



A separation result for systems with feedback constraints

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Abstract

In this paper we propose a certainty equivalence principle for the stabilization of nonlinear systems via measurement feedback, which clarifies the connection between the solution of a couple of Hamilton Jacobi inequalities (HJIs) and the design constraints imposed on the control and the estimation error fed back in the control loop by the observer itself. Once a solution of these HJIs has been found a measurement feedback controller can be directly implemented. This controller has different features with respect to classical controllers: in classical control schemes an observer consists of a “copy” of the system plus a term proportional to the error between the actual measurement and the “estimated” measurement. Here, we allow a term which is a *nonlinear* function of this error. This result is particularly powerful in conjunction with step-by-step strategies as illustrated by an application to feedforward systems.

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1. Introduction

Any n -dimensional system can be viewed as the interconnection of n one-dimensional systems, each one with a *state* z , a *control* v , a *endogenous measurement* μ , *uncertainty* Ψ , some *exogenous inputs* χ , some *exogenous measurements* v and a set of *constraints* \mathcal{M} among state, inputs and measurements. An *exogenous input* can be considered as an “exogenous disturbance” affecting the system itself and as such it cannot be manipulated to control the system. We will distinguish the inputs of a system into *control (or endogenous) inputs* v and *exogenous inputs* χ . We denote the inputs altogether, endogenous and exogenous, by ι .

An exogenous input of a system may be as well available as a feedback signal although not being a measurement of the system itself, since it is available as a measurement from other interconnected systems. Thus, we will distinguish the measurements of a system into *endogenous measurements* μ (available from the system itself) and *exogenous measurements* v (available from other systems). We denote the measurements altogether, endogenous and exogenous, by ζ .

It may happen that state, inputs and measurements are related to each other by a set of *constraints* \mathcal{M} such as input/measurement nonlinearities (saturation, dead zones,...) or differential constraints (differential equations satisfied by exogenous inputs or measurements,...). We will refer to the set of constraints by saying that $(x, \iota, \zeta) \in \mathcal{M}$.

Finally, model uncertainties can be accommodated into a vector Ψ , which denotes the “uncertainty” of a system.

Once a system is considered as the interconnection of simpler dynamics, the problem is to control each one of these simpler dynamics through simpler controllers and to design from these controller a more complex controller for the overall system. In this paper we give a certainty equivalence principle for the control via measurement feedback of a system of the form

$$\dot{x} = Ax + B_2v + B_1(\zeta)\Psi, \quad \mu = C_2x + C_1(\zeta)\Psi, \quad x(0) = x_0 \quad (1)$$

taking into account *feedback constraints*. The main problem is to give a characterization of the “correction term” $G(\mu - C_2\sigma)$ in an observer-based controller $v = F(\sigma, \zeta)$, $\dot{\sigma} = H(\sigma, \zeta) + G(\mu - C_2\sigma, \zeta)$ as a function of the solution of a Hamilton Jacobi inequality (HJI) as it has been done in the case of state feedback controllers (see [2] and Section 3 of this paper). Feedback constraints are important to take into account

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the dynamics of actuators and sensors in the control loop or to modulate the quantity of noise or uncertainty fed back in the control loop by the controller itself through the measurements and HJIs are a natural tool for coping with feedback constraints (see motivating examples in Section 3). In [3] a certainty equivalence strategy is adopted for a class of systems quite similar to (1). However, the measurement is equal to the first state and no feedback constraints and no uncertainty or noise are considered. Moreover, the Lyapunov functions are of the form $V(x) = z^T Pz + a(x_1)z + b(x_1)$, where z is the vector of unmeasured states. Here, we consider more general Lyapunov functions and give the relation between these Lyapunov functions and the “correction term” $G(\mu - C_2\sigma)$. Also in [2] no feedback constraints and no uncertainty are considered. Using the general framework introduced above, we illustrate in Section 4 how our certainty equivalence principle can be applied for designing controllers for dynamics like (1) and how these controllers can be “coordinated” for controlling complex interconnections of dynamics like (1).

2. Incremental rates

It is important for the controller design to evaluate the effect of the uncertainty Ψ on the system dynamics taking into account the systems constraints \mathcal{M} . This motivates the following definition.

Definition 2.1 (Incremental rate). We will say that a system Σ with states $x \in \mathbb{R}^n$, inputs $\iota \in \mathbb{R}^m \times \mathbb{R}^r$ and measurements $\varsigma \in \mathbb{R}^p \times \mathbb{R}^s$, uncertainty $\Psi \in \mathbb{R}^q$ and constraints \mathcal{M} has smooth incremental (from zero) rate γ^z if there exist a nonempty subset $\{z\}$ of $\{x, \iota\}$ and a smooth nonnegative function $\gamma^z : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^{\geq}$ such that $(\|\Psi\| - \|\Psi|_{z=0}\|)^2 \leq \gamma^z(x, \iota, \varsigma)\|z\|^2$ for all $(x, \iota, \varsigma) \in \mathcal{M}$ and $\forall t \geq 0$.

For a system Σ with states x , inputs ι and measurements ς , uncertainty Ψ and constraints \mathcal{M} we expect to have the following relation among Ψ , on one side, and x, ι, ς on the other:

$$\gamma^2(\varsigma)\|\Psi\|^2 \leq \gamma^x(x, \varsigma)\|x\|^2 + \gamma^v(\varsigma)\|v\|^2 + \sum_{j \in J} \gamma^{\chi_j}(x, \iota, \varsigma)\chi_j^2, \quad \forall (x, \iota, \varsigma) \in \mathcal{M}, \quad (2)$$

where χ_j is the j th element of χ , $j \in J := \{1, \dots, r\}$, and $\gamma^x : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^{\geq}$, $\gamma^v : \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^+$ and $\gamma^{\chi_j} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^{\geq}$ are smooth incremental rates (“rescaled” by the square of a smooth function $\gamma : \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^+$). The inequality (2) means that the uncertainty Ψ is known up to a nonlinear function of the states and the inputs, weighted by the corresponding incremental rates evaluated under some constraints \mathcal{M} . This approach is inspired to a \mathcal{H}_∞ -control problem formulation, with Ψ having the role of a “disturbance”, the right-hand side of (2) having the role of a “penalty index” and γ having the role of an “attenuation level” [4]. Note that the

incremental rate γ^v is assumed (with no loss of generality) to be a positive real function.

