A Sufficient Condition for Nonlinear Noninteracting Control with Stability Via Dynamic State Feedback

Stefano Battilotti

Abstract—**In this paper we give a sufficient condition to solve the problem of achieving local noninteracting control with asymptotic stability via dynamic state feedback for a nonlinear affine system. Our condition, which is automatically satisfied in the case of linear systems, although only sufficient, is indeed helpful in overcoming some of the geometric obstructions recently pointed out in the literature.**

I. Introduction

Let us consider a square nonlinear affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$
$$y_i = h_i(x) \qquad i = 1,\cdots,m \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$, f and g_i are smooth vector fields, $h_i: \mathbb{R}^n \to \mathbb{R}$ are smooth scalar functions, defined on some open subset of $\mathbb{R}^n$. Without loss of generality we shall assume that $x_0 = 0$ is an equilibrium point. Moreover, throughout the paper, we set

$$h(x) = (h_1(x) \cdots h_m(x))^T$$
$$g(x) = (g_1(x) \cdots g_m(x))$$
$$G = \text{span}\{g(x)\}$$
$$K_i = \ker\{dh_i(x)\}$$

and suppose that $\dim G = m$.

The class of state feedbacks we consider is given by static state feedbacks

$$u = \alpha(x) + \beta(x)v$$

with β square nonsingular matrix (such feedbacks are called *regular*), and dynamic state feedbacks

$$u = \alpha(x,w) + \beta(x,w)v$$
$$\dot{w} = \gamma(x,w) + \delta(x,w)v$$

where α, β, δ, and γ are smooth functions defined on suitable open subsets. The problem of modifying (1) by means of either static or dynamic feedback so that the ith scalar output depends on only the ith scalar input is generally known as the nonlinear noninteracting control problem. Historically, necessary and sufficient conditions for solving the nonlinear noninteracting control problem (without stability) by means of regular static state feedback were given in [4], [5], [6], [8], [11], [12], [17]. Dynamic extension has also been investigated in [7], [13], [18], [19].

The problem of stability and noninteraction has been completely solved in the case of linear systems in [1] and [3]. In the case that only static state feedback is allowed, Gilbert [1] has shown that any decoupling feedback induces a well-defined internal dynamics whose asymptotic properties are independent of the feedback itself. Therefore, internal stability and noninteraction can be attained via static state feedback only if these dynamics are asymptotically stable. In the case that also dynamic state feedback is allowed, Wonham and Morse [3] have given a geometric necessary and sufficient condition to solve the problem of stability and noninteraction, showing that if noninteraction can be achieved, then it also can be achieved with stability.

In the nonlinear case, the problem of local stability and noninteraction has been considered only recently by Isidori and Grizzle [20], who have extended the result of Gilbert [1] to the class of systems (1). They have shown that there exists a well-defined internal dynamics (called P^* dynamics) which is fixed with respect to any decoupling regular static state feedback, showing that the local asymptotic stability of this dynamics is a necessary condition to obtain both noninteraction and stability via regular static state feedback.

On the basis of these facts, in the case that the P^* dynamics is unstable, i.e., the problem of stability and noninteraction cannot be solved by means of any regular static state feedback, it is natural to ask if something similar to the result of Wonham and Morse can hold for nonlinear systems. A counterexample in [20] clearly shows that it is not possible, in general, to obtain noninteraction and stability by means of dynamic state feedback. The reason was actually shown by Wagner [22], who has found out that, for the class of systems considered in [20], there exists a well-defined dynamics (called Δ_{MIX} dynamics), which cannot be eliminated by any regular (in a sense to be specified) dynamic feedback which makes (1) noninteractive. Therefore, the Δ_{MIX} dynamics, which is contained in the P^* dynamics and is trivial in the case of linear systems (in perfect accordance with the result of Wonham and Morse), must be asymptoti-

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The author is with the Dipartimento di Informatica e Sistemistica, Via Eudossiana 18-00184-Roma, Italy.
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cally stable if noninteraction and stability is sought by means of dynamic state feedback.

In this paper, we give a sufficient condition, which is trivially satisfied in the case of linear systems, according to the Morse and Wonham's theory, to solve the problem of local noninteracting control with stability by means of regular (in a sense to be specified) dynamic feedback. Our result turns out to be useful in the case that the P^* dynamics is unstable, i.e., when noninteraction and stability cannot be attained by means of regular static state feedback. For simplicity, we shall consider the same class of systems considered in [20]. The general case will be treated elsewhere. We shall prove that, under some mild regularity assumptions, we can obtain local noninteraction and asymptotic stability by means of dynamic state feedback if the Δ_{MIX} dynamics is locally asymptotically stable and certain rank conditions are satisfied. As it will be clear, this gives a generalization of the result of Wonham and Morse for the class of systems considered. As in [3], the crucial point of our construction is the definition of some extended independent distributions, starting from some distributions which are not necessarily independent and are defined on the original system (Section IV).

The paper is organized in the following way. In Section II, some fundamental concepts are recalled. In Sections III through V, we assume that the $\Delta_{MIX} = 0$ and solve the problem in this particular case; then, in Section VII, we relax this assumption, giving a more general result. In Section VI, we consider also the possibility of reducing the dimension of the dynamic extension. Finally in Section VIII an example, for which our assumptions are satisfied, is given, clearly showing how to construct a decoupling and stabilizing dynamic state feedback and how our condition can weaken the requirement that the P^* dynamics be locally asymptotically stable.

