Stabilization in Probability of Nonlinear Stochastic Systems With Guaranteed Region of Attraction and Target Set

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***Abstract*—We deal with nonlinear dynamical systems, consisting of a linear nominal part perturbed by model uncertainties, nonlinearities and both additive and multiplicative random noise, modeled as a Wiener process. In particular, we study the problem of finding suitable measurement feedback control laws such that the resulting closed-loop system is stable in some probabilistic sense. To this aim, we introduce a new notion of *stabilization in probability*, which is the natural counterpart of the classical concept of *regional stabilization* for deterministic nonlinear dynamical systems and stands as an intermediate notion between *local* and *global stabilization in probability*. This notion requires that, given a *target set*, a trajectory, starting from some compact region of the state space containing the target, remains forever inside some larger compact set, eventually enters any given neighborhood of the target in finite time and remains thereinafter, all these events being guaranteed with some probability. We give a Lyapunov-based sufficient condition for achieving stability in probability and a separation result which splits the control design into a *state feedback* problem and a *filtering problem*. Finally, we point out constructive procedures for solving the state feedback and filtering problem with arbitrarily large region of attraction and arbitrarily small target for a wide class of nonlinear systems, which at least include *feedback linearizable systems*. The generality of the result is promising for applications to other classes of stochastic nonlinear systems. In the deterministic case, our results recover classical stabilization results for nonlinear systems.**

***Index Terms*—Nonlinear stabilization, output feedback, stochastic systems.**

I. INTRODUCTION

THE theory of stochastic processes provides enough mathematical tools to prove important results with extreme clarity and precision, especially in nonlinear filtering and stochastic control [23], [24], [26], [27]. The stochastic framework is particularly suitable for taking into account either randomly varying system parameters or stochastic exogenous inputs, such as turbulence in flight control problems. It is extremely important in the solution of engineering problems that the beauty and elegance of mathematics be supported by constructive and systematic design tools. Recently, in this direction, the stabilization of nonlinear stochastic systems has gained a renewed interest (see [20], [21], and [16]–[18]; see also the textbooks [13] and [19]). By stability, it is usually meant that [13]

- the probability that the trajectory, stemming from the initial state x_0, leaves an ϵ-ball around the origin goes to zero as x_0 tends to the origin;
- the trajectory, stemming from x_0, goes asymptotically to zero almost surely.

This stability, usually known as *stability in probability*, is either *local* or *global (in the large)* according if x_0 is in some (small) neighborhood of the origin or, respectively, it is *any point of the state space*. In [13] Lyapunov-based conditions are given for guaranteeing stability in probability and require the solution of partial differential inequalities (PDIs). In [15] and [17], it has been proved that a backstepping design technique can be successfully implemented for solving globally such PDIs, whenever the state is available for feedback. To our knowledge, the only paper devoted to satisfy such PDIs by *output feedback* is [16], in which the following class of nonlinear systems is considered:

$$\begin{aligned} dx_i(t) &= x_{i+1}(t)dt + \varphi_i^T(y(t))dw(t), \qquad i = 1, \cdots, n-1 \\ dx_n(t) &= u(t)dt + \varphi_n^T(y(t))dw(t) \\ y(t) &= x_1(t) \end{aligned} \tag{1}$$

where $w(t)$ is a Wiener process. Even in a deterministic framework, as shown through some counterexamples in [11], the class of systems for which global stabilization can be achieved using output feedback can be only slightly enlarged with respect to (1). Indeed, as shown in [11], the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_2^j(t) + u(t), \qquad j \geq 3 \\ y(t) &= x_1(t) \end{aligned} \tag{2}$$

cannot be globally stabilized by any C^0 finite-dimensional output feedback dynamic controller. On the other hand, the earlier works of Esfandiari and Khalil [8], [9] and Teel and Praly [12] have shown that feedback linearizable systems, such as for example (2), are instead *semiglobally* stabilizable via output feedback. *Semiglobal stabilization* was introduced in [4] and requires local asymptotic stability plus a region of attraction containing any *a priori* compact set of the state space. The basic ingredients for achieving semiglobal stability via output feedback are *input saturations* and *high-gain observers*: large

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values of the observer gain guarantee that the error between the state and its estimate, generated by the observer itself, goes to zero "sufficiently fast," while input saturations compensate for destabilizing effects such as *peaking* [6]. Peaking is a phenomenon occurring when one is trying to force some state variables to zero as fast as possible while some others show up an impulsive-like behavior.

In the presence of measurement noise, the aforementioned techniques fail to work. In this respect, consider the following class of systems:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2u(t) + B_1\Phi(t,\, u(t),\, x(t))\\ y(t) &= C_2x(t) + C_1\Phi(t,\, u(t),\, x(t))\end{aligned} \tag{3}$$

where $\Phi(t,\, u(t),\, x(t))$ is some vector of model uncertainties and nonlinearities of which nothing but some *bounds* are known. In [1], a general theorem is given on the stabilization of (3) with some region of attraction via *measurement* feedback, i.e., y is available for feedback (in general, we distinguish the measurements from the outputs, if any, to be regulated). The regional stabilization result of [1] is based on $\mathcal{H}_\infty$ *linear control tools* and splits the control design into a *state feedback* problem and a *filtering* problem. Moreover, it recovers in a unified framework many classical results on *semiglobal stabilization via output feedback* in the case of *uncorrupted outputs* (i.e., outputs not affected by noise or uncertainties) such as those contained in [8], [9], and [12].

In this paper, we want to extend the results of [1] to a stochastic framework, by considering the following class of nonlinear stochastic systems, derived from (3):

$$\begin{aligned}dx(t) &= (Ax(t) + B_2u(t) + B_1\Phi(t,\, u(t),\, x(t)))dt\\ &\quad + H(t,\, x(t))dw(t)\\ dy(t) &= (C_2x(t) + C_1\Phi(t,\, u(t),\, x(t)))dt\\ &\quad + K(t,\, x(t))dw(t)\end{aligned} \tag{4}$$

where $w(t)$ is a Wiener process (the more general case with K and H depending also upon u is treated in [2]). Note that the class (4) includes for instance (2) (see Section V for a related example). To this aim, given numbers $\alpha,\, \beta \in [0,\, 1)$ and a pair of compact sets $\mathcal{B}^e \subset \Omega^e$, containing the origin, we introduce a new notion of stabilization in probability ($(\Omega^e,\, \mathcal{B}^e,\, \alpha,\, \beta) - SP$ or *stabilization in probability with target* $\mathcal{B}^e$ and region of attraction Ω^e), which is the natural counterpart of the concept of regional stabilization of the set $\mathcal{B}^e$ for (3) and stands between the standard notions of *local stabilization in probability* and *stabilization in the large* [13]. This novel stability property requires that for sufficiently large k the trajectories of the closed-loop system, resulting from (4), with initial condition in Ω^e remain inside some compact set $\Omega^e(k) \supseteq \Omega^e$ of the state–space, eventually enter any given neighborhood of the target set $\mathcal{B}^e$ in finite time and remain thereinafter with probability greater or equal to any $\gamma < (1-\alpha)(1-\beta)$. The numbers α and β are given *risk margins*: the first one quantifies the risk of leaving $\Omega^e(k)$ with initial condition in Ω^e rather than getting close to the target, while the second one gives a risk margin for remaining close to the target. If $\mathcal{B}^e$ and Ω^e can be taken any *a priori* given compact set of the state–space and α and β any numbers in $[0,\, 1)$, our definition extends to a stochastic setting the notion of *semiglobal stabilization* as introduced in [4] and in what follows we will refer to this property as *semiglobal stabilization in probability*. If Ω^e is all the state–space and $\mathcal{B}^e$ can be taken any *a priori* given compact subset of Ω^e and α and β any *a priori* given numbers in $[0,\, 1)$, our definition gives a stochastic analogue of the concept of *practical stabilization*, which will be referred to as *practical stabilization in probability*. Note that our definition of stabilization does not require $x^e = 0$ be necessarily an equilibrium point for (4) (but only for the nominal system).

As a second step, we prove a sufficient condition for achieving stability in probability with some target set $\mathcal{B}^e$ and region of attraction Ω^e. This result is based on a *probabilistic invariance property* which extends to a stochastic setup the following well-known property for deterministic systems: if there exists a C^1 proper and positive definite function V_k^e such that, along the trajectories of the closed-loop system resulting from (3), $\dot{V}_k^e$ is definite negative on $\Omega^e(k) \setminus \mathcal{B}^e(k)$, where $\Omega^e(k) = \{z\colon V_k^e(z) \le k\}$ and $\{\mathcal{B}^e(k)\}$ is a sequence of open sets, containing the origin and contained in some level set of V_k^e, such that $\mathcal{B}^e(k) \subseteq \mathcal{B}^e$ for sufficiently large k, then any trajectory starting from $\Omega^e \subseteq \Omega^e(k)$ stays forever in $\Omega^e(k)$, eventually enters any given neighborhood of $\mathcal{B}^e$ in finite time and remains thereinafter. Note that if $\limsup_{k\to\infty}(V_k^e(z)/k) = 0$ for each z, any *a priori* given compact set can be included in $\Omega^e(k)$ for sufficiently large k. Our condition is based on satisfying some PDI *on the compact set* $\Omega^e(k) \setminus \mathcal{B}^e(k)$. From a constructive point of view, this is much simpler than solving this PDI *on all the state space* and leads in some cases, as it will be shown, to the choice of simple *quadratic* functions V_k^e and *linear* controllers (see Section IV-A. We show that the problem of finding V_k^e can be split into two lower dimensional problems: one is related to the case in which the *state* x *is available for feedback* and the other to the possibility of stabilizing the system through *output injection*. Furthermore, we show that the conditions of our theorems can be actually met with arbitrarily large region of attraction and arbitrarily small target set for a wide class of nonlinear stochastic systems with uncorrupted outputs, which include at least *feedback linearizable systems*, and we show that *input saturations* and *high gain observers* are still successfull tools for this task. We accomplish this into two steps. First, we give a semiglobal in probability backstepping design procedure for solving the state feedback problem. While this procedure generalizes the classical semiglobal backstepping design for deterministic systems [12], it stands as a *practical semiglobal* version of the corresponding *global* result proved in [15] and [17]. On the other hand, our step-by-step procedure is computationally simpler for the choice at each step of both the Lyapunov functions and the change of coordinates and it is more general in the fact that the control input may be affected by uncertainties. Finally, we give also a recursive procedure to solve the filtering problem and we conclude this paper by working out the main computations for a case study. We remark that our design procedures can be easily implemented also for the same class of systems considered in [16]. The generality of the result is promising for applications to other classes of stochastic nonlinear systems.

We would like to stress that most of the computations involved are algebraic, while this would be no longer the case if *global* stability were pursued. Moreover, as opposite to existing techniques based on *optimal filtering* [23], [24], [26], [27] our approach, while not guaranteeing that the state estimate is optimal in the sense of minimizing the error covariance, on the other hand allows to achieve *robust performance with respect to model uncertainties*.

This paper is organized as follows. In Section II, some basic notions on stochastic processes are recalled. In Section III, we introduce the notion of stabilization in probability $(\Omega^e, \mathcal{B}^e, \alpha, \beta) - SP$ and the control problem is formulated for the class of nonlinear stochastic systems (4) with measurement feedback laws. In Section III-A, sufficient conditions are given for achieving $(\Omega^e, \mathcal{B}^e, \alpha, \beta) - SP$ and, in Section III-B, a separation result is proved for ensuring these sufficient conditions via feedback. Finally, in Section IV, design tools are provided for the class of state feedback linearizable systems, concluding with an illustrative example in Section V. We refer to the recent [3] for a more general problem including inverse optimality constraints.

II. NOTATIONS AND BASICS ON STOCHASTIC PROCESSES

We give some notations extensively used throughout this paper.

- If $\|v\|$ denotes the two-norm of any given vector v, by $\|A\|$ we denote the induced two-norm of any given matrix A; by $\|v\|_A$ we denote the A-norm of v, i.e., $\|v\|_A = \sqrt{v^T A v}$; let col $(v_1, \cdots, v_n)$ be the column vector with ith entry equal to v_i.
- By $\mathcal{SP}^n$ (resp. $\mathcal{SN}^n$) we denote the set of $n \times n$ positive (resp. negative) definite symmetric matrices; by $\mathcal{SSP}^n$ we denote the set of $n \times n$ positive semidefinite symmetric matrices; $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{R}^{\geq}$ the set of nonnegative real numbers.
- For any vector-valued function η: $\mathbb{R}^s \to \mathbb{R}^r$, we denote by η_i (or $[\eta]_i$) its ith component.
- For any given set $\mathcal{S}$, we denote by $\overline{\mathcal{S}}$ its closure and by $\partial\mathcal{S}$ its boundary; moreover, given $\delta > 0$ and a set $\mathcal{S}$, by *δ-neighborhood* of $\mathcal{S}$ we denote the set $\mathcal{S}_\delta = \{z\colon \inf_{y\in\mathcal{S}} \|z-y\| < \delta\}$.