Definition 2.2 (Incremental rates and scaling of Σ). We will say that a system Σ with states $x \in \mathbb{R}^n$, inputs $\iota \in \mathbb{R}^m \times \mathbb{R}^r$ and measurements $\varsigma \in \mathbb{R}^p \times \mathbb{R}^s$, uncertainty $\Psi \in \mathbb{R}^q$ and constraints \mathcal{M} has smooth incremental rates γ^x , γ^v and γ^{χ_j} , $j \in J$, with scaling γ if (2) holds for some smooth nonnegative functions $\gamma^x : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^{\geq}$, $\gamma^{\chi_j} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^{\geq}$ and positive $\gamma^v, \gamma : \mathbb{R}^p \times \mathbb{R}^s \rightarrow \mathbb{R}^+$.

Throughout the paper, if $\Psi = \text{col}(\psi_s, \psi_m)$ we will also use the notations γ_h^x, γ_h^v and $\gamma_h^{\chi_j}$, $h = s, m$, to denote the incremental rates of ψ_h , $h = s, m$. Also when an incremental rate is not explicitly cited in the context, we consider it equal to zero.

Before going further, we give some notations extensively used throughout the paper: $\|v\| = \sqrt{v^T v}$ denotes the euclidean norm of any given vector v and $\|v\|_A := \sqrt{v^T A v}$ for any positive semidefinite matrix A ; by $\text{col}(v_1, \dots, v_s)$ we denote the column vector with components v_1, \dots, v_s . Let \mathbb{R}^s be the vector space of s -dimensional real column vectors; \mathbb{R}^+ (resp. \mathbb{R}^{\geq}) denotes the set of positive (resp. nonnegative) real numbers; $\mathbb{R}^{n \times n}$ denotes the set of $n \times n$ matrices. For any continuous function $f : \mathbb{R}^q \times \mathbb{R}^l \rightarrow \mathbb{R}^r$, $(s, r) \mapsto f(s, r)$, we denote by $f(z, r)$ or $f|_{s=z}$ the function $f(s, r)$ with $s = z$. A continuous function $\alpha : \mathbb{R}^{\geq} \rightarrow \mathbb{R}^{\geq}$ is said to be of class \mathcal{K} (or $\alpha \in \mathcal{K}$) if $\alpha(0) = 0$ and it is increasing; a function $\alpha : \mathbb{R}^{\geq} \rightarrow \mathbb{R}^{\geq}$ is said to be of class \mathcal{K}_∞ (or $\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and $\lim_{s \rightarrow +\infty} \alpha(s) = +\infty$. For any given functions $h_j : \mathbb{R}^q \rightarrow \mathbb{R}$, $j = 1, 2$, we say that $h_1(s)$ is of the same order as $h_2(s)$ if there exists $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 h_2(s) \leq h_1(s) \leq \kappa_2 h_2(s)$ for all s and we write $h_1 \sim h_2$. For any smooth functions $V : \mathbb{R} \rightarrow \mathbb{R}^{\geq}$, $s \mapsto V(s)$, and $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $r \mapsto \lambda(r)$, we denote by $\nabla_r V(\lambda(r))$ the partial derivatives of $V(\lambda(r))$ with respect to r and by $\nabla_s V|_{\lambda(r)}$ the derivative of $V(s)$ with respect to s evaluated for $s = \lambda(r)$. A function $V : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $\mathcal{N} \subset \mathbb{R}^s$, is proper and positive definite for each $v \in \mathcal{N}$ if for each $v \in \mathcal{N}$ there exist $\alpha_1^v, \alpha_2^v \in \mathcal{K}_\infty$ such that $\alpha_1^v(\|x\|) \leq V(x, v) \leq \alpha_2^v(\|x\|)$ for all x .

3. A certainty equivalence design with feedback constraints

Consider any system $\Sigma(x, \iota, \varsigma, \Psi, \mathcal{M}, \mathcal{U}, \mathcal{F})$ of the form (1) with smooth $B_1(\varsigma)$ and $C_1(\varsigma)$ such that $C_1^T(\varsigma)B_1(\varsigma) = 0$ and $R_2(\varsigma) := C_1(\varsigma)C_1^T(\varsigma) > 0$ for all ς , states $x \in \mathbb{R}^n$, inputs ι (controls $v \in \mathbb{R}^m$ and exogenous inputs $\chi \in \mathbb{R}^r$), measurements ς (one endogenous measurement $\mu \in \mathbb{R}$ and exogenous measurements $v \in \mathbb{R}^s$), uncertainty $\Psi \in \mathbb{R}^q$, constraints \mathcal{M} and smooth incremental rates γ^x , γ^v and γ^{χ_j} , $j \in J := \{1, \dots, r\}$, with scaling γ , a family \mathcal{U} of continuous functions $v : \mathbb{R}^{\geq} \rightarrow \mathbb{R}^m$ and a family \mathcal{F} of continuous pairs $(v(t), \chi(t))$ with $v : [0, T_v) \rightarrow \mathcal{N} \subset \mathbb{R}^s$ and $\chi : [0, T_\chi) \rightarrow \mathcal{X} \subset \mathbb{R}^r$ (the intervals $[0, T_v)$ and $[0, T_\chi)$ represent the right maximal domain of definition of $v(t)$ and $\chi(t)$, respectively). The assumption $C_1^T(\varsigma)B_1(\varsigma) = 0$ simplifies the involved calculations and can be always met by suitably redefining the system matrices

and the uncertainty vector (let $\tilde{C}_1 = (0 \ C_1)$, $\tilde{B}_1 = (B_1 0)$ and $\tilde{\Psi} = \text{col}(\Psi, \Psi)$ and thus $\dot{x} = Ax + B_2 v + \tilde{B}_1 \tilde{\Psi}$ and $\mu = C_2 x + \tilde{C}_1 \tilde{\Psi}$ with $\tilde{C}_1^T \tilde{B}_1 = 0$). The assumption $C_1(\zeta) C_1^T(\zeta) > 0$ “regularizes” the class of systems as in the \mathcal{H}_∞ -control problem [2] and can be always met by suitably redefining the system matrices and the uncertainty vector (if for example $C_1 = 0$ then let $\tilde{B}_1 = (\tilde{B}_1 0)$, $\tilde{C}_1 = (0 1)$ and $\tilde{\Psi} = \text{col}(\Psi, 0)$ and thus $\dot{x} = Ax + B_2 v + \tilde{B}_1 \tilde{\Psi}$ and $\mu = C_2 x + \tilde{C}_1 \tilde{\Psi}$ with $\tilde{C}_1^T \tilde{C}_1 = 1$). We assume that Ψ satisfies standard assumptions for the local existence and uniqueness of the trajectories of (1). By *trajectory* of (1) we mean the 4-tuple $(x(t), v(t), \nu(t), \chi(t))$, with $(x(t), \nu(t), \zeta(t)) \in \mathcal{M}$ for all $t \in [0, t_{\max})$, where $v \in \mathcal{U}$, $(\nu, \chi) \in \mathcal{F}$ and $x(t)$ is the solution of (1) over its right maximal domain of definition $[0, t_{\max})$. A starting point of this paper is to adopt for the controller design of (1) the following certainty equivalence principle: first, find a state feedback controller by assuming full state information, i.e. $\mu = x$, then design an observer to estimate on line the state $x(t)$ and replace $x(t)$ with the estimate $\sigma(t)$ in the state feedback controller. With full state information and reformulating the stabilization problem as a \mathcal{H}_∞ -control problem [4], it can be proved that if a smooth solution $V_s : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $(x, \nu) \mapsto V_s(x, \nu)$, proper and positive definite for each $\nu \in \mathcal{N}$, can be found to the following HJI:

$$\begin{aligned} & 2\nabla_\nu V_s(x, \nu) \dot{\nu} + \nabla_x V_s(x, \nu) Ax \\ & + x^T A^T \nabla_x^T V_s(x, \nu) + \gamma^x(x, \zeta) \|x\|^2 \\ & + \nabla_x V_s(x, \nu) [B_1(\zeta) B_1^T(\zeta) / \gamma^2(\zeta) - B_2 B_2^T / \gamma^v(\zeta)] \\ & \times \nabla_x^T V_s(x, \nu) \leq -2 \sum_{j=1}^n \varphi_{sj}(x, \zeta) x_j^2 \end{aligned} \quad (3)$$

for all $(x, \nu, \zeta) \in \mathcal{M}$ and for some continuous functions $\varphi_{sj} : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $j = 1, \dots, n$, with positive (definite) $\varphi_{js}(x, \zeta)$ for each $\zeta \in \mathbb{R} \times \mathcal{N}$, then

$$\begin{aligned} 2\dot{V}_s & \leq -2 \sum_{j=1}^n \varphi_{sj}(x, \zeta) x_j^2 - \gamma^2(\zeta) \|\Xi\|^2 \\ & + \gamma^v(\zeta) \|v - F(x, \zeta)\|^2 + \sum_{j \in J} \gamma^{\chi_j}(x, \nu, \zeta) \chi_j^2 \end{aligned} \quad (4)$$

holds true along the trajectories of (1) with

$$\begin{aligned} F(x, \zeta) & := - (B_2^T / \gamma^v(\zeta)) \nabla_x^T V_s(x, \nu), \quad \Xi = \Psi - \Psi^*, \\ \Psi^* & := (B_1^T(\zeta) / \gamma^2(\zeta)) \nabla^T V_s(x, \nu). \end{aligned} \quad (5)$$

Here Ψ^* represents the *worst-case uncertainty* and $v = F(x, \zeta)$ corresponds to the choice of the control v for which the right-hand side of (4) is minimum.

HJIs are a natural tool for taking into account feedback constraints. Consider $\dot{x} = u$ with the constraint $|v| \leq 1$ and write (3) for this system with Lyapunov function $V_s(x)$ and $\gamma = \gamma^v = 1$, i.e. $-(\nabla_x V_s(x))^2 < 0$ for $x \neq 0$. From (5) and the constraint $|v| \leq 1$ it follows that $|\nabla_x V_s(x)| \leq 1$ for all $x \in \mathbb{R}$. Thus, we select $V_s(x) = \sqrt{1 + x^2} - 1$.

We remark also that the solvability of (3) takes into account any case in which the uncertainty Ψ may cause a lack of “controllability”: for example in $\dot{x} = u + \Psi$ with $\Psi = -u$ we have $\gamma = \gamma^v = B_1 = B_2 = 1$, $\gamma^x = 0$ and $A = 0$ and (3) is simply $(\nabla V_s(x))^2 [1/\gamma^2 - 1/\gamma^v] = 0$, which is not negative definite whatever is $V_s(x)$.

We want also to remark the fact that $V_s(x, \nu)$ may depend on the exogenous measurements. This is quite natural in a context in which a system may depend on exogenous inputs or measurements. We discard any dependence of the Lyapunov function from the exogenous inputs since, as clear from (5), the controller would depend on the exogenous inputs and this would render any certainty equivalence strategy not feasible.

As to the observer design for (1), an observer can be thought of a “copy” of the system (of which the state is to be estimated) plus a function $G(\mu - C_2 \sigma, \zeta)$ of the *measurement estimation error* $\mu - C_2 \sigma$. The main problem is to give a characterization of the function $G(\mu - C_2 \sigma, \zeta)$ in terms of the solution of a HJI as we did for the state feedback controller $v = F(x, \zeta)$: we find in the literature only either a characterization in terms of HJIs which do not lead to any implementable observer design [2] or particular observer design procedures which are not related to the solution of a general HJI [3]. This characterization is a key point for taking into account inverse optimality and feedback constraints in the control design. Inverse optimality is important to relate the observer gains to the minimization of suitable cost functionals. Feedback constraints are important to take into account the dynamics of actuators and sensors in the control loop or to modulate the quantity of noise or uncertainty fed back in the control loop by the controller itself through the measurements. The importance of modulating the quantity of noise or uncertainty fed back in the control loop by the controller is illustrated by the following example.

Example 3.1. Consider

$$\begin{aligned} \dot{x}_1 & = x_2, \quad \mu_1 = x_1 + x_2^4 \sin(x_1) + x_2^2 w, \\ \dot{x}_2 & = u + u^2, \quad \mu_2 = x_2 + w \end{aligned} \quad (6)$$

and study the possibility of controlling it. For the presence of the nonlinear term $u + u^2$, we reasonably expect the control u be bounded. Write (6) in backstepping coordinates $z_1 = x_1$ and $z_2 = x_2 - \tilde{x}_2$

$$\begin{aligned} \dot{z}_1 & = \tilde{x}_2 + z_2, \quad \mu_1 = z_1 + (\tilde{x}_2 + z_2)^4 \sin(x_1) + (\tilde{x}_2 + z_2)^2 w, \\ \dot{z}_2 & = u + u^2 - \dot{\tilde{x}}_2, \quad \mu_2 = \tilde{x}_2 + z_2 + w, \end{aligned} \quad (7)$$

where \tilde{x}_2 acts as control for \dot{z}_1 and u as control for \dot{z}_2 and w is an exogenous disturbance. Let $\tilde{x}_2 = -k_1(\sigma_1)$, $\dot{\sigma}_1 = k_2(\sigma_1) + k_3(\mu_1 - \sigma_1)$ be an observer-based controller for the first system in (7) with smooth functions $k_1, k_2, k_3 : \mathbb{R} \rightarrow \mathbb{R}$. Thus,

$$\begin{aligned} \dot{z}_2 & = u + u^2 + \nabla_{\sigma_1} k_1(\sigma_1) [k_2(\sigma_1) + k_3(z_1 + (\tilde{x}_2 + z_2)^4 \sin(z_1) \\ & + (\tilde{x}_2 + z_2)^2 w - \sigma_1)]. \end{aligned}$$