II. A Review of Some Fundamental Concepts

The reader is referred for details and proofs to [23].

We say that (1) has some *relative degree* at x_0 if the two following conditions are satisfied: there exist integers $\{r_1, \cdots, r_m\}$, $1 \le r_i < \infty$, such that

1) $L_{g_i} L_f^k h_i = 0$ for $0 \le k < r_i - 1$, $j = 1, \cdots, m$ and $L_{g_i} L_f^{r_i - 1} h_i(x_0) \neq 0$ for some j ($L_\theta \phi$ is the Lie derivative of the smooth function ϕ along the smooth vector field θ);

2) the matrix (called *decoupling matrix*)

$$\begin{pmatrix} L_{g_1} L_f^{r_1 - 1} h_1 & \cdots & L_{g_m} L_f^{r_1 - 1} h_1 \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_m - 1} h_m & \cdots & L_{g_m} L_f^{r_m - 1} h_m \end{pmatrix}$$

is nonsingular at x_0.

In what follows, if θ is any smooth vector field and Δ is any smooth distribution, by $[\theta, \Delta]$ we mean the set $\{[\theta, \tau]: \tau \in \Delta\}$, where $[\cdot, \cdot]$ is the Lie bracket of any two vector fields.

A smooth distribution Δ is said to be locally invariant under a smooth vector field θ if $[\theta, \Delta] \subset \Delta$ for all x in a neighborhood of x_0.

A smooth distribution Δ is said to be locally controlled invariant if there exists a smooth feedback $u = \alpha(x) + \beta(x)v$ such that Δ is locally invariant under $\tilde{f} = f + g\alpha$ and $\tilde{g}_j = g\beta_j$, for $j = 1, \cdots, m$, where β_j is the jth column of β.

The family of invariant distributions under a set of smooth vector fields $\{\tau_1, \cdots, \tau_m\}$ and containing a given smooth distribution K is closed under intersection, so that the minimal element exists and it is a smooth distribution. We denote by $\langle \tau_1, \cdots, \tau_m \mid K \rangle$ this distribution, which can be computed by means of the following lemma.

Lemma II.1: Let us define the following sequence of distributions

$$S_0 = K$$

$$S_k = \sum_{i=1}^{m} [\tau_i, S_{k-1}] + S_{k-1}.$$

If there exists a k^* such that $S_{k^*} = S_{k^*+1}$ (in this case, we say that $\langle \tau_1, \cdots, \tau_m \mid K \rangle$ is *finitely computable*), then

$$S_{k^*} = \langle \tau_1, \cdots, \tau_m \mid K \rangle. \qquad \diamond$$

Actually, in the above definition, we implicitly suppose that S_k is a smooth distribution (see [23] for a more general setting). Note that if each S_k is a nonsingular (i.e., of constant dimension) distribution, then $\langle \tau_1, \cdots, \tau_m \mid K \rangle$ is finitely computable. The involutivity of $\langle \tau_1, \cdots, \tau_m \mid K \rangle$, i.e., $[\theta_1, \theta_2] \in \langle \tau_1, \cdots, \tau_m \mid K \rangle$ $\forall \theta_1, \theta_2 \in \langle \tau_1, \cdots, \tau_m \mid K \rangle$, is ensured by the next lemma.

Lemma II.2: Suppose that K is spanned by some of the vector fields $\{\tau_1, \cdots, \tau_m\}$. The distribution $\langle \tau_1, \cdots, \tau_m \mid K \rangle$ is involutive if it is nonsingular. $\diamond$

Finally, consider an involutive nonsingular distribution Δ, which is invariant under the vector field θ and suppose $\theta(0) = 0$. The restriction of θ to the leaf of Δ passing through x_0, which will be denoted by $\theta \mid \Delta$, defines a vector field on that leaf.

III. Fundamental Assumptions and Definitions

For simplicity, we consider the same class of systems as considered in [20].

Assumption 1: The system (1) has some relative degree at x_0.

Assumption 1 can be shown to imply that there exists a smooth regular static state feedback such that the resulting closed-loop system is locally noninteractive (but not necessarily asymptotically stable) [5], [8], [23], i.e., the ith scalar input does not affect the scalar jth output for $j \neq i$. In what follows, we can suppose that (1) has already been rendered noninteractive by means of regular static state feedback. We suppose also that x_0 is still an equilibrium point for that system.

Now let

$$P_i^* = \langle f, g_1, \cdots, g_m \mid \text{span}\{g_j : j \neq i\} \rangle \qquad i = 1, \cdots, m$$

$$R_i^* = \langle f, g_1, \cdots, g_m \mid \text{span}\{g_i\} \rangle \qquad i = 1, \cdots, m$$

$$P_0 = \langle f, g_1, \cdots, g_m \mid G \rangle$$

and

$$P^* = \bigcap_{i=1}^{m} P_i^*.$$

To compute P_i^*, R_i^*, and P_0 by means of standard algorithms (see Section II), we require that they are finitely computable (in general, it is not enough to suppose that they are nonsingular): to this purpose, we assume that each S_k (see Lemma II.1) is nonsingular in a neighborhood of x_0 and throughout the paper, by saying locally finitely computable, we implicitly mean this. Moreover, we suppose constancy of dimensions and if n is the dimension of the system dim $P_0 = n$, i.e., the system is strongly accessible at x_0 [2]. The latter condition is very natural in the case of linear systems, since it implies full controllability. We collect all these assumptions in the following.