We shortly recall some notions of stochastic processes, referring the reader for the basic concepts to [26] and [27]. We assume that the reader is familiar with the basic notions of probability theory and stochastic processes $\{x(t), t \in \mathbb{R}\}$ on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$ (we assume that the probability space and all the σ-algebras we consider are completed with all the subsets of sets having null measure). We denote by $\mathbf{E}\{\cdot\}$ the expectation and $\mathbf{P}\{\cdot|\cdot\}$ ($\mathbf{E}\{\cdot|\cdot\}$) the conditional probability (expectation).

An important definition regards the notion of *Markov time*. Let $\{\mathcal{F}_t, t \in \mathbb{R}\}$ be an increasing family of right continuous σ-algebras contained in $\mathcal{F}$ (*filtration*).

Definition 2.1: A nonnegative random variable τ, $\tau \leq +\infty$, is called an $\mathcal{F}_t$ Markov time if for all $t \geq 0 \{\omega\colon \tau(\omega) \leq t\} \in \mathcal{F}_t$ (i.e., it is $\mathcal{F}_t$ adapted). If $\mathbf{P}\{\tau < \infty\} = 1$, then τ is called a stopping time.

A stochastic process $\{x(t), t \in \mathbb{R}\}$ is **Gaussian** if every finite combination $\sum_{i=1}^N \alpha_i x_{t_i}$, $\alpha_i \in \mathbb{R}$, is a Gaussian random variable. A stochastic process $\{x(t), t \in \mathbb{R}\}$ is a **Wiener process** (with respect to $\{\mathcal{F}_t, t \in \mathbb{R}\}$) if it is sample continuous, $\mathbf{E}\{x(t)|\mathcal{F}_s\} = x(s)$ and $\mathbf{E}\{(x(t)-x(s))^2|\mathcal{F}_s\} = t-s$ for $t \geq s$. A stochastic process $\{x(t), t \in \mathbb{R}\}$ is a **Markov process** if for any collections $t_1 < \cdots < t_N$ and $r_1, \ldots, r_N$

$$\mathbf{P}\{x_{t_N} < r_N | x_{t_1} = r_1, \cdots, x_{t_{N-1}} = r_{N-1}\} = \mathbf{P}\{x_{t_N} < r_N | x_{t_{N-1}} = r_{N-1}\}. \quad (5)$$

For the corresponding definitions in the multidimensional case, we refer to [28].

By a *stochastic differential equation*, we mean the following equation:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dw(t) \quad (6)$$

with initial condition $x(t_0) = \overline{x}$, where $\{w(t), t \in \mathbb{R}\}$ is a Wiener process (with respect to $\{\mathcal{F}_t, t \in \mathbb{R}\}$). The solution $x(t, t_0, \overline{x})$ of (6), whenever it exists, is a Markov process satisfying

$$x(t, t_0, \overline{x}) = \overline{x} + \int_{t_0}^t f(x(s, t_0, \overline{x}), s)ds + \int_{t_0}^t g(x(s, t_0, \overline{x}), s)dw(s) \quad (7)$$

almost surely (a.s.). The last integral is called *Itô integral*. It is well known [13] that if

$$\|f(x_1, t) - f(x_2, t)\| + \|g(x_1, t) - g(x_2, t)\| \leq K\|x_1 - x_2\|$$
$$\|f(x, t)\| + \|g(x, x)\| \leq H(1 + \|x\|) \quad (8)$$

for all (x_1, t), (x_2, t) and (x, t) in $\mathcal{Z} \times [t_0, T]$, with $\mathcal{Z}$ a compact set containing $\overline{x}$, then there exists an a.s. unique stochastic process $x(t)$, sample continuous and satisfying (7) on $[t_0, \tau_{\mathcal{Z},T}(t)]$, where $\tau_{\mathcal{Z},T}(t) = \min(t, \tau_{\mathcal{Z}}, T)$ and $\tau_{\mathcal{Z}}$ is the Markov time (relatively to the σ-algebra generated by $\{x(s), s \leq t\}$) defined as the first time at which $x(t)$ reaches the boundary of $\mathcal{Z}$ [13].

An important property of solutions of stochastic differential equations is *regularity*. Consider a sequence of increasing bounded domains $\{\mathcal{Z}(n)\}$, containing the origin, such that the distance of the boundary from the origin goes to infinity as n tends to infinity and let $\{\tau_{\mathcal{Z}(n)}\}$ be the corresponding sequence of Markov times. Since $\{\tau_{\mathcal{Z}(n)}\}$ is nondecreasing, its limit exists. We will say that the solution is *regular* if $\lim_{n\to\infty} \tau_{\mathcal{Z}(n)} = \infty$ a.s. Any regular solution can be uniquely (a.s.) extended for all $t \geq t_0$.

Any solution $x(t)$ of (6) satisfies the following *strong Markov property* [27]:

$$\mathbf{P}\{x(t+\tau, t_0, \overline{x}) \in A\} = \int \mathbf{P}\{\tau \in ds;\ x(\tau, t_0, \overline{x}) \in dz\} \cdot \mathbf{P}\{x(t, s, z) \in A\} \quad (9)$$

where τ is any given Markov time (relatively to the σ-algebra generated by $\{x(s), s \leq t\}$). In (9) we can substitute $\mathbf{P}\{\cdot\}$

with its conditioned version $\mathbf{P}\{\cdot|\cdot\}$ as long $\mathbf{P}\{\cdot|\cdot\}$ is regular, i.e., it is a function $p(\omega, A)$, measurable for each fixed A and a probability for each fixed ω [25].

From now on, we will denote $x(t, t_0, \overline{x})$, if not otherwise stated, simply by $x(t)$. Given a C^2 (measurable) function $V: \mathbb{R}^n \to \mathbb{R}$, define

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x}(x)f(x, t) + \frac{1}{2}\mathbf{Tr}\{g^T(x, t)\frac{\partial^2 V}{\partial x^2}(x)g(x, t)\}. \tag{10}$$

*Proposition 2.1 (***Dynkin's Formula***):* Let $\overline{x} \in \mathcal{Z}$. The solution $x(t)$ of (6) satisfies on $[t_0, \tau_{\mathcal{Z}, T}(t)]$ the following equation:

$$\mathbf{E}\{V(x(\tau_{\mathcal{Z}, T}(t)))\} - V(\overline{x}) = \mathbf{E}\left\{\int_{t_0}^{\tau_{\mathcal{Z}, T}(t)} \mathcal{L}V(x(s))\, ds\right\}. \tag{11}$$

The integral appearing in the right-hand side of (11) is meant in the sense that

$$\int_{t_0}^{\tau_{\mathcal{Z},T}(t)} \mathcal{L}V(x(s))\, ds = \int_{t_0}^{t} \xi_{\tau_{\mathcal{Z},T}>t}\mathcal{L}V(x(s))\, ds$$

where $\xi_{\tau_{\mathcal{Z},T}>t}$ is the indicator function corresponding to the event $\{\tau_{\mathcal{Z},T} > t\}$.

Also, we will use extensively the following (generalized) **Čebyšev inequality**:

$$\mathbf{P}\{\eta \notin \mathcal{S}\} \leq \frac{\mathbf{E}\{V(\eta)\}}{\inf_{s\in\mathbb{R}^n\setminus\mathcal{S}}\{V(s)\}} \tag{12}$$

where $\mathcal{S} \subset \mathbb{R}^n$, $V(\cdot)$ is real nonnegative and η is a given random variable such that $\mathbf{E}\{V(\eta)\}$ exists. Finally, we recall the following fundamental formula of the differential calculus.

*Proposition 2.2 (***Itô rule***):* Given a C^2 function $\varphi: \mathbb{R}^n \to \mathbb{R}$ and if $x(t)$ is a solution of (6), then

$$d\varphi(x(t)) = \frac{\partial \varphi}{\partial x}(x(t))dx(t) + \tfrac{1}{2}\,\mathbf{Tr}\{g^T(x(t), t)\frac{\partial^2\varphi}{\partial x^2}(x(t))g(x(t), t)\}\, dt. \tag{13}$$

III. Problem Formulation and Main Results

Let us consider nonlinear stochastic systems Σ of the form (4), where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $w(t)$ is an s-dimensional Wiener process (with respect to the given filtration). Moreover, given a C^0 locally Lipschitz $\Xi: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $\Phi: \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$ is such that for each u, x and t

$$\Phi(t, u, x) \in \{v \in \mathbb{R}^r: \|v\| \leq \Xi(u, x)\}. \tag{14}$$

The class $\mathcal{D}_\Xi$ of functions Φ, defined as in (14), describes model uncertainties and nonlinearities, while the function Ξ determines the available information (i.e., through an upper bound) on $\mathcal{D}_\Xi$. In view of this,

$$\dot{x}(t) = Ax(t) + B_2u(t) \tag{15}$$

$$\dot{y}(t) = C_2x(t) \tag{16}$$

can be seen as the *nominal system* of (4) and with Φ varying in $\mathcal{D}_\Xi$, system (4) describes the family of dynamical systems we refer to in this paper.

The aim of this paper is to study under which conditions it is possible to modify the behavior of (4) in such a way to obtain stability in some "stochastic" sense through the following class of *admissible* controllers $\mathcal{C}(k)$, with integer k

$$u(t) = \eta(F(k)\sigma(t)) \tag{17}$$

$$d\sigma(t) = (L(k)\sigma(t) + B_2u(t))dt + G(k)dy(t) \tag{18}$$

where $\eta: \mathbb{R}^m \to \mathbb{R}^m$ is a locally Lipschitz function. Although from a mathematical point of view there are more general definitions of admissible controls (i.e., nonanticipative or progressively measurable), from the engineer's point of view it is more desirable to have an expression of the control law like (17) and (18), which is directly implementable. Moreover, although from (18) it seems that the measurement of dy is required, this is not the case since (17) and (18) can be rewritten as

$$u(t) = \eta(F(k)(\hat{\sigma}(t) + G(k)y(t)) \tag{19}$$

$$d\hat{\sigma}(t) = (L(k)(\hat{\sigma}(t) + G(k)y(t)) + B_2u(t))dt \tag{20}$$

where $\hat{\sigma} = \sigma - G(k)y$, which requires only the available measurement y.

Denote by $x_k^e(t, t_0, x_0^e) = col(x_k(t, t_0, x_0^e), \sigma_k(t, t_0, x_0^e))$ the trajectory of the closed-loop system $\Sigma \circ \mathcal{C}(k)$ at time $t \geq t_0$ stemming from $x_0^e = col(x_0, \sigma_0)$. With some abuse of notation, wherever there is no ambiguity, we will use $x_k^e(t)$ instead of $x_k^e(t, t_0, x_0^e)$. Moreover, we will assume that the sets we consider always contain the origin.

Definition 3.1: Let $\alpha, \beta \in [0, 1)$ and $\Omega^e, \mathcal{B}^e \subset \mathbb{R}^{2n}$ be compact sets. The system (4) is said to be $(\Omega^e, \mathcal{B}^e, \alpha, \beta)$-stabilizable in probability (or $(\Omega^e, \mathcal{B}^e, \alpha, \beta) - SP$) if there exist a sequence of admissible control laws $\{\mathcal{C}(k)\}$, a sequence of compact sets $\{\Omega^e(k)\}$ and open sets $\{\mathcal{B}^e(k)\}$ of $\mathbb{R}^{2n}$ such that

i) there exists k^* such that $\Omega^e(k) \supset \Omega^e \supset \mathcal{B}^e \supseteq \mathcal{B}^e(k)$ for all $k \geq k^*$;

ii) for each $\delta > 0$ and $\Phi \in \mathcal{D}_\Xi$

$$\liminf_{k\to\infty} \inf_{x_0^e \in \overline{\mathcal{B}}^e(k)} \mathbf{P}\{x_k^e(t) \in \overline{\mathcal{B}}_\delta^e\ \forall t \geq t_0\} \geq 1 - \beta; \tag{21}$$

iii) for each $\delta > 0$ and $\Phi \in \mathcal{D}_\Xi$

$$\begin{aligned}\liminf_{k\to\infty} \inf_{x_0^e \in \Omega^e\setminus\mathcal{B}^e(k)} &\mathbf{P}\{x_k^e(t) \in \Omega^e(k)\ \forall t \geq t_0 \\ &\text{and } x_k^e(t + \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)}) \in \mathcal{B}_\delta^e\ \forall t \geq 0 \\ &\text{and } \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \infty\} \geq (1-\alpha)(1-\beta).\end{aligned} \tag{22}$$

Note that the events in (21) and (22) are measurable by separability and measurability (on the product σ-algebra $\mathcal{F} \times \mathcal{I}$, where $\mathcal{I}$ is the Borel σ-algebra on the line) of the process $x_k^e(t)$ and being $\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} \leq t\}$ adapted to the σ-algebra generated by $\{x_k^e(s),\ \ s \leq t\}$.