Clearly, a bounded control u cannot compensate for the term $k_3(z_1 + (\tilde{x}_2 + z_2)^4 \sin(z_1) + (\tilde{x}_2 + z_2)^2 w - \sigma_1)$ if the function $k_3(s)$ is not subject to any restriction.

A smooth measurement feedback controller \mathcal{C} is defined as

$$\dot{\sigma} = H(\sigma, \varsigma) + G(\mu - C_2\sigma, \varsigma), \quad v = \Gamma(\sigma, \varsigma), \quad \sigma(0) = \sigma_0 \quad (8)$$

with $\sigma \in \mathbb{R}^n$, σ_j denoting as usual the j th element of σ , smooth $H : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^n$, $G : \mathbb{R} \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^n$ and $\Gamma : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^m$, vanishing for $\sigma = 0$ and $\mu = 0$, and constraints

$$\begin{aligned} \|\Gamma(\sigma, \varsigma)\| &\leq \Delta_f \in (0, \infty], \\ \|G(\mu - C_2\sigma, \varsigma)\| &\leq \Delta_m \in (0, \infty], \quad \forall \sigma, \varsigma. \end{aligned} \quad (9)$$

The numbers Δ_f and Δ_m characterize the maximum level allowed for the control input v and, respectively, for the estimation error $\mu - C_2\sigma$ fed back in the control loop by (8). We say that the controller (8) has control input level Δ_f and measurement estimation error level Δ_m (or simply levels (Δ_f, Δ_m)) if there exist $\Delta_f, \Delta_m \in (0, \infty]$ such that (9) holds true.

The connection between $G(s, \varsigma)$ in (8) and the solution of a HJI is assessed by the following theorem. Let $V_m : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $(e, v) \mapsto V_m(e, v) = \tilde{V}_m(\|e\|_{P_m(v)})$, smooth, proper and positive definite for each $v \in \mathcal{N}$ with $P_m(v)$ symmetric and positive definite for each $v \in \mathcal{N}$, be a solution (if any) of the following HJI:

$$\begin{aligned} 2\nabla_v V_m(e, v)\dot{v} + \nabla_e V_m(e, v)A_m(\varsigma, e) + A_m^T(\varsigma, e)\nabla_e^T V_m(e, v) \\ + \nabla_e V_m(e, v)B_1(\varsigma)B_1^T(\varsigma)\nabla_e^T V_m(e, v)/\gamma^2(\varsigma) \\ + \gamma^v(\varsigma)\|F(e, \varsigma)\|^2 \leq -2 \sum_{j=1}^n \varphi_{mj}(e, \varsigma)e_j^2 \end{aligned} \quad (10)$$

for all $e \in \mathbb{R}^n$ and $(x, \iota, \varsigma) \in \mathcal{M}$ and for some continuous functions $\varphi_{mj} : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $j=1, \dots, n$, with positive (definite) $\varphi_{jm}(e, \varsigma)$ for each $\varsigma \in \mathbb{R} \times \mathcal{N}$ and

$$\begin{aligned} A_m(e, \varsigma) &:= Ae + (B_1(\varsigma)B_1^T(\varsigma)/\gamma^2(\varsigma)) \\ &\quad \times \nabla_e^T V_s(e, v) - (\tfrac{1}{4})G(2C_2e, \varsigma), \\ G(s, \varsigma) &:= \gamma^2(\varsigma)P_m^{-1}(v)\nabla_s \tilde{V}_m(s)/R_2(\varsigma). \end{aligned} \quad (11)$$

The HJI (10) resembles classical HJI related to a \mathcal{H}_∞ -based observer design such as those used in [2] or [3]. Still, it has two distinctive features with respect to previous versions: more general Lyapunov functions $V_m(e, v)$ (parametrized Lyapunov functions) and innovation term $G(2C_2e, \varsigma)$ being a nonlinear function of $V_m(e, v)$.

Theorem 3.1. *Let $\Sigma(x, \iota, \varsigma, \Psi, \mathcal{M}, \mathcal{U}, \mathcal{F})$ be given as in (1) and $R_2(\varsigma) := C_1(\varsigma)C_1^T(\varsigma) > 0$ for all ς . If*

- (full state information) *there exist continuous $\varphi_{sj} : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $j=1, \dots, n$, with positive definite $\varphi_{js}(x, \varsigma)$ for each $\varsigma \in \mathbb{R} \times \mathcal{N}$, and smooth $V_s : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $(x, v) \mapsto V_s(x, v)$, proper and positive definite for each $v \in \mathcal{N}$, such that (3) holds for all $(x, \iota, \varsigma) \in \mathcal{M}$;*
- (observer design) *there exist a continuous $\varphi_{mj} : \mathbb{R}^n \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $j=1, \dots, n$, with positive definite $\varphi_{jm}(e, \varsigma)$ for*

each $\varsigma \in \mathbb{R} \times \mathcal{N}$, smooth $P_m : \mathcal{N} \rightarrow \mathbb{R}^{n \times n}$, with $P_m(v)$ symmetric and positive definite for each $v \in \mathcal{N}$, and smooth $V_m : \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $(e, v) \mapsto V_m(e, v) = \tilde{V}_m(\|e\|_{P_m(v)})$, with $\tilde{V}_m(s)$ proper and positive definite and $\nabla_s \tilde{V}_m(s)/s \in (0, 1]$ even and nonincreasing for all $s \geq 0$, such that (10) holds for all $(x, \iota, \varsigma) \in \mathcal{M}$ and, in addition, for all $e \in \mathbb{R}^n$, $\varsigma \in \mathbb{R} \times \mathcal{N}$ and $\Xi \in \mathbb{R}^q$

$$\begin{aligned} R_2(\varsigma) &> \|C_1^T(\varsigma)C_1(\varsigma)\| \|C_2e\| \|\nabla_s \tilde{V}_m\|_{\|e\|_{P_m(v)}} \\ &\quad \times \nabla_{sss}^3 \tilde{V}_m |_{C_2e+C_1(\varsigma)\Xi} / \|e\|_{P_m(v)}, \\ 1 &\geq \|\nabla_s \tilde{V}_m\|_{\|e\|_{P_m(v)}} \nabla_{ss}^2 \tilde{V}_m |_{C_2e+C_1(\varsigma)\Xi} / \|e\|_{P_m(v)}. \end{aligned} \quad (12)$$