Assumption 2: The distributions R_i^*, P_i^*, and P_0 are locally finitely computable and

$$P_0, P_i^*, P^*, P_i^* + \bigcap_{j\neq i} P_j^*, R_i^* \text{ and } \sum_{j\neq i} R_j^* \text{ and } \sum_{j=1}^{m} R_j^* \qquad i = 1,\cdots,m$$

have constant dimension in a neighborhood of x_0. Moreover, (1) is strongly accessible at x_0.

Remark 1: Setting $f = g_0$, since R_i^* locally finitely computable, from Lemma II.1 (see also [23, Lemma I.8.6]) it follows that there exist vector fields $\{X_{ik}: i = 1,\cdots, m; k = 1,\cdots, s_i\}$ in the set

$$W_i = \{\theta: \theta = g_i \text{ or } \theta = [g_{j_h},\cdots,[g_{j_1}, g_i]\cdots]], \quad 1 \le h \le n-1, \quad 0 \le j_h \le m, \quad \text{for } 1 \le k \le h\} \tag{2}$$

such that locally

$$R_i^* = \text{span}\{X_{ik}: 1 \le k \le s_i\} \qquad i = 1,\cdots, m \tag{3}$$

where $s_i = \dim R_i^*$.

Next, we denote by $\mathscr{I}$ the Lie ideal generated by the vector fields $\{[g_j, ad_f^k g_i]: i, j = 1,\cdots, m;\ k \ge 0 \text{ and } i \neq j\}$ in the Lie algebra generated by $\{f, g_1,\cdots, g_m\}$ and define

$$\Delta_{MIX} = \text{span}\{\tau: \tau \in \mathscr{I}\}.$$

It has been shown in [22] that, for the class of analytic systems which can be rendered noninteractive by means of regular static state feedback, there exists well-defined dynamics (called *Δ_{MIX} dynamics*) which is invariant (in a sense specified in Section VII) under the class of dynamic state feedbacks (which we call *regular dynamic noninteraction feedbacks*) such that the closed-loop system resulting from (1) still has some relative degree at (x_0, w_0) and is noninteractive.

Remark 2: One might ask if R_i^* and Δ_{MIX} depend on the regular static state feedback chosen to render (1) noninteractive. As a consequence of the results contained in [22] (see also Section VII), Δ_{MIX} is still the same if computed on two different noninteractive systems obtained from (1) via regular static state feedback. Moreover, since, whenever it exists, the maximal controllability distribution contained in $\bigcap_{j\neq i} K_j$ coincides with R_i^* (see [17] and [23, Lemma 7.4.2]), also R_i^* is independent of the regular static state feedback chosen to render (1) noninteractive.

Remark 3: Supposing $\Delta_{MIX} = 0$ and repeatedly using the Jacobi identity, it can be shown that X_{ik} and X_{jh} commute for $j, i = 1,\cdots, m;\ j \neq i;\ k = 1,\cdots, s_i$, and $h = 1,\cdots, s_j$.

It is shown in [20] that under Assumptions 1 and 2 there exist a local coordinate change, which for simplicity will be denoted by $x = (x_1^T \cdots x_{m+1}^T)^T$, such that

$$P_i^* = \text{span}\{\partial/\partial x_j: j \neq i\}$$
$$P^* = \text{span}\{\partial/\partial x_{m+1}\}$$
$$P_0 = \text{span}\{\partial/\partial x_i: i = 1,\cdots, m+1\}$$

and the system (1) is locally expressed by

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_{ii}(x_i)u_i \qquad i = 1,\cdots,m \\ \dot{x}_{m+1} &= f_{m+1}(x) + \sum_{j=1}^{m} g_{m+1,j}(x)u_j \\ y_i &= h_i(x_i) \qquad i = 1,\cdots,m \end{aligned} \tag{4}$$

(in general x_i, $i = 1,\cdots, m+1$, is vector valued). Moreover, $P_i^* \cap G = \text{span}\{g_j: j \neq i\}$. In [20], [23] it is also shown that the dynamics (called *P^* dynamics*)

$$\dot{x}_{m+1} = f_{m+1}(0,\cdots,0, x_{m+1}) = f \mid P^*$$

is fixed under any regular static feedback which renders (1) noninteractive. We suppose in what follows that the system (1) has been put in the form (4).