The set Ω^e gives the *guaranteed region of attraction* of the closed-loop system $\Sigma \circ \mathcal{C}(k)$, while $\mathcal{B}^e$ represents the *target set* (see Fig. 1).

Property ii) is a *local* property with respect to $\mathcal{B}^e$ (see Fig. 2): for each δ-neighborhood $\mathcal{B}_\delta^e$ of $\mathcal{B}^e$, the probability that the trajectories $x_k^e(t)$ of the closed-loop system $\Sigma\circ\mathcal{C}(k)$, starting from $\overline{\mathcal{B}}^e(k)$ (see point A in Fig. 2), stay forever in $\overline{\mathcal{B}}_\delta^e$ is at least $1-\beta$ for sufficiently large k. Property iii) is a property *in the large* with respect to Ω^e (see Fig. 2): the trajectories of $\Sigma \circ \mathcal{C}(k)$ starting inside Ω^e (see point x_0^e in Fig. 2) remain inside $\Omega^e(k)$,

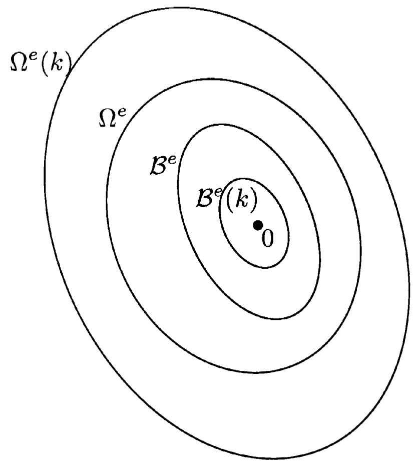

Fig. 1. Sequences of sets $\Omega^e(k)$ and $\mathcal{B}^e(k)$ and their relation with the target set $\mathcal{B}^e$ and the region of attraction Ω^e.

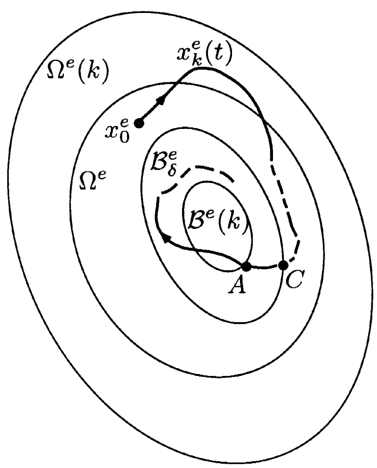

Fig. 2. Systems trajectories and their relation with the sets Ω^e, $\Omega^e(k)$, $\mathcal{B}_\delta^e$ and $\mathcal{B}^e(k)$.

eventually enter any given δ-neighborhood $\mathcal{B}_\delta^e$ of the target set $\mathcal{B}^e$ in finite time (see point C in Fig. 2) and remain thereinafter with probability at least $(1-\alpha)(1-\beta)$ for sufficiently large k.

Note also that iii) requires that $\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \infty$ a.s. As will be clear in the next section, under the standard assumptions of local existence and uniqueness a.s. of trajectories, each Markov time $\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)}$, conditioned to $x_k(t) \in \Omega^e(k)$ for all $t \geq t_0$, is finite a.s. and $\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} \to \infty$ as $k \to \infty$ as long as $\mathcal{B}^e(k)$ can be taken so that $\lim_{k\to\infty} \mathcal{B}^e(k) = \{0\}$.

The numbers α and β are given risk margins: the first one quantifies the risk of leaving the compact set $\Omega^e(k)$ with initial condition in Ω^e rather than getting close to the target, while the second one gives a risk margin for remaining close to the target.

The role of the risk margins, region of attraction and target set is peculiar of our setup and become unessential in the classical definitions given in [13]. According to the different choices of Ω^e, $\mathcal{B}^e$, α, β and $\mathcal{C}(k)$ we have different types of stability, among which we recover the classical ones.

- If $\mathcal{B}^e$ and Ω^e can be taken any *a priori* given compact set of $\mathbb{R}^{2n}$, α and β any *a priori* given numbers in $[0, 1)$ and $\{\mathcal{C}(k)\} = \mathcal{C}$ does not depend on $\mathcal{B}^e$, α, β and the sequences $\mathcal{B}^e(k)$ and $\Omega^e(k)$, Definition 3.1 recovers the classical definition of *asymptotic stability in probability in the large* [13].
- If $\mathcal{B}^e$ and Ω^e can be taken any *a priori* given compact set of $\mathbb{R}^{2n}$, α and β any *a priori* given numbers in $[0, 1)$, we get the stochastic analogue of the concept of *semiglobal stabilization*, as introduced in [4] and we will denote it as *semiglobal stabilization in probability*). In this case, the controller does depend on k.

A. Sufficient Conditions for Achieving Stabilization in Probability With Some Target Set

A technical condition for ensuring stabilization in probability with some target set $\mathcal{B}^e$ is given by the following theorem. The proof is based upon some key ideas found in [13] (see also [22]) and a shorter version can be found also in [3]. Throughout the proof, the operator $\mathcal{L}$ is defined as in (10) *mutatis mutandis* according to the stochastic differential equation considered for x^e.

Theorem 3.1: The system (4) is $(\Omega^e, \mathcal{B}^e, \alpha, \beta) - SP$ if there exist a sequence of admissible control laws $\{\mathcal{C}(k)\}$, a sequence of (at least) C^2, positive–definite and proper functions $\{V_k^e(x^e)\}$, a sequence of C^0, positive–definite functions $\{Q_k^e(x^e)\}$ and open sets $\{\mathcal{B}^e(k)\}$ of $\mathbb{R}^{2n}$ such that

iv) there exists k^* such that $\Omega^e(k) \supset \Omega^e \supset \mathcal{B}^e \supseteq \mathcal{B}^e(k)$ for all $k \geq k^*$, where

$$\Omega^e(k) = \{z \in \mathbb{R}^{2n} : V_k^e(z) \leq k\};$$

v) $\mathcal{L}V_k^e(x^e) \leq -Q_k^e(x^e)$ for all k, $\Phi \in \mathcal{D}_\Xi$ and $x^e \in \Omega^e(k) \setminus \mathcal{B}^e(k)$;

vi)

$$\limsup_{k\to\infty} \sup_{x^e\in\Omega^e\setminus\mathcal{B}^e(k)} \frac{V_k^e(x^e)}{k} \leq \alpha$$

$$\limsup_{k\to\infty} \frac{\sup\limits_{z\in\partial\mathcal{B}^e(k)} V_k^e(z)}{\inf\limits_{z\in\mathbb{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)} \leq \beta \qquad \forall\,\delta > 0.$$

Remark 3.1: The distinctive feature of conditions v)-vi) with respect to similar ones given in [13] and, more recently, in [22], is that they require $\mathcal{L}V_k^e$ be negative definite on some *compact set* $\Omega^e(k) \setminus \mathcal{B}^e(k)$, which on turn varies with k and that the risk margins be in some relation with the level sets of V_k^e. In particular, the first condition appearing in vi) requires that, for sufficiently large k, $\Omega^e \setminus \mathcal{B}^e(k)$ (dashed set in Fig. 3) is included in the level set $k\alpha$ of V_k^e or, roughly speaking, that $\Omega^e \setminus \mathcal{B}^e(k)$ is included in $\Omega^e(k)$ at most by some factor $\alpha \leq 1$.

On the other hand, the second condition appearing in vi) requires that the ratio between the maximal value of V_k^e on $\partial\mathcal{B}^e(k)$ (i.e., the last level curve of V_k^e hitting from outside the boundary of $\mathcal{B}^e(k)$) and the infimum value of V_k^e on the set $\mathbb{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e$ (i.e., the first level curve of V_k^e hitting from inside the boundary of $\mathcal{B}_\delta^e$) is at most $\beta \leq 1$ (see Fig. 4).

Note that, as a consequence of the above remarks, if α can be taken any small number then any *a priori* given compact set can be included in the region of attraction (*semiglobal stabilization in probability*). On the other hand, if β can be taken any small

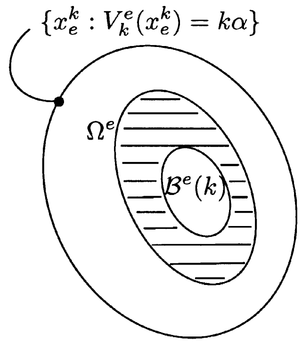

Fig. 3. Geometric interpretation of risk margin α.

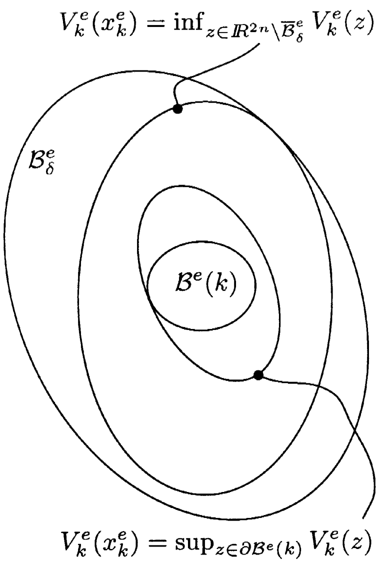

Fig. 4. Geometric interpretation of risk margin β.

number then any *a priori* given compact set can be chosen as the target set (*practical stabilization in probability*).

The proof of Theorem 3.1 is based on a *probabilistic invariance property* which extends to a stochastic setup the following well-known property: if there exists a C^1 proper and positive definite function $V_k^e\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that, along the trajectories $x_k^e(t)$ of the closed-loop system $\Sigma \circ \mathcal{C}(k)$, $\dot{V}_k^e$ is definite negative on $\Omega^e(k) \setminus \mathcal{B}^e(k)$, then any trajectory $x_k^e(t)$ starting from $\Omega^e \subseteq \Omega^e(k)$ stays forever in $\Omega^e(k)$, eventually enters any given δ-neighborhood of $\mathcal{B}^e$ in finite time and remains thereinafter. In our setting, this invariance property corresponds to an event which occurs with probability at least $(1-\alpha)(1-\beta)$. For the above reasons, α and β can be thought of as *risk margins*. $\square$

Proof (of Theorem 3.1): Throughout the proof, $\tau_{\mathcal{S}}(t) = \min\{t, \tau_{\mathcal{S}}\}$, where $\tau_{\mathcal{S}}$ is the Markov time (relatively to the σ-algebra generated by $\{x_k^e(s),\, s \le t\}$) defined as the first time at which the trajectory of (4) reaches the boundary of $\mathcal{S}$. Fix any $\Phi \in \mathcal{D}_\Xi$ and assume $k \ge k^*$ so that $\Omega^e(k) \supset \Omega^e \supset \mathcal{B}^e \supseteq \mathcal{B}^e(k)$ according to iv).

We have to show only ii) and iii) of Definition 3.1, since iv) implies i). As a consequence of the Dynkin's formula (with $\mathcal{Z} = \Omega^e(k) \setminus \mathcal{B}^e(k)$ and $T = \infty$), since $\mathcal{L}V_k^e$ is negative definite on $\Omega^e(k) \setminus \mathcal{B}^e(k)$

$$\mathbf{E}\{V_k^e(x_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(t),\, t_0,\, x_0^e))\} \le V_k^e(x_0^e) \tag{23}$$

for all $x_0^e \in \partial\mathcal{B}^e(k)$. We claim that

$$\begin{aligned}&\mathbf{P}\{x_k^e(t,\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \ \text{ for some } \ t \ge t_0\}\\ &\quad\le \frac{V_k^e(x_0^e)}{\inf_{z\in\mathbf{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)}\end{aligned} \tag{24}$$

for all $x_0^e \in \partial\mathcal{B}^e(k)$. Indeed, since the trajectories are sample continuous

$$\begin{aligned}&\mathbf{P}\{x_k^e(r,\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \text{ for some rational } r \ge t_0\}\\ &\quad= \mathbf{P}\{x_k^e(t,\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \text{ for some, } \qquad t \ge t_0\}\end{aligned} \tag{25}$$

and since for any finite sequence $r_1, \cdots, r_N \ge t_0$

$$\begin{aligned}&\lim_{N\to\infty} \mathbf{P}\{x_k^e(r_j,\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \text{ for some } r_j \ge t_0\\ &\qquad j \in \{1, \cdots, N\}\}\\ &\quad= \mathbf{P}\{x_k^e(r_j,\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \text{ for some } r_j \ge t_0\\ &\qquad j \in \{1, 2, \cdots\}\}\end{aligned} \tag{26}$$

it is enough to prove (24) with t replaced by r, with $r \in \{r_1, \cdots, r_N\} \ge t_0$. From (23), we have for any $\lambda > 0$ and for at least one r

$$\begin{aligned}V_k^e(x_0^e) &\ge \mathbf{E}\{V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r))\}\\ &\ge \int_{\max_r V_k^e > \lambda} V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r)))d\mathbf{P}\\ &\quad+ \int_{\max_r V_k^e \le \lambda} V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r)))d\mathbf{P}\\ &\ge \lambda\mathbf{P}\{\max_r V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r))) > \lambda\}\\ &\quad+ \int_{\max_r V_k^e \le \lambda} V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r)))\, d\mathbf{P}\end{aligned} \tag{27}$$

where for simplicity $V_k^e(s) = V_k^e(x_k^e(s,\, t_0,\, x_0^e))$ for any s. From (27) with $\lambda = \inf_{z\in\mathbf{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)$

$$\begin{aligned}&\mathbf{P}\{x_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r),\, t_0,\, x_0^e))\\ &\qquad \notin \overline{\mathcal{B}}_\delta^e \text{ for some } r \ge t_0,\ \ r \in \{r_1, \cdots, r_N\}\}\\ &\quad\le \mathbf{P}\{\max_r V_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r))) > \inf_{z\in\mathbf{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)\}\\ &\quad\le \frac{V_k^e(x_0^e)}{\inf_{z\in\mathbf{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)}.\end{aligned} \tag{28}$$