- (feedback constraints) *there exists $\Delta_f, \Delta_m \in (0, \infty]$ such that $\|F(x, \varsigma)\| \leq \Delta_f$ and $\|G(s, \varsigma)\| \leq \Delta_m$ for all $x \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $\varsigma \in \mathbb{R} \times \mathcal{N}$, with $F(x, \varsigma)$ as in (5) and $G(s, \varsigma)$ as in (11);*
- (nonlinear coupling) *for all $(x, \iota, \varsigma) \in \mathcal{M}$, $e \in \mathbb{R}^n$ and $\varsigma \in \mathbb{R} \times \mathcal{N}$*

$$\begin{aligned} 2\nabla_e V_m(e, v)B_1(\varsigma)B_1^T(\varsigma)/\gamma^2(\varsigma)[\nabla_x V_s(x, v) \\ - \nabla_{x-e} V_s(x - e, v) - \nabla_e V_s(e, v)]^T \\ + \gamma^v(\varsigma)(\|F(x - e, \varsigma) - F(x, \varsigma)\|^2 - \|F(e, \varsigma)\|^2) \\ \leq \sum_{j=1}^n [\varphi_{sj}(x, \varsigma)x_j^2 + \varphi_{mj}(e, \varsigma)e_j^2]. \end{aligned} \quad (14)$$

Under the above assumptions, there exists a smooth measurement feedback controller (8) with levels (Δ_f, Δ_m) such that along the trajectories of (1)–(8)

$$\begin{aligned} \dot{W} &\leq - \sum_{j=1}^n [\varphi_{sj}(x, \varsigma)x_j^2 + \varphi_{mj}(x - \sigma, \varsigma)(x_j - \sigma_j)^2] \\ &\quad + \sum_{j \in J} \gamma^{z_j}(x, \iota, \varsigma)\chi_j^2, \end{aligned} \quad (15)$$

where $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{N} \rightarrow \mathbb{R}^{\geq}$, $W(x, \sigma, v) = 2[V_s(\|x\|_{P_f(v)}) + V_m(\|x - \sigma\|_{P_m(v)})]$, is smooth and proper and positive definite for each $v \in \mathcal{N}$.

Proof. Throughout the proof, in $\nabla_r V_m(r, v)$ and $\nabla_r V_s(r, v)$ we drop the argument and assume $(x, \iota, \varsigma) \in \mathcal{M}$, $e \in \mathbb{R}^n$, $\varsigma \in \mathbb{R} \times \mathcal{N}$ and $\Xi \in \mathbb{R}^q$. Define our controller (8) with

$$\begin{aligned} \Gamma(\sigma, \varsigma) &= F(\sigma, \varsigma) := -\frac{B_2^T}{\gamma^v(\varsigma)} \nabla_\sigma^T V_s, \\ H(\sigma, \varsigma) &= A\sigma + \frac{B_1(\varsigma)B_1^T(\varsigma)}{\gamma^2(\varsigma)} \nabla_\sigma^T V_s + B_2v \end{aligned} \quad (16)$$

and $G(s, \varsigma)$ as in (11). This controller has levels (Δ_f, Δ_m) by our assumptions. We will use conditions (12)–(13) to prove for each e and ς the existence of a global maximum $\Xi = \Xi^*$ for the function

$$D(e, \varsigma, \Xi) = 2\nabla_e V_m[B_1(\varsigma)\Xi - G(z(e, \varsigma, \Xi), \varsigma)] - \gamma^2(\varsigma)\|\Xi\|^2$$

with $z(e, \varsigma, \Xi) = C_2 e + C_1(\varsigma)\Xi$. This will allow to obtain an upper bound for $D(e, \varsigma, \Xi)$ independent of Ξ . By (12)

$$\nabla_{\Xi}^2 D(e, \varsigma, \Xi) < 0 \quad (17)$$

for all Ξ, ς, e and, since $\nabla_s \tilde{V}_m(s)/s \in (0, 1]$ for all $s \geq 0$ and $\gamma(\varsigma) > 0$ for all ς , then

$$\lim_{\|\Xi\| \rightarrow \infty} D(e, \varsigma, \Xi) = -\infty \quad (18)$$

for each e and ς . From (17) and (18) it follows for each e and ς the existence of a unique $\Xi^*(e, \varsigma) \in \mathbb{R}^q$ such that $\nabla_{\Xi} D|_{\Xi^*(e, \varsigma)} = 0$ and

$$D(e, \varsigma, \Xi) \leq D(e, \varsigma, \Xi^*(e, \varsigma)), \quad \forall \Xi \quad (19)$$

i.e. a global maximum for $D(e, \varsigma, \Xi)$. From $\nabla_{\Xi} D|_{\Xi^*(e, \varsigma)} = 0$ we get

$$\begin{aligned} \Xi^*(e, \varsigma) &= (1/\gamma^2(\varsigma))[B_1^T(\varsigma) - (\gamma^2(\varsigma)/R_2(\varsigma)) \\ &\quad \times C_1^T(\varsigma)C_2 P_m^{-1}(v) \nabla_{ss}^2 \tilde{V}_m|_{z(e, \varsigma, \Xi)}] \nabla_e^T V_m \end{aligned}$$

for all ς and e . Since $C_1(\varsigma)B_1^T(\varsigma) = 0$ for all ς and as a consequence of (13) we get

$$|z(e, \varsigma, \Xi^*(e, \varsigma))| \leq 2|C_2 e| \quad (20)$$

for all ς, e . Since $1 \geq 4a[1-a]$ for all $a \in \mathbb{R}$, $C_1(\varsigma)B_1^T(\varsigma) = 0$ for all ς and $\nabla_s \tilde{V}_m(s)/s \in (0, 1]$ is even and nonincreasing for all $s \geq 0$, from (20) and (19) we obtain after some manipulations the following upper bound for $D(e, \varsigma, \Xi)$ independent of Ξ :