Remark 4: If (1) is not strong accessible, we have the more general form

$$\begin{aligned} \dot{x}_i &= f_i(x_i, x_{m+2}) + g_{ii}(x_i, x_{m+2})u_i \qquad i = 1,\cdots,m \\ \dot{x}_{m+1} &= f_{m+1}(x_1,\ldots,x_{m+2}) \\ &\quad + \sum_{j=1}^{m} g_{m+1,j}(x_1,\cdots,x_{m+2})u_j \\ \dot{x}_{m+2} &= f_{m+2}(x_{m+2}) \\ y_i &= h_i(x_i, x_{m+2}) \qquad i = 1,\cdots,m. \end{aligned} \tag{4.1}$$

In this case, if $\dot{x}_{m+2} = f_{m+2}(x_{m+2})$ is locally asymptotically stable, we can consider the system obtained by setting $x_{m+2} = 0$ and solve the problem of noninteraction with stability for this system.

Remark 5: Recalling (3) and using the form (4), by induction it is possible to show that

$$X_{ik} = (\partial/\partial x_i)Y_{ik}(x_i) + (\partial/\partial x_{m+1})Z_{ik}(x) \qquad i = 1,\cdots,m \tag{5}$$

where Y_{ik} and Z_{ik} denote, respectively, the components x_i and x_{m+1} of X_{ik}.

We give here a result, which will be useful in the following sections.

Lemma III.1: Under Assumption 2,

a) $\sum_{j\neq i} R_j^* = P_i^* = \text{span}\{\partial/\partial x_j: j \neq i\} \qquad i = 1,\cdots,m$

b) $\sum_{j=1}^{m} R_j^* = P_0 = \operatorname{span}\{\partial/\partial x_i : i = 1,\cdots, m+1\}$

c) $R_i^* \subset \bigcap_{j \neq i} P_j^* \subset \bigcap_{j \neq i} K_j \qquad i = 1,\cdots, m.$ ◇

Proof: This can be proven as in [17]. Here we give a very simple proof. We show first that $\sum_{j\neq i} R_j^* \subset P_i^*$. Since by definition $R_j^* \subset P_i^*$ for $j \neq i$, then also $\sum_{j\neq i} R_j^* \subset P_i^*$. Conversely, $\sum_{j\neq i} R_j^*$ is constant dimensional, thus invariant under f and g_j for $j = 1,\cdots, m$ (since R_i^* is), and contains span $\{g_j: j \neq i\}$, then by definition $P_i^* \subset \sum_{j\neq i} R_j^*$ so that a) immediately follows. In exactly the same way we can prove b). On the other hand, c) is a consequence of the fact that $\sum_{h\neq j} R_h^* \subset P_j^* \subset K_j$ and $R_i^* \subset \bigcap_{j\neq i}\sum_{h\neq j} R_h^*$. ●

IV. The Dynamic Extension

We now, first, define an extended system of suitable dimension and a set of extended distributions defined on it, starting from the distributions R_i^*. Then, we show that, under the assumption $\Delta_{MIX} = 0$, these extended distributions are locally independent, involutive, nonsingular, and controlled invariant for the extended system. In Section V, we show that there exists a regular static feedback, defined on the extended system, which renders these distributions simultaneously invariant. It will be seen that the resulting closed-loop system is noninteractive with respect to the partition $\{u_1,\cdots, u_m\}$. Finally, under certain rank conditions (see (8)), it will be seen that there exists a *nonregular* static feedback, defined on the extended system, which stabilizes the system without destroying the above noninteraction property.

Let us start by defining

$$n_i = \dim x_i \qquad i = 1,\cdots, m$$
$$n_0 = \dim x_{m+1}$$
$$n_{wi} = n_i + n_0 \qquad i = 1,\cdots, m$$
$$n_w = \sum_{i=1}^{m} n_{wi}.$$

Consider the extended system

$$\dot{x} = f(x) + g(x)u$$
$$\dot{w} = u_w$$
$$y = h(x)$$

where $\dim w = n_w$. Having set

$$w_i = \begin{pmatrix}\lambda_i \\ \mu_i\end{pmatrix}, \quad \dim \lambda_i = n_i, \quad \dim \mu_i = n_0 \qquad i = 1,\cdots, m$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix},$$

and

$$x^e = \begin{pmatrix} x \\ w \end{pmatrix}, \quad u^e = \begin{pmatrix} u \\ u_w \end{pmatrix}$$

then we can rewrite the extended system in the compact form

$$\dot{x}^e = f^e(x^e) + g^e(x^e)u^e \qquad (7)$$
$$y^e = h^e(x^e)$$

with

$$f^e(x^e) = \begin{pmatrix} f(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g^e(x^e) = \left(g_1^e(x^e) \cdots g_{2m}^e(x^e)\right)$$
$$h^e(x^e) = h(x)$$

where the ith zero block has dimension n_{wi}, $i = 1,\cdots, m$, and

$$g_i^e(x^e) = \begin{pmatrix} g_i(x) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g_{m+i}^e(x^e) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I_{n_{wi}\times n_{wi}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$i = 1,\cdots, m.$$

Moreover, let $x_0^e = (x_0^T\ w_0^T)^T = 0$. In what follows we need also the following notations

$$G_e = \operatorname{span}\{g_i^e: i = 1,\cdots, m\},\ G_{wi} = \operatorname{span}\{g_{m+i}^e\} \qquad i = 1,\cdots, m$$

$$G_w = \bigoplus_{i=1}^{m} G_{wi} \quad \text{and} \quad G^e = \operatorname{span}\{g^e\}.$$