Moreover, by continuity of $x_k^e(t, \cdot, \cdot)$

$$\begin{aligned}&\mathbf{P}\{x_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(r),\, t_0,\, x_0^e)\\ &\qquad \notin \overline{\mathcal{B}}_\delta^e \text{ for some } r \ge t_0,\ \ r \in \{r_1, \ldots, r_n\}\}\\ &\quad= \mathbf{P}\{x_k^e(\tau_{\mathcal{B}^e(k)}(r),\, t_0,\, x_0^e)\\ &\qquad \notin \overline{\mathcal{B}}_\delta^e \text{ for some } r \ge t_0,\ \ r \in \{r_1, \cdots, r_n\}\}\end{aligned} \tag{29}$$

and

$$\begin{aligned}0 &= \mathbf{P}\left\{x_k^e(\tau_{\mathcal{B}^e(k)},\, t_0,\, x_0^e) \notin \overline{\mathcal{B}}_\delta^e\right\}\\ &\le \frac{V_k^e(x_0^e)}{\inf_{z\in\mathbf{R}^{2n}\setminus\overline{\mathcal{B}}_\delta^e} V_k^e(z)}.\end{aligned} \tag{30}$$

Using (28)–(30) we obtain (24) with t replaced by r, with $r \in r_1, \ldots, r_N \ge t_0$ and our claim is proved.

To prove iii), we implicitly assume that $x_0^e \in \Omega^e \setminus \mathcal{B}^e(k)$. First of all, we prove

$$\mathbf{P}\left\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} < \infty\right\} = 1 \tag{31}$$

in the case that $x_k^e(t, t_0, x_0^e)$ is regular, since otherwise it is trivially true. Since $Q_k^e(x^e)$ is continuous on its domain, we have $\mathcal{L}V_k^e \leq -\nu(k) < 0$ for all $x^e \in \Omega^e(k) \setminus \mathcal{B}^e(k)$ and for some $\nu(k) > 0$. Directly from Dynkin's formula (with $\mathcal{Z} = \Omega^e(k) \setminus \mathcal{B}^e(k)$ and $T = \infty$) we obtain

$$\nu(k)\mathbf{E}\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(t) - t_0\} \leq V_k^e(x_0^e). \tag{32}$$

Thus, by Čebyšev inequality (with $\mathcal{S} = (-\infty, r - t_0)$, with $r \geq t_0$ ranging over the rationals, and $V(\eta) = \eta = \tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} - t_0$)

$$\begin{aligned}&\mathbf{P}\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} \geq r\}\\ &\quad = \mathbf{P}\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} - t_0 \geq r - t_0\} \leq \frac{V_k^e(x_0^e)}{\nu(k)(r - t_0)}.\end{aligned} \tag{33}$$

Since $V_k^e(x_0^e)/\nu(k)(r - t_0) \to 0$ as $r \to \infty$, from (33) and

$$\lim_{r\to\infty} \mathbf{P}\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} \geq r\} = \mathbf{P}\{\lim_{r\to\infty}\{\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)} \geq r\}\}$$

we obtain (31).

Next, we show that

$$\liminf_{k\to\infty} \inf_{x_0^e \in \Omega^e\setminus\mathcal{B}^e(k)} \mathbf{P}\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \tau_{\Omega^e(k)}\} \geq 1 - \alpha. \tag{34}$$

From Čebyšev inequality (with $\mathcal{S} = (-\infty, 1)$ and $V(\eta) = \eta = \dfrac{V_k^e(x_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}(t), t_0, x_0^e))}{k}$) and (23) and (31), by letting $t \to \infty$

$$\begin{aligned}&\mathbf{P}\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} > \tau_{\Omega^e(k)}\}\\ &\quad \leq \mathbf{P}\left\{\frac{V_k^e(x_k^e(\tau_{\Omega^e(k)\setminus\mathcal{B}^e(k)}, t_0, x_0^e))}{k} \geq 1\right\}\\ &\quad \leq \frac{V_k^e(x_0^e)}{k}.\end{aligned} \tag{35}$$

By (35) and vi), we get

$$\begin{aligned}&\limsup_{k\to\infty} \sup_{x_0^e\in\Omega^e\setminus\mathcal{B}^e(k)} \mathbf{P}\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} > \tau_{\Omega^e(k)}\}\\ &\quad \leq \limsup_{k\to\infty} \sup_{x_0^e\in\Omega^e\setminus\mathcal{B}^e(k)} \frac{V_k^e(x_0^e)}{k} \leq \alpha.\end{aligned} \tag{36}$$

By continuity of $x_k^e(t, \cdot, \cdot)$ and since $\mathcal{B}^e(k) \subset \Omega^e(k)$

$$\mathbf{P}\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} = \tau_{\Omega^e(k)}\} = 0.$$

This, together with (36) and passing to the complementary events, implies (34).

By (24) and vi) and using the continuity of $x_k^e(t, \cdot, \cdot)$, for each $\epsilon, \delta > 0$

$$\mathbf{P}\{x_k^e(t, s, z) \notin \overline{\mathcal{B}}_\delta^e \text{ for some } t \geq s\} < \beta + \epsilon \tag{37}$$

for all $z \in \partial\mathcal{B}^e(k)$. By the strong Markov property, for each $\epsilon, \delta > 0$ as shown in (38) at the bottom of the page, which, passing to the complementary events, implies for each $\delta > 0$

$$\begin{aligned}&\liminf_{k\to\infty} \inf_{x_0^e\in\Omega^e\setminus\mathcal{B}^e(k)}\\ &\mathbf{P}\left\{x_k^e(t + \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)}, t_0, x_0^e)\right.\\ &\qquad \left.\in \overline{\mathcal{B}}_\delta^e\ \forall\, t \geq 0 \,\middle|\, \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \tau_{\Omega^e(k)}\right\} \geq 1 - \beta.\end{aligned} \tag{39}$$

Property iii) of Definition 3.1 follows directly from (31), (34), (39), and the Bayes formula. $\square$

Remark 3.2: From the proof of Theorem 3.1, it is easy to see that iv)–vi) guarantee the properties i)–iii) of Definition 3.1 to hold *uniformly* with respect to t_0. $\square$

B. A Separation Result

Conditions of Theorem 3.1 require the knowledge of the sequences $\{V_k^e\}$, $\{Q_k^e\}$, $\{\mathcal{B}^e(k)\}$ and $\{\mathcal{C}(k)\}$. For the sake of simplicity, we will restrict ourselves to the class of systems (4) for which $B_1 = 0$ and $C_1 = 0$ (i.e., no uncertainties) and $H(t, 0) = 0$ and $K(t, 0) = 0$ (i.e., no additive noise): the more general case with uncertainties and additive noise can be worked out as well but with more involved conditions (see [2]). For this class of systems, we want to prove a *separation result*, which extends to a stochastic framework the one given in [1] and allows to find the above sequences by solving two simpler problems: one is related to the case in which the *state* x *is available for feedback* and the other to the possibility of stabilizing the system through *output injection*. We note that, since $H(t, 0) = 0$ and $K(t, 0) = 0$, the target set can be taken any closed neighborhood of the origin, $\{\mathcal{B}^e(k)\}$ any arbitrary sequence of open neighborhoods of the origin such that $\lim_{k\to\infty} \mathcal{B}^e(k) = 0$ and β any given number in $[0, 1)$. In this case, more simply we will say that the system is (Ω^e, α)-stabilizable in probability (or $(\Omega^e, \alpha) - SP$).

Let

$$\begin{aligned}H(t, x) &= (\, H_1(t, x) \quad \cdots \quad H_r(t, x)\,)\\ K(t, x) &= (\, K_1(t, x) \quad \cdots \quad K_r(t, x)\,)\end{aligned} \tag{40}$$

(i.e., H_j and K_j are the jth columns of H and K) and, without loss of generality, assume that $H^T(t, x)K(t, x) = 0$ for all t and x.

$$\begin{aligned}&\mathbf{P}\left\{x_k^e(t + \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)}, t_0, x_0^e) \notin \overline{\mathcal{B}}_\delta^e \text{ for some } t \geq 0 \,\middle|\, \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \tau_{\Omega^e(k)}\right\}\\ &\quad = \int_{t_0}^{\infty} \int_{z\in\partial\mathcal{B}^e(k)} \left(\mathbf{P}\{\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} \in ds;\ x_k^e(\tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)}, t_0, x_0^e) \in dz \,\middle|\, \tau_{\mathbb{R}^{2n}\setminus\mathcal{B}^e(k)} < \tau_{\Omega^e(k)}\}\right.\\ &\qquad \left.\cdot\, \mathbf{P}\{x_k^e(t, s, z) \notin \overline{\mathcal{B}}_\delta^e \text{ for some } t \in [s, \infty)\}\right) < \beta + \epsilon\end{aligned} \tag{38}$$

Theorem 3.2: Assume the following.

- **[State Feedback (SF)]** there exist $P_{SF}(k), Q_{SF}(k) \in \mathcal{SP}^n$ and $R_1(k) \in \mathcal{SP}^m$ such that

$$x^T\left[A^T P_{SF}(k)+P_{SF}(k)A-P_{SF}(k)B_2R_1^{-1}(k)B_2^TP_{SF}(k)\right]x + \mathbf{Tr}\{H^T(t,x)P_{SF}(k)H(t,x)\} \le -\|x\|^2_{Q_{SF}(k)} \tag{41}$$

  for all k, $x \in \Omega_{SF}(k)$ and t, with $\Omega_{SF}(k) = \{v \in \mathbb{R}^n : \|v\|^2_{P_{SF}(k)} \le k\}$.
- **[Output Injection (OI)]** there exist $P_m(k), Q_m(k) \in \mathcal{SP}^n$, a C^1 function $\delta : \mathbb{R} \to (0, 1]$ such that

$$\varphi(s) = \int_0^s \delta(\vartheta)d\vartheta \to \infty, \qquad s \to \infty$$
$$\frac{\partial^2\varphi}{\partial s^2}(s) \le \frac{\partial\varphi}{\partial s}(s) \qquad \forall\, s \ge 0 \tag{42}$$

  and for all k, $x \in \Omega_{SF}(k)$ and t

$$R_2(k) \ge K(t,x)K^T(t,x) \tag{43}$$

  and

$$\begin{aligned}P_m(k)A + A^TP_m(k) + P_m(k)H(t,x)H^T(t,x)P_m(k)\\ - C_2^TR_2^{-1}(k)C_2 + P_{SF}(k)B_2R_1^{-1}(k)B_2^TP_{SF}(k)\\ \le -Q_m(k).\end{aligned} \tag{44}$$

- **[Coupling (C)]** there exists a C^0 locally Lipschitz function $\eta : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\begin{aligned}&\|\eta(F(k)(x-e)) - F(k)x\|^2_{R_1(k)}\\ &\quad + \sum_{j=1}^r \Big[H_j^T(t,x)P_m(k)H_j(t,x) + K_j^T(t,x)R_2^{-1}(k)\\ &\quad \times C_2P_m^{-1}(k)C_2^TR_2^{-1}(k)K_j(t,x)\Big] - \frac{1}{2}\|x\|^2_{Q_{SF}(k)}\\ &\quad - \delta(\|e\|^2_{P_m(k)})\Big[\|F(k)e\|^2_{R_1(k)} + \frac{1}{2}\|e\|^2_{Q_m(k)}\Big]\\ &\le 0\end{aligned} \tag{45}$$

  for all t, $x \in \Omega_{SF}(k)$ and $e \in \mathbb{R}^n$, with $F(k) = -R_1^{-1}(k)B_2^TP_{SF}(k)$.
- **[Risk Margin (RM)]** for some compact set $\Omega^e \subset \mathbb{R}^{2n}$ there exists k^* such that

$$\Omega^e \subset \Omega^e(k) \tag{46}$$

  for all $k \ge k^*$ and

$$\limsup_{k\to\infty} \sup_{(x,\sigma)\in\Omega^e} \frac{V_k^e(x,\sigma)}{k} \le \alpha \tag{47}$$

  with

$$\begin{aligned}V_k^e(x,\sigma) &= \|x\|^2_{P_{SF}(k)} + \varphi(\|e\|^2_{P_m(k)})\\ \Omega^e(k) &= \left\{(x,\sigma) \in \mathbb{R}^n \times \mathbb{R}^n : V_k^e(x,\sigma) \le k\right\}.\end{aligned} \tag{48}$$

Under the aforementioned assumptions, there exists an admissible controller which renders (4) $(\Omega^e, \alpha) - SP$.