$$\begin{aligned} D(e, \varsigma, \Xi) &\leq -\frac{1}{2} \nabla_e V_m G(2C_2 e, v) \\ &\quad + \nabla_e V_m B_1(\varsigma) \frac{B_1^T(\varsigma)}{\gamma^2(\varsigma)} \nabla_e^T V_m \end{aligned} \quad (21)$$

for all Ξ, ς, e . Let $e = x - \sigma$, e_j be the j th element of e and set Ξ as in (5). From (10) and (21) one has along the trajectories of (1)–(8)

$$\begin{aligned} 2\dot{V}_m + \gamma_v(\varsigma) \|F(e, \varsigma)\|^2 - \gamma^2(\varsigma) \|\Xi\|^2 \\ \leq -2 \sum_{j=1}^n \varphi_{mj}(e, \varsigma) e_j^2 \\ + (2/\gamma^2(\varsigma)) \nabla_e V_m B_1(\varsigma) B_1^T(\varsigma) \\ \times [\nabla_x V_s - \nabla_{x-e} V_s - \nabla_e V_s]^T. \end{aligned} \quad (22)$$

On the other hand, from (3) one obtains (4) along the trajectories of (1). Summing up this inequality, with $v = F(\sigma, \varsigma)$, and (22) from (14) we get (15). \square

We remark that the solvability of (10) takes into account any case in which the uncertainty Ψ may cause a lack of “observability”: for example in $\dot{x} = u$, $\mu = x + \Psi$, with $\Psi = -x$ we have $\gamma = \gamma^v = B_2 = C_2 = C_1 = 1$, $B_1 = 0$, $\gamma^x = 1$ and

$A = 0$ and (10) is simply $-(\frac{1}{2})\nabla V_m(e)\nabla_s \tilde{V}_m|_{s=2e}/[R_2 P_m] + (\nabla V_s(e))^2$, which is not negative definite whatever are $V_s(e)$, $V_m(e) = \tilde{V}_m(P_m|e|)$ and $R_2 \geq 1$ such that $\tilde{V}_m(s)/s \in (0, 1]$ for all $s \geq 0$ and $(\nabla V_s(e))^2 < e^2$ (by (3)).

The condition $\nabla_s \tilde{V}_m(s)/s \in (0, 1]$ and nonincreasing for all $s \geq 0$ gives a restriction on the first derivative $\nabla_s \tilde{V}_m$ (in particular, $\tilde{V}_m(s)$ should be globally Lipschitz with Lipschitz constant less or equal to one) while (12)–(13) give restrictions on the second and third derivatives $\nabla_{ss}^2 \tilde{V}_m$ and $\nabla_{sss}^3 \tilde{V}_m$: these conditions are crucial for achieving the worst-case upper bound (19). Note also that if $\tilde{V}_m(s) = (\frac{1}{2})s^2$, and thus $V_m(e, v) = (\frac{1}{2})\|e\|_{P_m(v)}^2$, then all these conditions are trivially satisfied. More generally, if $|\nabla_{sss}^3 \tilde{V}_m(s)|, |\nabla_{ss}^2 \tilde{V}_m(s)|, |\nabla_s \tilde{V}_m(s)| \leq a$ for all $s \geq 0$ and for some $a > 0$ then it is reasonable to expect that (12)–(13) are satisfied by a constant rescaling of $\tilde{V}_m(s)$. Note also that if $V_s(x, v) = (\frac{1}{2})\|x\|_{P_s(v)}^2$ then (14) is trivially satisfied. Thus, the nonlinear coupling condition is always satisfied when $\nabla_x V_s(x, v)$ is linear with respect to x .

We remark also the main differences with previous works such as [2,3]. In [3] a certainty equivalence strategy is adopted for a class of systems quite similar to (1). However, the measurement is equal to the first state and no feedback constraints and no uncertainty or noise are considered. Moreover, the Lyapunov functions are of the form $V(x) = z^T P z + a(x_1)z + b(x_1)$, where z is the vector of unmeasured states. However, this is a severe restriction when $n \geq 3$ and it is much more natural to look for Lyapunov functions of the form $V(x) = z^T P(x_1)z + a(x_1)z + b(x_1)$. However, the term $P(x_1)$ gives place in the time derivative of V to cubic terms in z which are hard to compensate for through a measurement feedback control. Thus, it is more appropriate to think of using a “filtered” implementation of $P(x_1)$, i.e. the output of a filter, and, since the control design is based on such a Lyapunov functions, of using controllers with “filtered” gains. For this reason we consider in Theorem 3.1 Lyapunov functions of the form $V(x, v)$ or $V(e, v)$ where v is an exogenous measurement which may represent a “filtered” implementation of $P(x_1)$ (see [5] for applications). In [2] no feedback constraints and no uncertainty are considered. In both [2,3] the restrictions $\nabla_s \tilde{V}_m(s)/s \in (0, 1]$ even and nonincreasing for all $s \geq 0$ and (12)–(13) on $\nabla_{ss}^2 \tilde{V}_m$ and $\nabla_{sss}^3 \tilde{V}_m$ are satisfied since $G(s) = \gamma^2(\varsigma)P_m^{-1}C_2^T s/[2R_2(\varsigma)]$ in (10) and thus $\nabla_s \tilde{V}_m(s) = s$, which in our setting corresponds to select a *quadratic* $\tilde{V}_m(s)$.

The Lyapunov functions used in Theorem 3.1 have parameters which may satisfy differential equations, which is quite natural for a system with exogenous inputs and measurements and some of these exogenous signals may be the states of some other interconnected system (“filtered” Lyapunov functions). If W is independent of χ and v and φ_{sj} and φ_{mj} are independent of ς and γ^{kj} are independent of x, v and ς , our filtered Lyapunov functions are ISS [9] or iISS [1] Lyapunov functions. Here we need a more general definition of Lyapunov function since the stability margins φ_{sj} and φ_{mj} as well as the incremental rates γ^{kj} may depend on the measurements which may on turn be functions of the states and the controls. Moreover, our Lyapunov functions may be parametrized by the exogenous

inputs and measurements. A rigorous definition of “filtered” Lyapunov functions is given in [6].