Let us now construct the following extended vector fields

$$X_{ik}^e(x^e) = \begin{pmatrix} X_{ik}(x) \\ 0 \\ \vdots \\ X_{ik}^*(x_i, \mu_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1,\cdots, m$$

where (see (5) for the definition of Y_{ik} and Z_{ik})

$$X_{ik}^*(x_i, \mu_i) = \left.\begin{pmatrix} Y_{ik}(x_i) \\ Z_{ik}(x) \end{pmatrix}\right|_{x_j = 0 \text{ for } j\neq i;\ x_{m+1} = \mu_i}$$

and define the extended distributions

$$R_i^e = \operatorname{span}\{X_{ik}^e: 1 \leq k \leq s_i\}.$$

Clearly, $R_i^e \cap G_w = 0$, since otherwise the vector fields $\{X_{ik}: i = 1,\cdots, m;\ k = 1,\cdots, s_i\}$ would not be independent in a neighborhood of x_0. Note also that $s_i \leq n_{wi}$. The dimension of the added dynamics can be actually reduced (see Section VI). The following two lemmas are crucial for our result to hold.

Lemma IV.1: Suppose that $\Delta_{MIX} = 0$. Under Assumption 2, the distributions R_i^e, $i = 1, \cdots, m$, are a) independent, b) involutive, and c) have constant dimension s_i in a neighborhood of x_0^e. ◇

Proof: a) follows from the fact that the matrix $(X_{i1}^* \cdots X_{is_i}^*)$ has full-column rank and c) follows by construction. Let us now prove b). Since R_i^* are nonsingular at x_0, then they are also involutive from Lemma II.2; thus, we can write

$$[X_{ik}, X_{ih}] = \sum_{t=1}^{s_i} c_{ikht} X_{it} \qquad i = 1, \cdots, m;\ h, k = 1, \cdots, s_i$$

where c_{ikht} is a set of unique smooth functions. It can be shown that c_{ikht} are functions only of x_i. As a matter of fact, since $\Delta_{MIX} = 0$ and the $\{X_{ik}\}$ are in the set (2), we have

$$0 = [[X_{ik}, X_{ih}], X_{jr}] = \sum_{t=1}^{s_i} (-L_{X_{jr}} c_{ikht}) X_{it} \qquad i \neq j;\ i, j = 1, \cdots, m;\ h, k = 1, \cdots, s_i;\ r = 1, \cdots, s_j$$

which, from the independence of $\{X_{it}: 1 \le t \le s_i\}$, implies $L_{X_{jr}} c_{ikht} = 0$ for $j \neq i$; in turn, this, along with Lemma III.1, implies our thesis. Now, note that $[X_{ik}, X_{ih}]$ has the form (5), so that we can define $[X_{ik}, X_{ih}]^*(x_i, \mu_i)$ in the same way as we did for $X_{ik}^*(x_i, \mu_i)$. By means of straightforward computations, it can be shown that

$$[X_{ik}, X_{ih}]^* = [X_{ik}^*, X_{ih}^*] \qquad i = 1, \cdots, m;\ k, h = 1, \cdots, s_i.$$

This, in turn, since the c_{ikht} depend only on x_i, implies $[X_{ik}^*, X_{ih}^*] = \sum_{t=1}^{s_i} c_{ikht} X_{it}^*$ so that

$$[X_{ik}^e, X_{ih}^e] = \begin{pmatrix} [X_{ik}, X_{jh}] \\ 0 \\ \vdots \\ [X_{ik}^*, X_{jh}^*] \\ \vdots \\ 0 \end{pmatrix} = \sum_{t=1}^{s_i} c_{ikht} X_{it}^e \qquad i = 1, \cdots, m;\ k, h = 1, \cdots, s_i,$$

i.e., involutivity of R_i^e. ●

Remark 6: One might ask if the choice of the set $\{X_{ik}: 1 \le k \le s_i\}$ can influence the construction of R_i^e. It can be shown that, no matter how we choose $\{X_{ik}: 1 \le k \le s_i\}$ in the set (2), under the assumption $\Delta_{MIX} = 0$, we always obtain the same distribution R_i^e. As a matter of fact, denoting by $\{X_{ik}^1: 1 \le k \le s_i\}$ and $\{X_{ik}^2: 1 \le k \le s_i\}$ any two choices in the set (2), we have

$$X_{ik}^1 = \sum_{h=1}^{s_i} c_{ih} X_{ih}^2 \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i.$$

As in the proof of Lemma IV.1, since $\Delta_{MIX} = 0$, it can be shown that the c_{ih} depend only x_i. Thus, denoting by X_{ik}^{1e} and X_{ik}^{2e} the extended vector fields constructed as above, we have

$$X_{ik}^{1e} = \sum_{h=1}^{s_i} c_{ih} X_{ih}^{2e} \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i.$$

Remark 7: It is worth noting that if $[X_{ik}, X_{jh}] = 0$ for $j \neq i$, $k, h = 1, \cdots, s_i$ and all x in a neighborhood of x_0, it follows by construction that

$$[X_{ik}^e, X_{jh}^e] = 0 \qquad j \neq i;\ k, h = 1, \cdots, s_i$$

for all x^e in a neighborhood of x_0^e.