Proof: Throughout the proof, unless otherwise stated, we will omit k and the arguments of the functions involved. Moreover, we assume $k \ge k^*$ so that $\Omega^e(k) \supset \Omega^e$ by (46). Let

$$u = \eta(F(k)\sigma) \tag{49}$$
$$d\sigma = (L(k)\sigma + B_2u)dt + G(k)dy \tag{50}$$

with

$$\begin{aligned}F(k) &= -R_1^{-1}(k)B_2^TP_{SF}(k)\\ L(k) &= A - G(k)C_2\\ G(k) &= P_m^{-1}(k)C_2^TR_2^{-1}(k)\end{aligned} \tag{51}$$

be our candidate controller. The closed-loop system is

$$\begin{aligned}dx &= (Ax + B_2u)dt + Hdw\\ de &= Ledt + (H - GK)dw\end{aligned} \tag{52}$$

where $e = x - \sigma$ and $u = \eta(F\sigma)$.

Let $V_{SF}(x) = x^TP_{SF}x$. By (10)

$$\begin{aligned}\mathcal{L}\varphi &= \frac{\partial\varphi}{\partial e}Le + \frac{1}{2}\mathbf{Tr}\left\{(H-GK)^T\frac{\partial^2\varphi}{\partial e^2}(H-GK)\right\}\\ \mathcal{L}V_{SF} &= 2x^TP_{SF}\Big[Ax + B_2u\Big] + \mathbf{Tr}\Big\{H^TP_{SF}H\Big\}\end{aligned} \tag{53}$$

(where $\mathcal{L}\varphi$ means $\mathcal{L}\varphi(\|e\|^2_{P_m})$). Moreover

$$\frac{\partial^2\varphi}{\partial e^2} = 2\frac{\partial^2\varphi}{\partial s^2}\Big|_{s=e^TP_me}P_mee^TP_m + 2\frac{\partial\varphi}{\partial s}\Big|_{s=e^TP_me}P_m. \tag{54}$$

Since $HK^T = 0$ and $\mathbf{Tr}(AB) = \mathbf{Tr}(BA)$, by (42), (43), and (54) one has

$$\begin{aligned}&\frac{1}{2}\mathbf{Tr}\left\{(H-GK)^T\frac{\partial^2\varphi}{\partial e^2}(H-GK)\right\}\\ &\quad\le \frac{\partial\varphi}{\partial s}\Big|_{s=e^TP_me}\Big[\mathbf{Tr}\{H^TP_mH + K^TG^TP_mGK\}\\ &\qquad + e^TP_m(HH^T + GKK^TG^T)P_me\Big]\end{aligned} \tag{55}$$

for all t, $x \in \Omega_{SF}(k)$ and $e \in \mathbb{R}^n$.

Since $HK^T = 0$ and $0 < \delta(s) \le 1$ for all s, using (44) and (55), we have for all t, $x \in \Omega_{SF}(k)$ and $e \in \mathbb{R}^n$

$$\begin{aligned}&\mathcal{L}\varphi + \delta(\|e\|^2_{P_m})\|Fe\|^2_{R_1}\\ &\quad\le \delta(\|e\|^2_{P_m})\Big[2e^TP_mLe + \|Fe\|^2_{R_1}\\ &\qquad + e^TP_m\Big(HH^T + GKK^TG^T\Big)P_me\\ &\qquad + \mathbf{Tr}\{H^TP_mH + K^TG^TP_mGK\}\Big]\\ &\quad\le \delta(\|e\|^2_{P_m})\Big[e^T(L^TP_m + P_mL)e + \|Fe\|^2_{R_1}\\ &\qquad + e^TP_m\Big(HH^T + GR_2G^T\Big)P_me\\ &\qquad + \mathbf{Tr}\{H^TP_mH + K^TG^TP_mGK\}\Big]\\ &\quad\le -\delta(\|e\|^2_{P_m})\Big[\|e\|^2_{Q_m}\\ &\qquad - \mathbf{Tr}\{H^TP_mH + K^TG^TP_mGK\}\Big]\end{aligned} \tag{56}$$

where G is defined as in (51).

Moreover, by completing the square and using (41)

$$\begin{aligned}\mathcal{L}V_{SF} \leq& \|u - Fx\|^2_{R_1}\\ &+ x^T\Big[A^T P_{SF} + P_{SF}A - P_{SF}B_2R_1^{-1}B_2^T P_{SF}\Big]x\\ &+ \mathbf{Tr}\{H^T(t, x)P_{SF}H(t, x)\}\\ \leq& -\|x\|^2_{Q_{SF}} + \|u - Fx\|^2_{R_1} \end{aligned} \tag{57}$$

for all u, t and $x \in \Omega_{SF}(k)$.

From (45), summing up together (56) and (57), we conclude that

$$\mathcal{L}V^e \leq -\frac{1}{2}\Big[\|x\|^2_{Q_{SF}} + \delta(\|e\|^2_{P_m})\|e\|^2_{Q_m}\Big] \tag{58}$$

for all t and $(x, \sigma) \in \Omega^e(k)$. Our result follows from (58), (RM), and Theorem 3.1. □

Remark 3.3: Consider the class of systems (4) with $H_j(t, x) = H_j x$, $K_j(t, x) = K_j x$ for all j, t and x. Pick $\eta(s) = s$ (i.e., *linear controllers*) and $\delta(s) = 1$ (i.e., *quadratic* Lyapunov functions). With our positions, while (41) and (44) become *algebraic matrix inequalities*, (45) boils down to the following inequality:

$$\begin{aligned}&\frac{1}{2}Q_{SF}(k) - \sum_{j=1}^{r}\\ &\quad \Big(H_j^T P_m(k) H_j + K_j^T R_2^{-1}(k) C_2 P_m^{-1}(k) C_2^T R_2^{-1}(k) K_j\Big)\\ &\geq 0.\end{aligned} \tag{59}$$

In particular, when $K_j = 0$ for all j, we recover the stabilization result of [14], but in a weaker sense since stability in probability is implied by mean square stability. □

IV. Design Tools

The conditions of Theorem 3.2 do not provide any constructive procedure to find $P_{SF}(k)$, $P_m(k)$ and the functions η and δ. In the next two sections, we want to outline algorithms for accomplishing this task for a wide class of nonlinear stochastic system with *uncorrupted outputs*, which include at least feedback linearizable systems (further generalizations can be worked out as well but with more complicate algorithms). First, we give a *semiglobal in probability backstepping* design procedure for solving the state feedback problem, then a recursive procedure to solve the filtering problem. We will prove that *input saturations* and *high-gain observers*, as in the deterministic case (see [8], [9], and [12]), are key tools to handle the problem. For simplicity, we will omit the argument t throughout this section, if not otherwise explicitly stated. Moreover, the results of this section can be extended to the more general case of uncertainties and additive noise at the price of extra complexity in the design (see [2]).

A. Backstepping Design

In this section, we will show that the following class of nonlinear stochastic systems:

$$\begin{aligned}dx_1 &= (a_{11}x_1 + x_2)dt + h_1(t, x)dw_1\\ dx_2 &= (a_{21}x_1 + a_{22}x_2 + x_3)dt + h_2(t, x)dw_2\\ \vdots &= \cdots\end{aligned}$$

$$dx_n = \left(\sum_{j=1}^{n} a_{nj}x_j + u\right)dt + h_n(t, x)dw_n \tag{60}$$

where $x = col(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$, $a_{ji} \in \mathbb{R}$, $h_j(t, 0) = 0$ for all t and $w_1, \cdots, w_n$ are independent Wiener processes, is *semiglobally stabilizable in probability*, as long as a suitable *triangularity property* is satisfied. To state our assumptions, we need the following definitions.

Definition 4.1: We will say that a function $F(t, x)$ satisfies the property TS(j) if F is norm-bounded from above by a locally Lipschitz function of $x_1, \ldots, x_j$, uniformly w.r.t. $x_{j+1}, \ldots, x_n$ and t.

We are ready to give the main result of this section, which gives a *semiglobal* version of the *global* result proved in [15].

Theorem 4.1: Consider system (60); if h_j, $j = 1, \ldots, n$, satisfies the $TS(j)$ property, then (60) is semiglobally stabilizable in probability through a linear state feedback controller. □

As a first step toward the proof of Theorem 4.1, consider the following interconnected systems:

$$\begin{aligned}d\pi_{j-1} =& (A_{j-1}(k)\pi_{j-1} + B_{j-1,2}(k)x_j)dt\\ &+ H_{j-1}(t, x_j, \varrho_{j-1})dw_{j-1}\end{aligned} \tag{61}$$

$$dx_j = x_{j+1}dt + \widetilde{h}_j(t, x_j, \varrho_{j-1})d\widetilde{w}_j \tag{62}$$

with states $\pi_{j-1} \in \mathbb{R}^{j-1}$ and $x_j \in \mathbb{R}$, control input $x_{j+1} \in \mathbb{R}$, exogenous inputs $\varrho_{j-1} = \mathrm{col}(\pi_{j-1}, x_{j+1}, \ldots, x_n)$ and noises $w_{j-1} \in \mathbb{R}^{j-1}$ and $\widetilde{w}_j \in \mathbb{R}$. The key idea of the proposed control design is to consider x_j as a control input so to stabilize in probability the π_{j-1}-dynamics and use the obtained control law to define a change of coordinate on x_j and the control x_{j+1} for stabilizing in probability the overall system (62). Throughout this section, the operator $\mathcal{L}$ is defined as in (10) *mutatis mutandis* according to the associated stochastic differential equations.

Definition 4.2: We will say that (61) satisfies the property $DI(j-1)$ if H_{j-1} satisfies the $TS(j-1)$ property and there exist C^0 functions $P_{j-1}, Q_{j-1} : \mathbb{R}^+ \to \mathcal{SP}^{j-1}$, $R_{j-1,1} : \mathbb{R}^+ \to \mathbb{R}^+$ and $\Gamma_{j-1} : \mathbb{R}^+ \to \mathbb{R}^{1\times(j-1)}$ such that

- for all t, x_j and ϱ_{j-1} such that

  $$\pi_{j-1} \in \Omega_{j-1}(k) = \{v \in \mathbb{R}^{j-1} : \|v\|^2_{P_{j-1}(k)} \leq k\}$$

  one has

  $$\begin{aligned}&\mathcal{L}V_{k,j-1} + \|x_j - \Gamma_{j-1}(k)\pi_{j-1}\|^2_{R_{j-1,1}(k)}\\ &\quad \leq -\|\pi_{j-1}\|^2_{Q_{j-1}(k)} + \|x_j - F_{j-1}(k)\pi_{j-1}\|^2_{R_{j-1,1}(k)}\end{aligned} \tag{63}$$

  where

  $$V_{k,j-1}(\pi_{j-1}) = \|\pi_{j-1}\|^2_{P_{j-1}(k)} \tag{64}$$

  $$\begin{aligned}F_{j-1}(k) =& -R_{j-1,1}^{-1}(k)B_{j-1,2}^T(k)P_{j-1}(k)\\ &+ \Gamma_{j-1}(k).\end{aligned} \tag{65}$$

We state the following intermediate lemma.

Lemma 4.1: Assume that (61) satisfies $DI(j-1)$ and $\widetilde{h}_j$ of (62) satisfy $TS(j)$. Under the above assumptions, there exists a C^0 function $\lambda_j : \mathbb{R}^+ \to (0, 1)$ such that

$$d\pi_j = (A_j(k)\pi_j + B_{j2}(k)x_{j+1})dt + H_j(t, x_{j+1}, \varrho_j)dw_j \tag{66}$$

satisfies $DI(j)$, with the conditions shown at the bottom of the page.

Proof: For simplicity, throughout the proof and whenever there is no ambiguity, we omit the arguments of the functions involved.