4. Example

In this section, using the general framework introduced in the previous sections and Theorem 3.1, we derive a measurement feedback controller for (6). The results of [10,7,8] cannot be applied to our case since the measurements μ_j , $j=1, 2$, may differ significantly from the states x_j , $j=1, 2$, due to (possibly) unbounded noise w and large uncertainty and the feedforward coordinates, through which the control law is designed (see [8, Section 6.2]), are functions of the noise and uncertainty and cannot be implemented as a function of the measurements. Moreover, even a control design based on using the state feedback control laws proposed in [10,7,8] in conjunction with an observer is not feasible since the design of an observer for (6) is highly nontrivial. We rather implement a backstepping design in which \tilde{x}_{j+1} , with $\tilde{x}_3 := u$ and $\tilde{x}_1 := 0$, is used as control for each dynamics \dot{z}_j , $j=1, 2$, where $z_j := x_j - \tilde{x}_j$ is the backstepping coordinate, and for each system \dot{z}_j we apply Theorem 3.1. Moreover, it is clear that, since $\dot{x}_3 = u + u^2$, the stabilization is feasible only if $|u| \leq \Delta$, with sufficiently small $\Delta > 0$. By the integrator chain, this constraint propagates up through the control $\tilde{x}_n, \dots, \tilde{x}_2$ so that finally we impose that $|\tilde{x}_j| \leq \Delta$ for all $j=2, 3$, and for sufficiently small $\Delta > 0$.

First, we split the system to obtain three one-dimensional systems in a standard form. To do this we change coordinate x_j into *backstepping coordinate* $z_j := x_j - \tilde{x}_j$, $j=1, 2$, $\tilde{x}_1 := 0$, to use \tilde{x}_2 as control for \dot{z}_1 instead of using x_2 (which is a state of (6)) for \dot{x}_1 , and also we change measurement μ_j into $\tilde{\mu}_j := \mu_j - \tilde{x}_j$, $j=1, 2$, to have z_j as “nominal part” of the measurement $\tilde{\mu}_j$. It is easy to see after some calculations that (6), in *backstepping coordinates* z_j , $j=1, 2$, and with a *change of measurements* $\tilde{\mu}_j$, $j=1, 2$, can be split into two one-dimensional systems of the form (1). More precisely, we get

$$\Sigma_1^{ff} : \dot{z}_1 = \tilde{x}_2 + B_{11}\Psi_1, \quad \tilde{\mu}_1 = z_1 + C_{11}\Psi_1 \quad (23)$$

with $B_{11} = (1 \ 0)$, $C_{11} = (0 \ 1)$, state z_1 , control \tilde{x}_2 , exogenous inputs z_2, \tilde{x}_3 , exogenous measurements \tilde{x}_3 , uncertainties $\Psi_1 = (\tilde{\psi}_{1s} \ \tilde{\psi}_{1m})^T$ where

$$\begin{aligned} \tilde{\psi}_{1s} &= z_2, \\ \tilde{\psi}_{1m} &= (z_2 + \tilde{x}_2)^4 \sin z_1 + (z_2 + \tilde{x}_2)^2 w \end{aligned} \quad (24)$$

constraints and incremental rates

$$\begin{aligned} \mathcal{M}_1 &= \{|\tilde{x}_{i+1}| \leq \Delta_i \in (0, 1], i=1, 2\}, \\ \gamma_{1s}^{z_2} &\sim 1, \quad \gamma_{1m}^{z_2} \sim 1 + z_2^8, \\ \gamma_{1m}^{\tilde{x}_2} &\sim \Delta_1^4, \quad \gamma_{1m}^w \sim z_2^4 + 1 \end{aligned} \quad (25)$$

and

$$\Sigma_2^{ff} : \dot{z}_2 = \tilde{x}_3 + B_{21}\Psi_2, \quad \tilde{\mu}_2 = z_2 + C_{21}\Psi_2 \quad (26)$$

with $B_{21} = (1 \ 0)$, $C_{21} = (0 \ 1)$, state z_2 , control \tilde{x}_3 , exogenous inputs z_1, \tilde{x}_2 , exogenous measurements \tilde{x}_2 , uncertainties $\Psi_2 = (\tilde{\psi}_{2s} \ \tilde{\psi}_{2m})^T$ where

$$\tilde{\psi}_{2s} = u + u^2, \quad \tilde{\psi}_{2m} = w \quad (27)$$

constraints and incremental rates

$$\gamma_{2s}^{z_3}, \gamma_{2s}^{\tilde{x}_2}, \gamma_{2m}^w \sim 1, \quad \gamma_{2s}^{\tilde{x}_3} \sim \Delta_1^2. \quad (28)$$

For Σ_1^{ff} we select $\Delta_1 \in (0, 1)$ and $h_{1m}(\Delta_1), h_{1s}(\Delta_1) > 0$ such that $h_{1s}(\Delta_1) \leq \frac{1}{80}$ and $\gamma_{1s}^{\tilde{x}_2} + h_{1m}(\Delta_1)\gamma_{1m}^{\tilde{x}_2} \leq h_{1s}(\Delta_1)$ (in our case $\gamma_{1s}^{\tilde{x}_2} = 0$). With $\Delta_f = \Delta_m = \Delta_1$ and $V_s(s) = V_m(s) = (\frac{1}{2})[1 + s^2]^{1/2}$ it can be seen that the assumptions of Theorem 3.1 are satisfied for any $R_{11} \geq R'_{11} := \max\{(1/\Delta_1), 1/\sqrt{h_{1m}}\}$. The need of square root Lyapunov functions is motivated by having bounded controllers, characterized by the derivative of these Lyapunov functions, and by satisfying the constraints on the first three derivatives of Lyapunov functions themselves (see Theorem 3.1). Moreover, the relation $\gamma_{1s}^{\tilde{x}_2} + h_{1m}(\Delta_1)\gamma_{1m}^{\tilde{x}_2} \leq h_{1s}(\Delta_1)$ is motivated by having an incremental rate of the control input \tilde{x}_2 which can be rendered as small as desired by suitably selecting the “control amplitude” Δ_1 . Thus, by application of Theorem 3.1 a controller \mathcal{C}_1^{ff} for Σ_1^{ff} is given by

$$\begin{aligned} \mathcal{C}_1^{ff} : \tilde{x}_2 &= -\frac{1}{2R_{11}} \frac{\sigma_1}{\sqrt{1 + \sigma_1^2}}, \\ \dot{\sigma}_1 &= \frac{1}{2R_{11}} \left[\frac{(h_{1s} - 1)\sigma_1}{\sqrt{1 + \sigma_1^2}} + \frac{\tilde{\mu}_1 - \sigma_1}{h_{1s}\sqrt{1 + (\tilde{\mu}_1 - \sigma_1)^2}} \right]. \end{aligned} \quad (29)$$

