{11}x_1 \\ \tilde{x}_3 + \tilde{a}_{22}\tilde{x}_2 + \tilde{a}_{21}x_1 \\ \tilde{a}_{33}\tilde{x}_3 + \tilde{a}_{32}\tilde{x}_2 + \tilde{a}_{31}x_1 + u_3(x_1, \tilde{\Sigma}_3) \\ F_{21}(y, \tilde{\Sigma}_2) + F_{22}(y, \tilde{\Sigma}_3) \\ F_{31}(y, \tilde{\Sigma}_3) \end{pmatrix}$$

$$+\frac{1}{2}\frac{\partial V_3}{\partial \tilde{X}_3}(\tilde{X}_3, \tilde{\Sigma}_3)T_3T_3^{\mathrm{T}}\frac{\partial V_3^{\mathrm{T}}}{\partial \tilde{X}_3}(\tilde{X}_3, \tilde{\Sigma}_3) + k\tilde{\delta}^2(x_1) \leqslant -\alpha_3 V_3(\tilde{X}_3, \tilde{\Sigma}_3)$$

with $\alpha_3 = \min\{\alpha_{S3}, \varepsilon_3\}$ and for some smooth $F_{22}(y, \tilde{\Sigma}_3)$ and $F_{31}(y, \tilde{\Sigma}_3)$. Any following step can be performed as in step #3.

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