Using the fact that $\widetilde{h}_j$ satisfies TS(j) and by $DI(j-1)$

$$\mathbf{Tr}\{H_{j-1}^T F_{j-1}^T(k) F_{j-1}(k) H_{j-1}\} + 2\widetilde{h}_j^2\Big|_{x_j=F_{j-1}(k)\pi_{j-1}} \leq \|\pi_{j-1}\|^2_{\widetilde{M}_j(k)} \tag{67}$$

for all t and $\pi_{j-1} \in \Omega_{j-1}(k)$, where $\widetilde{h}_j\Big|_{x_j=F_{j-1}(k)\pi_{j-1}}$ denotes $\widetilde{h}_j$ evaluated for $x_j = F_{j-1}(k)\pi_{j-1}$ and for some C^0 function $\widetilde{M}_j : \mathbb{R}^+ \to \mathcal{SSP}^{j-1}$. Since $\widetilde{M}_j(k)$ can be chosen as a function of $P_{j-1}(k)$ (or, what is the same, $\Omega_{j-1}(k)$) only, we can select $\lambda_j : \mathbb{R}^+ \to (0, 1)$ such that

$$\lambda_j^2(k)\widetilde{M}_j(k) \leq \frac{Q_{j-1}(k)}{4}. \tag{68}$$

Pick any C^0 function $\widetilde{P}_j : \mathbb{R}^+ \to (0, 1)$ and define

$$\Omega_j(k) = \{(v_1, v_2) \in \mathbb{R}^{j-1}\times\mathbb{R} : \|v_1\|^2_{P_{j-1}(k)} + \widetilde{P}_j(k)v_2^2 \leq k\}. \tag{69}$$

Find a C^0 function $\widetilde{Q}_j : \mathbb{R}^+ \to \mathbb{R}^+$

$$\frac{\widetilde{Q}_j(k)}{2} \geq \frac{R_{j-1,1}(k)}{\lambda_j(k)}. \tag{70}$$

By the Itô rule

$$d\zeta_j = \lambda_j(k)(x_{j+1} - \Gamma_j(k)\pi_j)dt + \overline{h}_j dw_j \tag{71}$$

where

$$\overline{h}_j = \lambda_j(k)\begin{pmatrix} -F_{j-1}(k)H_{j-1}(k) & \widetilde{h}_j \end{pmatrix}$$

and

$$\Gamma_j(k) = F_{j-1}(k)\begin{pmatrix} A_{j-1}(k) + B_{j-1,2}(k)F_{j-1}(k) & \dfrac{B_{j-1,2}(k)}{\lambda_j(k)} \end{pmatrix}.$$

First, we solve the state feedback problem for (71) and, finally, we use the $DI(j-1)$ property to prove the lemma. To this aim, using the fact that $\widetilde{h}_j$ satisfies $TS(j)$ and by $DI(j-1)$ and (67) we can write

$$\begin{aligned}\mathbf{Tr}\{\overline{h}_j^T\overline{h}_j\} &= \lambda_j^2(k)\Big[\mathbf{Tr}\{H_{j-1}^T F_{j-1}^T(k)F_{j-1}(k)H_{j-1}\} + \widetilde{h}_j^2\Big] \\ &\leq \lambda_j^2(k)\Big[\mathbf{Tr}\{H_{j-1}^T F_{j-1}^T(k)F_{j-1}(k)H_{j-1}\} \\ &\quad + 2(\widetilde{h}_j - \widetilde{h}_j\Big|_{\zeta_j=0})^2 + 2\widetilde{h}_j^2\Big|_{\zeta_j=0}\Big] \\ &\leq \lambda_j^2(k)\Big[\|\pi_{j-1}\|^2_{\widetilde{M}_j(k)} + \widetilde{N}_j(k)\zeta_j^2\Big]\end{aligned} \tag{72}$$

for all t, x_{j+1} and ϱ_j such that $\pi_j \in \Omega_j(k)$ and for some C^0 function $\widetilde{N}_j : \mathbb{R}^+ \to \mathbb{R}^{\geq}$. Find a C^0 function $\widetilde{R}_{j1} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$-\frac{\lambda_j^2(k)\widetilde{P}_j^2(k)}{\widetilde{R}_{j1}(k)} + \lambda_j^2(k)\widetilde{P}_j(k)\widetilde{N}_j(k) = -\widetilde{Q}_j(k) \tag{73}$$

and let $\widetilde{V}_{kj}(\zeta_j) = \widetilde{P}_j(k)\zeta_j^2$ and $\widetilde{F}_j(k) = -\lambda_j(k)\widetilde{R}_{j1}^{-1}(k)\widetilde{P}_j(k)$. From (73)

$$\begin{aligned}&\mathcal{L}\widetilde{V}_{kj} + \|x_{j+1} - \Gamma_j(k)\pi_j\|^2_{\widetilde{R}_{j1}(k)} \\ &\quad \leq -\|\zeta_j\|^2_{\widetilde{Q}_j(k)} + \|x_{j+1} - \Gamma_j(k)\pi_j - \widetilde{F}_j(k)\zeta_j\|^2_{\widetilde{R}_{j1}(k)}.\end{aligned} \tag{74}$$

If

$$P_j(k) = \begin{pmatrix} P_{j-1}(k) & 0 \\ 0 & \widetilde{P}_j(k) \end{pmatrix} \tag{75}$$

we have

$$\begin{aligned}&\mathbf{Tr}\{H_j^T P_j(k) H_j\} \\ &\quad = \mathbf{Tr}\{H_{j-1}^T P_{j-1}(k) H_{j-1}\}\Big|_{x_j=\frac{\zeta_j}{\lambda_j(k)}+F_{j-1}(k)\pi_{j-1}} \\ &\quad\quad + \lambda_j^2(k)\widetilde{P}_j(k)\Big[\mathbf{Tr}\{H_{j-1}^T F_{j-1}^T(k)F_{j-1}(k)H_{j-1}\} \\ &\quad\quad + \widetilde{h}_j^2\Big]\Big|_{x_j=\frac{\zeta_j}{\lambda_j(k)}+F_{j-1}(k)\pi_{j-1}}.\end{aligned} \tag{76}$$

$$\begin{aligned}\pi_j &= \begin{pmatrix} \pi_{j-1} \\ \zeta_j \end{pmatrix} \\ \zeta_j &= \lambda_j(k)(x_j - F_{j-1}(k)\pi_{j-1}) \\ w_j &= \begin{pmatrix} w_{j-1} \\ \\ \widetilde{w}_j \end{pmatrix} \\ A_j(k) &= \begin{pmatrix} A_{j-1}(k) + B_{j-1,2}(k)F_{j-1}(k) & B_{j-1,2}(k) \\ -\lambda_j(k)F_{j-1}(k)(A_{j-1}(k) + B_{j-1,2}(k)F_{j-1}(k)) & -F_{j-1}(k)B_{j-1,2}(k) \end{pmatrix} \\ B_{j2}(k) &= \lambda_j(k)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ H_j(t, x_{j+1}, \varrho_j) &= \begin{pmatrix} H_{j-1}\Big|_{x_j=\frac{\zeta_j}{\lambda_j(k)}+F_{j-1}(k)\pi_{j-1}} & 0 \\ -\lambda_j(k)F_{j-1}(k)H_{j-1}\Big|_{x_j=\frac{\zeta_j}{\lambda_j(k)}+F_{j-1}(k)\pi_{j-1}} & \lambda_j(k)\widetilde{h}_j\Big|_{x_j=\frac{\zeta_j}{\lambda_j(k)}+F_{j-1}(k)\pi_{j-1}} \end{pmatrix}.\end{aligned}$$

From (68) and (76), summing up together (63) and (74), it follows that

$$\begin{aligned}\mathcal{L}V_{k,j-1} &+ \|x_j - \Gamma_{j-1}(k)\pi_{j-1}\|^2_{R_{j-1,1}(k)} \\ &\quad + \mathcal{L}\widetilde{V}_{kj} + \|x_{j+1} - \Gamma_j(k)\pi_j\|^2_{\widetilde{R}_{j1}(k)} \\ &= \mathcal{L}V_{kj} + \|x_{j+1} - \Gamma_j(k)\pi_j\|^2_{R_{j1}(k)} \\ &\le -\|\pi_j\|^2_{Q_j(k)} + \|x_{j+1} - F_j(k)\pi_j\|^2_{R_{j1}(k)} \qquad (77)\end{aligned}$$

with

$$\begin{aligned}R_{j1}(k) &= \widetilde{R}_{j1}(k) \\ Q_j(k) &= \frac{1}{2}\begin{pmatrix} Q_{j-1}(k) & 0 \\ 0 & \widetilde{Q}_j(k)\end{pmatrix} \\ F_j(k) &= \begin{pmatrix} 0 & \widetilde{F}_j(k)\end{pmatrix} + \Gamma_j(k) \\ &= -R_{j1}^{-1}(k)B_{j2}^T P_j(k) + \Gamma_j(k). \qquad (78)\end{aligned}$$

□

Now, we use Lemma 4.1 to prove Theorem 4.1. First, we show that the system, obtained from (60) by applying repeatedly Lemma 4.1, is semiglobally stabilizable in probability through a linear controller. We prove the existence of C^0 functions $A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$, $B_2 : \mathbb{R}^+ \to \mathbb{R}^{n\times 1}$, $\Gamma : \mathbb{R}^+ \to \mathbb{R}^{1\times n}$, $P_{SF}, Q_{SF} : \mathbb{R}^+ \to \mathcal{SP}^n$, $R_1 : \mathbb{R}^+ \to \mathbb{R}^+$, and $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that

$$d\pi = (A(k)\pi + B_2(k)u)dt + H(t, \pi)dw \qquad (79)$$

and for all $t, \pi \in \Omega_{SF}(k) = \{v \in \mathbb{R}^n : \|v\|^2_{P_{SF}(k)} \le k\}$

$$\begin{aligned}\mathcal{L}V_{k,SF} &+ \|u - \Gamma(k)\pi\|^2_{R_1(k)} \\ &\le -\|\pi\|^2_{Q_{SF}(k)} + \|u - F(k)\pi\|^2_{R_1(k)} \qquad (80)\end{aligned}$$

with $V_{k,SF}(\pi) = \|\pi\|^2_{P_{SF}(k)}$ and $F(k) = -R_1^{-1}(k)B_2^T(k)P_{SF}(k) + \Gamma(k)$. Indeed, we can proceed by steps. Let $\Gamma_1(k) = -a_{11}$, $\lambda_1(k) = 1$, $\varrho_1 = \mathrm{col}(\pi_1, x_3, \cdots, x_n)$ and $\pi_1 = x_1$. Following the proof of Lemma 4.1, we find C^0 functions $\widetilde{P}_1, \widetilde{Q}_1, \widetilde{R}_{11} : \mathbb{R}^+ \to \mathbb{R}^+$, $\widetilde{N}_1 : \mathbb{R}^+ \to \mathbb{R}^{\ge}$ such that

$$-\frac{\widetilde{P}_1^2(k)}{\widetilde{R}_{11}(k)} + \widetilde{P}_1(k)\widetilde{N}_1(k) = -\widetilde{Q}_1(k) \qquad (81)$$

and for all t, x_2, ϱ_1, and

$$\pi_1 \in \Omega_1(k) = \left\{v \in \mathbb{R} : v^2 \le \frac{k}{\widetilde{P}_1(k)}\right\}$$

$$h_1^2(t, x) \le \widetilde{N}_1(k)\pi_1^2. \qquad (82)$$

Let $\widetilde{V}_{k1}(\pi_1) = \widetilde{P}_1(k)\pi_1^2$, $\widetilde{F}_1(k) = -\widetilde{R}_{11}^{-1}(k)\widetilde{P}_1(k)$. From (81), for all t, x_2 and ϱ_1 such that $\pi_1 \in \Omega_1(k)$ we obtain

$$\mathcal{L}V_{k1} + x_2^2 R_{11}(k) \le -x_1^2 Q_1(k) + (x_2 - F_1(k)\pi_1)^2 R_{11}(k) \qquad (83)$$

with $P_1(k) = \widetilde{P}_1(k)$, $Q_1(k) = \widetilde{Q}_1(k)$, $V_{k1} = \widetilde{V}_{k1}$, $R_{11}(k) = \widetilde{R}_{11}(k)$, $F_1(k) = \widetilde{F}_1(k)$. The other steps follow directly by applying repeatedly Lemma 4.1, with $P_{SF}(k) = \mathrm{diag}\{\widetilde{P}_1(k), \ldots, \widetilde{P}_n(k)\}$.

Let

$$\begin{aligned}\pi_j &= \pi_j(x_1, \cdots, x_j) \\ &= col(\zeta_1(x_1), \cdots, \zeta_j(x_1, \cdots, x_j)) \\ &\qquad j = 1, \cdots, n \\ \zeta_1(x_1) &= x_1 \\ \zeta_j(x_1, \cdots, x_j) &= \lambda_j(k)(x_j - F_{j-1}(k)\pi_{j-1}(x_1, \cdots, x_{j-1})) \\ &\qquad j = 2, \cdots, n. \qquad (84)\end{aligned}$$

Since $\widetilde{P}_j(k) \in (0, 1)$, then

$$\limsup_{k\to\infty} \sup_{\pi\in\Omega_{SF}} \frac{V_{k,SF}(\pi)}{k} = 0 \qquad (85)$$

for each $\pi \in \mathbb{R}^n$ and for each compact set $\Omega_{SF} \subset \mathbb{R}^n$.