In a similar way we calculate the parameters h_{2s}, h_{2m} and the controller \mathcal{C}_2^{ff} for Σ_2^{ff}

$$\begin{aligned} \mathcal{C}_2^{ff} : \tilde{x}_3 &= -\frac{1}{2R_{21}} \frac{\sigma_2}{\sqrt{1 + \sigma_2^2}}, \\ \dot{\sigma}_2 &= \frac{1}{2R_{21}} \left[\frac{(h_{2s} - 1)\sigma_2}{\sqrt{1 + \sigma_2^2}} + \frac{\tilde{\mu}_2 - \sigma_2}{h_{2s}\sqrt{1 + (\tilde{\mu}_2 - \sigma_2)^2}} \right]. \end{aligned} \quad (30)$$

We take \mathcal{C}_j^{ff} , $j=1, 2$, as measurement feedback controller for (6). After some calculations with Lyapunov functions

$$W_j(z_j, \sigma_j) = [1 + z_j^2]^{1/2} + [1 + (z_j - \sigma_j)^2]^{1/2} - 2, \quad j=1, 2,$$

we obtain along the trajectories of $\Sigma_1^{ff} - \mathcal{C}_1^{ff}$ and, respectively, $\Sigma_2^{ff} - \mathcal{C}_2^{ff}$

$$\dot{W}_1 \leq -\varphi_{1s}z_1^2 - \varphi_{1m}e_1^2 + \gamma_1^{z_2}z_2^2 + \gamma_1^w w^2$$

with

$$\begin{aligned} \varphi_{1s} &\sim 1/[R_{11}(1 + z_1^2)], \quad \varphi_{1m} \sim 1/[R_{11}(1 + e_1^2)], \\ \gamma_1^{z_2} &\sim R_{11}[1 + z_2^8], \quad \gamma_1^w \sim R_{11}[1 + z_2^4] \end{aligned} \quad (31)$$

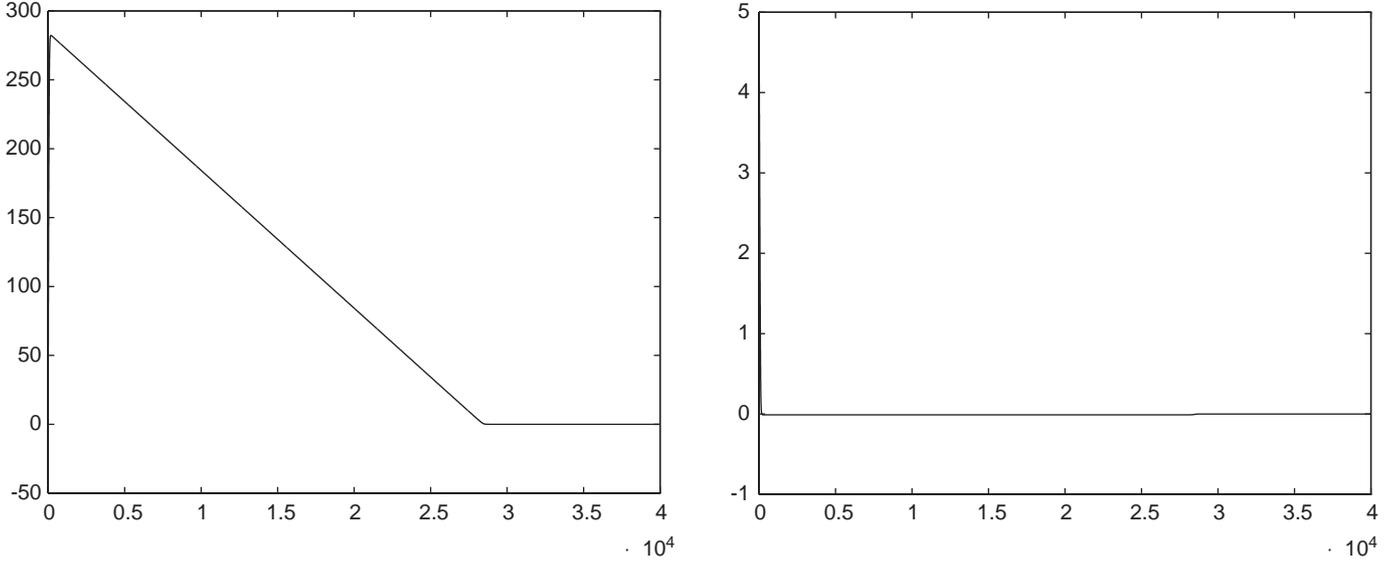


Fig. 1.

and

$$\dot{W}_2 \leq -\varphi_{2s} z_2^2 - \varphi_{2m} e_2^2 + \gamma_2^{z_1} z_1^2 + \gamma_2^{e_1} e_1^2 + \gamma_2^w w^2$$

with

$$\varphi_{2s} \sim 1/[R_{21}(1+z_2^2)], \quad \varphi_{2m} \sim 1/[R_{21}(1+e_2^2)],$$

$$\gamma_2^{z_1} \sim R_{21}/[R_{11}^4(1+z_1^2)],$$

$$\gamma_2^{e_1} \sim R_{21}/[R_{11}^4(1+e_1^2)], \quad \gamma_2^w \sim R_{21}.$$

It is shown in [6] how to select for a general class of feedforward systems the parameters R_{j1} , $j = 1, 2$, in such a way that the closed loop system is iISS with respect to square integrable disturbances w [1]. See Fig. 1. More specifically, the parameters R_{j1} , $j = 1, 2$, are chosen so that the stability margins φ_{js} and φ_{jm} compensate for the incremental rate $\gamma_i^{z_j}$ for $j \neq i$ and $j, i = 1, 2$. The iISS property is proved by proving that the closed-loop system is GAS and UBEBS (see [1]). In the case of (6), we select $R_{11} = 50$, $R_{21} = 10$ and $h_{1s} = h_{2s} = 0.01$. As shown by the simulations below with $w(t) = e^{-t}$, $x_1(0) = 4$, $x_2(0) = 5$, $\sigma_1(0) = 0$, $\sigma_2(0) = 0$ (from the left the graphics of $x_1(t)$ and $x_2(t)$ are shown), the convergence of the state trajectories to zero, although slow due to the small values of the controller gains, is guaranteed despite the square integrable disturbance $w(t)$ and the measurement uncertainties.

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