Remark 8: It can be shown that, if $\Delta_{MIX} = 0$, then

$$R_i^e \cap G^e = \text{span}\{p_i^e\} \qquad i = 1, \cdots, m,$$

where

$$p_i^e = \begin{pmatrix} g_i(x) \\ 0 \\ \vdots \\ g_i^*(x_i, \mu_i) \\ \vdots \\ 0 \end{pmatrix}$$

and g_i^* is defined as X_{ik}^* (it is well-defined since g_i is of the form (5)). As a matter of fact, using the same arguments as in the proof of Lemma IV.1, it can be shown that $p_i^e \in R_i^e$. Our thesis follows from the fact that $R_i^* \cap G = \text{span}\{g_i\}$ and $R_i^e \cap G_w = 0$.

Remark 9: By construction, if follows that

$$[f^e, R_i^e] \subset G_w + R_i^e \subset G^e + R_i^e \qquad i = 1, \cdots, m$$

$$[g_j^e, R_i^e] \subset G_w + R_i^e \subset G^e + R_i^e \qquad i = 1, \cdots, m;\ j = 1, \cdots, 2m.$$

Thus, if $\Delta_{MIX} = 0$, then each R_i^e is involutive (Lemma IV.1.c)) and thus controlled invariant for (7) [23].

The following lemma will be needed in the next section.

Lemma IV.2: The distributions $\sum_{j \neq i} R_j^e$ for $i = 1, \cdots, m$ and $\sum_{i=1}^m R_i^e$ are nonsingular and involutive in a neighborhood of x_0^e.

Proof: Immediate from Remark 7 and Lemma IV.1. ●

V. Noninteraction and Stability

In what follows by invariance under g_{m+i}^e (or g^e) we mean invariance under each column of these matrices by the bracket of a smooth vector field with g_{m+i}^e we mean the bracket of the vector field with each column of that matrix, by the Lie derivative of a vector (or a matrix) of smooth functions along a smooth vector field we mean the Lie derivative of each function along that vector field and finally by the product of g_{m+i}^e with a column vector (or a matrix) we mean the usual product of a matrix with that column vector (or that matrix).

We prove in this section a result, which gives a sufficient condition for local noninteracting control with stability via regular dynamic state feedback. Anyway, this result has to be regarded as an intermediate step to prove a more general result in Section VII.

Theorem V.1: Suppose that Assumptions 1 and 2 hold. Then the problem of local noninteracting control with stability by means of regular dynamic state feedback

$$u = \alpha(x^e) + \beta(x^e)v$$
$$\dot{w} = \gamma(x^e) + \delta(x^e)v$$

is solvable if $\Delta_{MIX} = 0$ and

$$R_i^* = \text{span}\{g_i, \cdots, ad_f^{s_i-1}g_i\} \qquad i = 1, \cdots, m \quad (8)$$

in a neighborhood of x_0.

Remark 10: In the case of linear systems, $\Delta_{MIX} = 0$ and also (8) is trivially true, since R_i^* by definition is a controllability subspace. Thus, Theorem V.1 is a generalization of the result contained in [3] for the class of nonlinear systems here considered. From (8) it also follows that $R_i^*(0)$ (i.e., the vector subspace of $\mathbb{R}^n$ assigned to the point $x_0 = 0$ by the distribution R_i^*) is a controllability subspace for the linear approximation of (4) in a neighborhood of x_0 (for, note that $(-1)^k A^k b_i = ad_f^k g_i(0)$, where $A = (\partial f/\partial x)(0)$ and $b_i = g_i(0)$) and that for each i the system

$$\dot{x}_j = f_j(x_j) + g_{jj}(x_j)u_j \qquad j \neq i$$
$$\dot{x}_{m+1} = f_{m+1}(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{m+1})$$
$$+ \sum_{j \neq i}^{m} g_{m+1,j}(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{m+1})u_j \quad (9)$$

is locally exponentially stabilizable by means of a linear state feedback. As a matter of fact, from Lemma III.1.a) and (8) we have

$$\text{span}\{(\partial/\partial x_j)_0 : j \neq i\} = P_i^*(0) = \sum_{j \neq i} R_j^*(0)$$
$$= \text{span}\{b_j, \cdots, A^{s_j-1}b_j : j \neq i\}$$

where $(\partial/\partial x_j)_0$ means $\partial/\partial x_j$ evaluated at 0 (it can be identified with a matrix having an identity matrix as jth block element).

Remark 11: From (8) and recalling Remark 6, without loss of generality and for simplicity, we can choose

$$X_{ik} = ad_f^{k-1}g_i \qquad i = 1, \cdots, m; \ k = 1, \cdots, s_i$$

and throughout the paper we implicitly suppose this.

Remark 12: The requirement $\Delta_{MIX} = 0$ is actually too strong. We need it only to prove Theorem V.1 and we shall relax it in Section VII.

Before proving Theorem V.1, we need some preliminary results.