From (80), it follows that, if $u = F(k)\pi$, then

$$\mathcal{L}V_{k,SF} \le -\frac{1}{2}\|\pi\|^2_{Q_{SF}(k)}, \ \forall\, \pi \in \Omega_{SF}(k). \qquad (86)$$

We conclude from (85), (86), and Theorem 3.1, with x_k^e replaced by π, V_k^e by $V_{k,SF}$, Ω^e by Ω_{SF} and with $\mathcal{B}^e(k)$ any sequence of open neighborhoods of the origin in $\mathbb{R}^n$ converging to the origin, that (79) is *semiglobally stabilizable in probability*. However, this in general does not guarantee this property also for the original system (60), since in x-coordinates the region of attraction may shrink. Note that

$$\begin{aligned} x &= \begin{pmatrix} \zeta_1 \\ \frac{\zeta_2}{\lambda_2} + F_1\zeta_1 \\ \frac{\zeta_3}{\lambda_3} + F_2\pi_2 \\ \vdots \\ \frac{\zeta_n}{\lambda_n} + F_{n-1}\pi_{n-1}\end{pmatrix} \\ &= \begin{pmatrix} \zeta_1 \\ \frac{\zeta_2}{\lambda_2} + \widetilde{F}_1\zeta_1 \\ \frac{\zeta_3}{\lambda_3} + \widetilde{F}_2\zeta_2 \\ \vdots \\ \frac{\zeta_n}{\lambda_n} + \widetilde{F}_{n-1}\zeta_{n-1}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Gamma_2\pi_2 \\ \vdots \\ \Gamma_{n-1}\pi_{n-1}\end{pmatrix} \qquad (87)\end{aligned}$$

and $V_{k,SF}(\pi)$ can be rewritten as follows:

$$\begin{aligned}&V_{k,SF}(\pi) \\ &\quad = \widetilde{P}_1(k)\zeta_1^2 + \widetilde{P}_2(k)\lambda_2(k)(x_2 - \widetilde{F}_1(k)\zeta_1)^2 + \cdots \\ &\qquad + \widetilde{P}_n(k)\lambda_n(k)(x_n - \widetilde{F}_{n-1}(k)\zeta_{n-1} - \Gamma_{n-1}\pi_{n-1})^2 \le k. \qquad (88)\end{aligned}$$

Since $\lambda_j(k)$, $\widetilde{F}_{j-1}(k)$ and $\Gamma_j(k)$ are functions only of $\widetilde{P}_1(k), \ldots, \widetilde{P}_{j-1}(k)$, it easily follows that one can guarantee an arbitrary region of attraction in x-coordinates by suitably choosing the functions $\widetilde{P}_j(k)$, $j = 1, \ldots, n$.

B. $\mathcal{H}_\infty$ Filtering

Assume that in (4) $y, u \in \mathbb{R}$, $H(t, x) = B_2\varrho(x)$, with smooth $\varrho(x)$ such that $\varrho(0) = 0$, $K(t, x) = 0$ and (A, B_2, C_2) is invertible with no invariant zeroes (the case $y \in \mathbb{R}^p$ and

$u \in \mathbb{R}^m$ can be treated in a similar way). It is known [10] that under these assumptions there exists a change of coordinates $z = Zx$ such that (4) reads out in the new coordinates

$$\begin{aligned} dz &= (\widehat{A}z + \widehat{B}_2 u)dt + \varrho(Z^{-1}z)\widehat{B}_2 dw \\ dy &= \widehat{C}_2 z dt \end{aligned} \tag{89}$$

where

$$\widehat{A} = \begin{pmatrix} a_{11} & 1 & 0 & \cdots & 0 & 0 \\ a_{12} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & 0 & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

$$\widehat{B}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\widehat{C}_2 = (1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0.)$$

In the previous section, we have shown how to satisfy (SF) (with (41) replaced by (57)) in π-coordinates for the class of systems (89). Since to (80) there corresponds a formal equivalent expression in x-coordinates, we consider for simplicity $(P_{SF}(k), Q_{SF}(k))$ and $(\Omega_{SF}(k), \Omega_{SF})$ directly in x-coordinates.

To simplify notations, we will omit the hats and assume directly $z = x$. We want to show that also (OI), (C) and (RM) of Theorem 3.2 can be met for (89). In particular, this happens if

- there exist $P_m(k), Q_m(k) \in \mathcal{SP}^n$ such that

$$\begin{aligned} &P_m(k)A + A^T P_m(k) + \varrho^2(x) P_m(k) B_2 B_2^T P_m(k) \\ &\quad - R_2^{-1}(k) C_2^T C_2 + F^T(k) R_1(k) F(k) \le -Q_m(k) \end{aligned} \tag{90}$$

  with $F(k) = -R_1^{-1}(k) B_2^T P_{SF}(k)$, for all k, for some $R_2(k) > 0$ and for all $x \in \Omega_{SF}(k)$;
- there exist a C^1 function $\delta : \mathbb{R} \to (0, 1]$ and a bounded locally Lipschitz function $\eta : \mathbb{R}^m \to \mathbb{R}^m$ such that, if φ is defined as in Theorem 3.2, then $\dfrac{\partial^2 \varphi}{\partial s^2}(s) \le \dfrac{\partial \varphi}{\partial s}(s)$ for all $s \ge 0$

$$\limsup_{k\to\infty} \sup_{(x,\sigma)\in\Omega^e} \frac{\|x\|^2_{P_{SF}(k)} + \varphi(\|e\|^2_{P_m(k)})}{k} = 0 \tag{91}$$

  and

$$\begin{aligned} &\|\eta(-F(k)(x-e)) + F(k)x\|^2_{R_1(k)} - \frac{\|x\|^2_{Q_{SF}(k)}}{2} \\ &\quad + \varrho^2(x) B_2^T P_m(k) B_2 \\ &\quad \le \delta(\|e\|^2_{P_m(k)}) \Big[\|F(k)e\|^2_{R_1(k)} + \frac{1}{2}\|e\|^2_{Q_m(k)} \Big] \end{aligned} \tag{92}$$

  for all $x \in \Omega_{SF}(k)$, $e \in \mathbb{R}^n$ and k.

The key idea of the filter design, similar to the one used in [3], is to choose η bounded and φ logarithmic in such a way to satisfy (90) and (92) by increasing the observer gain while enlarging the region of attraction. Throughout the remaining part of this section, we consider k fixed. First, let us prove (90). Let

$$\begin{aligned} P_n(k) &= [P]_{nn}(k) \\ P_{n-1}(k) &= \begin{pmatrix} [P]_{n-1,n-1}(k) & [P]_{n-1,n}(k) \\ [P]_{n,n-1}(k) & [P]_{nn}(k) \end{pmatrix} \\ P_j(k) &= \begin{pmatrix} [P]_{jj}(k) & v_j(k) \\ v_j^T(k) & P_{j+1}(k) \end{pmatrix} \qquad 1 \le j \le n-2 \end{aligned} \tag{93}$$

where $[P]_{ji}(k) = [P]_{ij}(k)$ and $v_j(k) = \mathrm{row}([P]_{j,j+1}(k), 0, \dots, 0)$. Moreover, define

$$\begin{aligned} Q_n(k) &= 2[P]_{n,n-1}(k) + \omega_n(P_n(k)) \\ Q_j(k) &= \begin{pmatrix} 2[P]_{j,j-1}(k) + \omega_j(P_j(k)) & \xi_j^T(P_j(k)) \\ \xi_j(P_j(k)) & Q_{j+1}(k) \end{pmatrix} \\ &\qquad j = 2, \dots, n-1 \\ Q_1(k) &= \begin{pmatrix} -L(k) + \omega_1(P_1(k)) & \xi_1^T(P_1(k)) \\ \xi_1(P_1(k)) & Q_2(k) \end{pmatrix} \end{aligned} \tag{94}$$

where $L : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^0 function and $\omega_j(P_j(k))$ and $\xi_j(P_j(k))$ are linear functions of the entries of $P_j(k)$.

By direct calculations

$$Q_1(k) = P_1(k)(J+I) + (J+I)^T P_1(k) - L(k) C_2^T C_2. \tag{95}$$

We claim that for each $[P]_{nn}(k) > 0$ such that

$$\frac{\|x\|^2_{Q_{SF}(k)}}{8} \ge [P]_{nn}(k)\varrho^2(x) = \varrho^2(x) B_2^T P_m(k) B_2 \tag{96}$$

there exist C^0 functions $L^* : \mathbb{R}^+ \to \mathbb{R}^+$ and $P_1 : \mathbb{R}^+ \to \mathcal{SP}^n$ such that $Q_1(k) \in \mathcal{SN}^n$ for all C^0 functions $L(k) \ge L^*(k)$. Indeed

- pick $[P]_{n,n-1} < 0$ such that $Q_n < 0$ and $[P]_{n-1,n-1} > 0$ such that $P_{n-1} \in \mathcal{SP}^2$;
- pick $[P]_{n-1,n-2}(k) < 0$ such that $Q_{n-1}(k) \in \mathcal{SN}^2$ and $[P]_{n-2,n-2}(k) > 0$ such that $P_{n-2}(k) \in \mathcal{SP}^3$;
- at step i, fix $[P]_{n-i,n-1-i}(k) < 0$ such that $Q_{n-i}(k) \in \mathcal{SN}^{i+1}$ and $[P]_{n-1-i,n-1-i}(k) > 0$ such that $P_{n-1-i}(k) \in \mathcal{SP}^{i+2}$;
- finally, fix $[P]_{11}(k) > 0$ such that $P_1(k) \in \mathcal{SP}^n$ and $L^*(k) > 0$ such that $Q_1(k) \in \mathcal{SN}^n$ for all C^0 functions $L(k) \ge L^*(k)$.

Define

$$\begin{aligned} Q_m(k) &= \epsilon^2(k) P_m(k) \\ P_m(k) &= \widetilde{P}_m(\epsilon(k)) \\ &= \mathrm{diag}\{\epsilon^{2(n-1)}(k), \epsilon^{2(n-2)}(k), \cdots, 1\} P_1(k) \\ &\qquad \cdot \mathrm{diag}\{\epsilon^{2(n-1)}(k), \epsilon^{2(n-2)}(k), \dots, 1\} \end{aligned} \tag{97}$$

where $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is some C^0 function (to be specified later) such that $\lim_{k\to\infty} \epsilon(k) = \infty$.

In what follows, for sake of simplicity we will omit the argument k when there is no ambiguity. We claim that there exists a C^0 function $\epsilon_1^* : \mathbb{R}^+ \to \mathbb{R}^+$ such that Q_m and P_m, defined in (97), solve (90) with $R_2 \le \dfrac{1}{L\epsilon^{2(2n-1)}}$, for all C^0 functions $\epsilon \ge \epsilon_1^*$. Indeed, substituting in (90), left and right-multiplying by $\mathrm{diag}\{\epsilon^{-2(n-1)}, \epsilon^{-2(n-2)}, \cdots, 1\}$ and dividing both members by ϵ^2, we obtain

$$P_1(J+S_1(\epsilon))+(J+S_1(\epsilon))^T P_1 - \frac{R_2^{-1}C_2^T C_2}{\epsilon^{2(2n-1)}} + S_2(\epsilon) \le -P_1 \tag{98}$$

where $S_1, S_2 : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ are C^0 functions such that $\lim_{\epsilon\to\infty} S_j(\epsilon)=0$, $j=1, 2$, and

$$J=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0\end{pmatrix}.$$

Choose $\epsilon_1^* : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{k\to\infty}\epsilon_1^*(k)=\infty$ and

$$P_1 S_1(\epsilon) + S_1^T(\epsilon)P_1 + S_2(\epsilon) \le P_1 \tag{99}$$

for all $x\in\Omega_{SF}(k)$ and C^0 functions $\epsilon \ge \epsilon_1^*$. By (95) and (99), it follows that (98) (and, thus, (90)) is indeed true. Note also that, since $G=P_m^{-1}C_2^T R_2^{-1} \ge L P_m^{-1}\epsilon^2\epsilon^{4(n-1)}$, the gain G grows at least as ϵ^2, which gives as a result *high-gain observer*.

Next, we prove (92). Choose $\delta(s)=\dfrac{1}{\epsilon(1+s)}$ if $s\ge 0$,

$$\eta(s)=\begin{cases} s, & \text{if } |s|\le \Delta(k)\\ \dfrac{s}{|s|}\Delta(k), & \text{otherwise}\end{cases} \tag{100}$$

where $\Delta(k)=\max_{x\in\Omega_{SF}(k)}|F(k)x|$. Note that with our choices $\dfrac{\partial^2\varphi}{\partial s^2}\le\dfrac{\partial\varphi}{\partial s}$ for all $s\ge 0$, where $\varphi(s)=\dfrac{1}{\epsilon}\ln(1+s)$, and the function η is a *saturation function*. Moreover, by (96) it follows that (92) is satisfied for all $x\in\Omega_{SF}(k)$ and $e\in\mathbb{R}^n$ if there exists a C^0 function $\epsilon_2^*:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\frac{\epsilon^2\|e\|^2_{\widetilde{P}_m(\epsilon)}+\|Fe\|^2_{R_1}}{\epsilon(1+\|e\|^2_{\widetilde{P}_m(\epsilon)})} - \|\eta(-F(x-e))+Fx\|^2_{R_1} + \frac{1}{4}\|x\|^2_{Q_{SF}} \ge 0 \tag{101}$$

for all C^0 functions $\epsilon\ge\epsilon_2^*$, $x\in\Omega_{SF}(k)$ and $e\in\mathbb{R}^n$.

In order to prove (101)), find a covering $\cup_{j=1}^3\mathcal{M}_j$ of $\{(x, e)\in\Omega_{SF}(k)\times\mathbb{R}^n\}$, with

$$\begin{aligned}\mathcal{M}_1&=\left\{(x, e)\in\Omega_{SF}(k)\times\mathbb{R}^n : \|x-e\|\le\vartheta_1;\ \|e\|\le\frac{\vartheta_1}{2}\right\}\\ &\qquad \vartheta_1>0\\ \mathcal{M}_2&=\{(x, e)\in\Omega_{SF}(k)\times\mathbb{R}^n : \|x-e\|\ge\vartheta_1;\ \|e\|\le\vartheta_2\}\\ &\qquad \vartheta_2\le\frac{\vartheta_1}{2}\\ \mathcal{M}_3&=\{(x, e)\in\Omega_{SF}(k)\times\mathbb{R}^n : \|e\|\ge\vartheta_2\}\end{aligned} \tag{102}$$

for some $\vartheta_2>0$ and $\vartheta_1>0$ such that $\eta(-F(x-e))=-F(x-e)$ for all $\|x-e\|\le\vartheta_1$.

First of all, it is easy to see that there exists a C^0 function $\epsilon_3^*:\mathbb{R}^+\to\mathbb{R}^+$ such that (101) holds for all C^0 functions $\epsilon\ge\epsilon_3^*$ and $(x, e)\in\mathcal{M}_1$.

Moreover, (101) holds for all $(x, e)\in\mathcal{M}_2$ for some $\vartheta_2\le\frac{\vartheta_1}{2}$ and for all $k>0$. Indeed, for all $(x, e)\in\mathcal{M}_2$, since $\vartheta_1>\vartheta_2$ we have $\frac{1}{4}\|x\|^2_{Q_{SF}}>0$. It follows that for any such a x by continuity there exists $e_x>0$ such that (101) holds for all $\|e\|\le e_x$ and for all C^0 functions $\epsilon\ge\epsilon_3^*$. Since $\mathcal{N}=\{x\in\Omega_{SF}(k) : \|x\|\ge\frac{\vartheta_1}{2}\}$ is compact and $\vartheta_1>0$, one can take $\vartheta_2=\min\{\frac{\vartheta_1}{2}, \min_{x\in\mathcal{N}} e_x\}$.

We are left with proving that there exists a C^0 function $\epsilon_4^*:\mathbb{R}^+\to\mathbb{R}^+$ such that (101) holds for all $(x, e)\in\mathcal{M}_3$ and for all C^0 functions $\epsilon\ge\epsilon_4^*$. On the other hand, this readily follows by the boundedness of η and since

$$\lim_{\epsilon\to\infty}\inf_{\|e\|\ge\vartheta_2}\frac{\epsilon\|e\|^2_{\widetilde{P}_m(\epsilon)}}{1+\|e\|^2_{\widetilde{P}_m(\epsilon)}}=\infty. \tag{103}$$

Pick $\epsilon_2^*\ge\max\{\epsilon_3^*, \epsilon_4^*\}$.

Finally, to conclude the proof, we need only to satisfy (91). Since

$$\lim_{s\to 0} s\ln s^{-r}=0 \qquad \forall\, r\ge 0$$

then, by definition of $P_m(k)$, there exists a C^0 function $\epsilon_5^*:\mathbb{R}^+\to\mathbb{R}^+$ such that for all C^0 functions $\epsilon\ge\epsilon_5^*$

$$\lim_{k\to\infty}\frac{1}{k}\left[\|x\|^2_{P_{SF}(k)}+\frac{1}{\epsilon}\ln(1+\|e\|^2_{\widetilde{P}_m(\epsilon)})\right]=0 \tag{104}$$

for each $(x, e)\in\Omega_{SF}(k)\times\mathbb{R}^n$, which proves (91). Finally, we conclude that (90)–(92) hold as long as $\epsilon:\mathbb{R}^+\to\mathbb{R}^+$ is any C^0 function such that $\epsilon\ge\max\{\epsilon_1^*, \epsilon_2^*, \epsilon_5^*\}$.

Theorem 4.2: Any system (89) is semiglobally stabilizable in probability via measurement feedback. □

Proof: (89) can be put in the form (60) after a linear change of coordinates. Finally, combine Theorem 4.1 with the previous results. □

V. EXAMPLE

We conclude by the following example:

$$\begin{aligned} dx_1&=x_2dt\\ dx_2&=udt+x_2^l dw\\ dy&=x_1dt\end{aligned} \tag{105}$$

which, as pointed out in the introduction, if $l\ge 3$ it is not globally stabilizable via C^0 dynamic output feedback with deterministic disturbances. However, it is in the form (89) and, thus, Theorem 4.2 applies.

We begin by applying the semiglobal backstepping algorithm proposed in Section IV-A. At the first step, we have

$$\begin{aligned} f_1(t, x)&=h_1(t, x)=0\\ \Omega_1(k)&=\left\{v\in\mathbb{R} : |v|\le\sqrt{\frac{k}{\widetilde{P}_{SF,1}(k)}}\right\}\\ \lambda_1(k)&=1\\ \widetilde{Q}_1(k)&=\widetilde{P}_1^2(k)\\ \widetilde{N}_1(k)&=0\\ \widetilde{R}_{11}(k)&=1\\ \widetilde{F}_1(k)&=-\widetilde{P}_1(k)\\ A_1&=0\\ B_{12}&=1\\ H_1&=0\end{aligned} \tag{106}$$

where $0 < \widetilde{P}_1(k) < 1$ and $\lim_{k\to\infty} \widetilde{P}_1(k)k^{-1} = 0$.

From the second step of the algorithm, we get

$$\begin{aligned} f_2(t,x) &= 0 \\ h_2(t,x) &= x_2^j \\ \Omega_2(k) &= \left\{(v_1, v_2) \in \mathbb{R}^2 : v_1^2\widetilde{P}_1(k) + v_2^2\widetilde{P}_2(k) \le k\right\} \\ \lambda_2^2(k) &= \frac{1}{8}\left(\frac{\widetilde{P}_1(k)}{k}\right)^{l+1} \\ \zeta_2 &= \lambda_2(x_2 + \widetilde{P}_1(k)x_1) \\ \pi_2 &= \mathrm{col}(\pi_1, \zeta_2) \\ \Gamma_2(k) &= -\widetilde{P}_1(k)\begin{pmatrix} -\widetilde{P}_1(k) & \lambda_2^{-1}(k)\end{pmatrix} \\ \widetilde{h}_2^2 &= 2(x_2^l - x_2^l|_{\zeta_2=0})^2 + 2x_2^{2l}|_{\zeta_2=0} \\ \widetilde{M}_2(k) &= 2\left(\frac{k}{\widetilde{P}_1(k)}\right)^{l-1} \\ \widetilde{N}_2(k) &= \max_{\pi_2\in\Omega_2(k)} \\ &\quad \cdot \frac{2\left[\left(\frac{\zeta_2}{\lambda_2(k)} - \widetilde{P}_1(k)\right)^l - \widetilde{P}_1^2(k)(\zeta_2 - \widetilde{P}_1\pi_1)^l\right]^2}{\zeta_2^2}. \end{aligned} \tag{107}$$

Moreover

$$\widetilde{R}_{21}(k) = \left[\frac{64\left(\frac{k}{\widetilde{P}_1(k)}\right)^{\frac{3(l+1)}{2}}}{\widetilde{P}_2(k)} + \widetilde{N}_2(k)\right]^{-1}\widetilde{P}_2(k)$$

$$F_2(k) = \Gamma_2(k) - \widetilde{R}_{21}^{-1}(k)\lambda_2(k)\widetilde{P}_2(k)\begin{pmatrix}0 & 1\end{pmatrix} \tag{108}$$

with $0 < \widetilde{P}_2(k) < 1$ such that

$$\lim_{k\to\infty} \frac{\widetilde{P}_2(k)\lambda_2^2(k)(1+\widetilde{P}_1^2(k))}{k} = 0. \tag{109}$$

Next, we apply to (105) in coordinates (x_1, ζ_2) the results of Section IV-B. We have

$$\widehat{A} = \begin{pmatrix} \widetilde{F}_1(k) & 1 \\ -\lambda_2(k)\widetilde{F}_1^2(k) & -\widetilde{F}_1(k)\end{pmatrix}$$
$$\widehat{B}_2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{110}$$

It is easy to find positive numbers $\widetilde{\varrho}_1$, $\widetilde{\varrho}_2$ and positive smooth function $\widetilde{\varrho}_3(\pi)$ such that

$$\varrho^2(\pi) \le (x_1^2\widetilde{\varrho}_1 + \zeta_2^2\widetilde{\varrho}_2)\widetilde{\varrho}_3(\pi). \tag{111}$$

A positive–definite solution $P_1(k)$ of (95) is given by

$$P_1(k) = \begin{pmatrix} [P]_{11}(k) & [P]_{12}(k) \\ [P]_{21}(k) & [P]_{22}(k)\end{pmatrix}$$

with

$$0 < [P]_{22}(k) \le \frac{1}{8}\min_{\pi_2\in\Omega_2(k)} \frac{x_1^2\widetilde{Q}_1(k) + \zeta_2^2\widetilde{Q}_2(k)}{(x_1^2\widetilde{\varrho}_1 + \zeta_2^2\widetilde{\varrho}_2)\widetilde{\varrho}_3(\pi)} \tag{112}$$

$$[P]_{21}(k) < (\widetilde{F}_1(k) - 1)[P]_{22}(k) \tag{113}$$

$$[P]_{11}(k) > \frac{[P]_{12}^2(k)}{[P]_{22}(k)} \tag{114}$$

$$\begin{aligned} &\frac{\left([P]_{11}(k) + 2[P]_{12}(k) - [P]_{22}(k)\widetilde{F}_1^2(k)\right)^2}{2\left|(1-\widetilde{F}_1(k))[P]_{22}(k) + 2[P]_{21}(k) + [P]_{12}(k)\right|} \\ &\quad + 2[P]_{11}(k)(\widetilde{F}_1(k)+1) \\ &\quad - 2[P]_{12}\widetilde{F}_1^2(k) < L(k). \end{aligned} \tag{115}$$

Define

$$P_m(k) = \begin{pmatrix}\epsilon^2(k) & 0 \\ 0 & 1\end{pmatrix} P_1(k) \begin{pmatrix}\epsilon^2(k) & 0 \\ 0 & 1\end{pmatrix} \tag{116}$$

$$Q_m(k) = \epsilon^2(k)P_m(k) \tag{117}$$

$$S_1(\epsilon(k)) = \begin{pmatrix} \frac{\widetilde{F}_1(k)}{\epsilon^2(k)} & 0 \\ -\frac{\widetilde{F}_1^2(k)}{\epsilon^4(k)} & -\frac{\widetilde{F}_1(k)}{\epsilon^2(k)}\end{pmatrix} \tag{118}$$

$$\begin{aligned} S_2(\epsilon(k)) = \frac{1}{\epsilon^2(k)}\Bigg[&\begin{pmatrix}\epsilon^{-2}(k) & 0 \\ 0 & 1\end{pmatrix} \\ &\cdot F_2^T(k)R_{21}(k)F_2(k)\begin{pmatrix}\epsilon^{-2}(k) & 0 \\ 0 & 1\end{pmatrix} \\ &+ P_1(k)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}P_1(k)\Bigg]. \end{aligned} \tag{119}$$

Finally, choose a C^0 function $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ in such a way to satisfy (99), (101), and (104).

VI. Conclusion

We have introduced a novel and general notion of stabilization in probability, which includes the classical ones. This notion extends to a stochastic setting the classical notion of semiglobal stabilization. The main difference with respect to the deterministic case is the presence of two risk margins: the first one gives the risk of going to infinity (in finite time) rather than getting close to the target, while the second one gives the chances of remaining close to the target. For a general class of stochastic uncertain systems we have shown a Lyapunov-based sufficient condition, which recovers classical results in the case of deterministic systems. Also, we have proved a separation result for guaranteeing this sufficient condition, which splits the control design into a *state feedback* problem and a *filtering problem*. Finally, we pointed out constructive procedures for solving the state feedback and filtering problem with arbitrarily large region of attraction and arbitrarily small target for a wide class of nonlinear systems, which includes feedback linearizable systems: for the state feedback problem we used classical ideas of backstepping and for the filtering problem we used high-gain observers and control saturation techniques.

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