Lemma V.1: Let U and V be, respectively, open sets in $\mathbb{R}^q$ and $\mathbb{R}^r$ and $\Delta = \text{span}\{Y_1, \cdots, Y_d\}$, $d \leq q$, be a nonsingular and involutive distribution for every x in U with $\dim \Delta = d$. Let $\{\nu_{hkt}: t, h, k = 1, \cdots, d\}$ be the unique set of smooth functions such that $[Y_h, Y_k] = \sum_{t=1}^{d} \nu_{hkt} Y_t$. Let also $\Gamma_1, \cdots, \Gamma_d$ be smooth functions $\Gamma_i: U \to \mathbb{R}^{r \times r}$. Consider now the set of partial differential equations

$$L_{Y_k}\alpha = \Gamma_k \alpha \qquad 1 \leq k \leq d \quad (10)$$

where α denotes a function $\alpha: U \to V$. Given a point $x_0 \in U$, there exist a neighborhood U_0 of x_0 in U and a unique smooth function $\alpha: U_0 \to V$, which satisfies the equations (10) and, on a submanifold $M \ni x_0$ complementary to the leaf of Δ passing through x_0, coincides with a given smooth function $\bar{\alpha}$ defined on M, if and only if the functions $\Gamma_1, \cdots, \Gamma_d$ satisfy the integrability conditions

$$\Gamma_k \Gamma_h - \Gamma_h \Gamma_k + L_{Y_h}\Gamma_k - L_{Y_k}\Gamma_h = \sum_{t=1}^{d} \nu_{hkt}\Gamma_t$$
$$1 \leq h, k \leq d \quad (11)$$

for all x in U. ◇

Proof: It can be proven essentially in the same way as in [23, Lemma VI.2.3]. ●

In Remark 9, we have seen that each R_i^e is separately controlled invariant for (7). The following lemma states that the distributions R_i^e are *compatible*, i.e., can be rendered simultaneously invariant. In what follows, we implicitly suppose that the assumptions of Theorem V.1 hold.

Lemma V.2: There exists a smooth state feedback $u^e = \alpha^e(x^e) + \beta^e(x^e)v^e$, defined in a neighborhood of x_0^e, such that $\alpha^e(0) = 0$, β^e is nonsingular at 0 and the distributions R_i^e, $i = 1, \cdots, m$, are locally invariant under $\tilde{f}^e = f^e + g^e\alpha^e$ and $\tilde{g}^e = g^e\beta^e$. Moreover, denoting by β_i^e the ith column of β^e, we have $g^e\beta_i^e = p_i^e$ for $i = 1, \cdots, m$. ◇

Remark 13: This result is a nonlinear analog of the well-known fact that a set of independent controlled invariant subspaces can be made simultaneously invariant under $A + BF$. It is worth noting that Lemma V.2 can be proven neither as in [17, Theorem 3.1] nor as in [24], because in our case we have $\oplus_{i=1}^{m}(R_i^e \cap G^e) \neq G^e$.

Proof of Theorem V.2: We split up the proof in three steps.

a) Since the R_i^* are by definition invariant under f, it is easy to see that by construction we can write

$$[f^e, X_{ik}^e] = g_{m+i}^e c_{ik} + \delta_{ik}^e$$
$$i = 1, \cdots, m; \ k = 1, \cdots, s_i \quad (12)$$

where c_{ik} is a set of unique smooth (vector) functions, which depend only from x_i and μ_i, and

$$\delta_{ik}^e = \begin{pmatrix} [f, X_{ik}] \\ 0 \\ \vdots \\ [f, X_{ik}]^* \\ \vdots \\ 0 \end{pmatrix}$$

where $[f, X_{ik}]^*$ is defined as in Section IV (it is well-defined since $[f, X_{ik}]$ has the form (5)). It can be proven that $\delta_{ik}^e \in R_i^e$. As a matter of fact, since R_i^* is by definition invariant under f, we can write $[f, X_{ik}] = \sum_{t=1}^{s_i} d_{ikt}X_{it}$, where d_{ikt} is a set of unique smooth functions. But since $\Delta_{MIX} = 0$, we have $[[f, X_{ik}], X_{jh}] = 0$ for $i \neq j$; using arguments similar to those in the proof of Lemma IV.1, it can be shown that d_{ikt} depends only on x_i and thus $\delta_{ik}^e = \sum_{t=1}^{s_i} d_{ikt}X_{it}^e \in R_i^e$.

Since c_{ik} depends only from x_i and μ_i, then

$$L_{X_{jh}^e} c_{ik} = 0 \qquad j \neq i;\ j, i = 1, \cdots, m;\ k = 1, \cdots, s_i; \quad h = 1, \cdots, s_j. \tag{13}$$

Moreover, it is easy to see that

$$[g_{m+i}^e, X_{ik}^e] = g_{m+i}^e \gamma_{ik} \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i \tag{14}$$

where γ_{ik} are $n_{wi} \times n_{wi}$ matrices of unique smooth functions, which depend only on x_i and μ_i, and

$$[g_{m+i}^e, X_{jh}^e] = 0 \qquad i = 1, \cdots, m;\ j \neq i;\ h = 1, \cdots, s_j. \tag{15}$$

From (14) it follows

$$L_{X_{jh}^e} \gamma_{ik} = 0 \qquad j, i = 1, \cdots, m;\ j \neq i;\ k = 1, \cdots, s_i; \quad h = 1, \cdots, s_j. \tag{16}$$

Now, let $\tilde{f}^e = f^e + \sum_{i=1}^m g_{m+i}^e \alpha_i$. We shall prove that there exist smooth (vector) functions $\alpha_1, \cdots, \alpha_m$, defined in a neighborhood of x_0^e, such that

$$[\tilde{f}^e, R_i^e] \subset R_i^e \qquad i = 1, \cdots, m. \tag{17}$$

For, making some computations, we get

$$[\tilde{f}, X_{ik}^e] = g_{m+i}^e (c_{ik} - L_{X_{ik}^e} \alpha_i + \gamma_{ik} \alpha_i) + - \sum_{j \neq i} g_{m+j}^e (L_{X_{ik}^e} \alpha_j) + \delta_{ik}^e \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i.$$

We have that (17) can be satisfied if and only if the following set of partial differential equations are satisfied

$$L_{X_{jh}^e} \alpha_i = 0 \qquad i, j = 1, \cdots, m;\ j \neq i;\ h = 1, \cdots, s_j \tag{18.1}$$

and

$$L_{X_{ik}^e} \alpha_i = c_{ik} + \gamma_{ik} \alpha_i \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i. \tag{18.2}$$

It is clear that (18.1) and (18.2) can be thought of as m separate problems, so that for each $i = 1, \cdots, m$ we can think of (18.1) and (18.2) rewritten in the following way

$$L_{X_{jh}^e} \begin{pmatrix} \alpha_i \\ \alpha_{m+1} \end{pmatrix} = \Gamma_{jh} \begin{pmatrix} \alpha_i \\ \alpha_{m+1} \end{pmatrix} \qquad j = 1, \cdots, m;\ h = 1, \cdots, s_j \tag{19}$$

where

$$\Gamma_{jh} = \begin{cases} \begin{pmatrix} \gamma_{jh} & c_{jh} \\ 0 & 0 \end{pmatrix} & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

and $\alpha_{m+1} = 1$ on a submanifold $M^e \ni x_0^e$ complementary to the leaf of R_i^e passing through x_0^e (since from (19) $L_{X_{jh}^e} \alpha_{m+1} = 0$, then $\alpha_{m+1} = 1$ in a neighborhood of x_0^e).

Since R_i^* is involutive, we can write $[X_{ih}, X_{ik}] = \sum_{t=1}^{s_i} \nu_{ihkt} X_{it}$ for $i = 1, \cdots, m$. Applying Lemma V.1 for each $i = 1, \cdots, m$ to (19) with $\Delta = \sum_{i=1}^m R_i^e$, which is locally nonsingular and involutive from Lemma IV.2, it follows that for each i there exists α_i solving (19) and such that $\alpha_i(0) = 0$ (in this case α_i is not unique) if and only if the following conditions are satisfied:

$$\Gamma_{ik}\Gamma_{ih} - \Gamma_{ih}\Gamma_{ik} + L_{X_{ih}^e}\Gamma_{ik} - L_{X_{ik}^e}\Gamma_{ih} = \sum_{t=1}^{s_i} \nu_{ihkt}\Gamma_{it} \qquad k, h = 1, \cdots, s_i \tag{20}$$

and

$$\Gamma_{ik}\Gamma_{jh} - \Gamma_{jh}\Gamma_{ik} + L_{X_{jh}^e}\Gamma_{ik} - L_{X_{ik}^e}\Gamma_{jh} = 0 \qquad j \neq i,\ k = 1, \cdots, s_i;\ h = 1, \cdots, s_j. \tag{21}$$

We shall prove first (21). Clearly, we have $\Gamma_{ik}\Gamma_{jh} = \Gamma_{jh}\Gamma_{ik} = 0$ for $i \neq j$ so that (21) reduces to $0 = L_{X_{jh}^e}\Gamma_{ik}$ for $j \neq i$ but this is true from (13) and (16). On the other hand, relations (20) can be easily derived from (12), (14), and the following Jacobi identities:

$$-[[f^e, X_{ik}^e], X_{ih}^e] + [[f^e, X_{ih}^e], X_{ik}^e] = [f^e, [X_{ih}^e, X_{ik}^e]] \qquad h, k = 1, \cdots, s_i$$

$$-[[g_{m+i}^e, X_{ik}^e], X_{ih}^e] + [[g_{m+i}^e, X_{ih}^e], X_{ik}^e] = [g_{m+i}^e, [X_{ih}^e, X_{ik}^e]] \qquad h, k = 1, \cdots, s_i.$$

So we have shown that (17) can be satisfied with $\alpha^e = (0^T \alpha_1^T \cdots \alpha_m^T)^T$, where the zero block has dimension m.

b) By means of a smooth nonsingular matrix

$$\beta_1^e = \begin{pmatrix} I_{m \times m} & 0 \\ * & I_{n_w \times n_w} \end{pmatrix}$$

it is possible to rearrange the columns of the matrix g^e so that, denoting by β_{1i}^e the ith column of β_1^e, we have (see also Remark 8)

$$R_i^e \ni \tilde{g}_i^e = g^e \beta_{1i}^e = p_i^e = X_{i1}^e \qquad i = 1, \cdots, m.$$

From involutivity of R_i^e (see Lemma IV.1) and Remark 7, it follows that

$$[\tilde{g}_j^e, R_i^e] \subset R_i^e \qquad i, j = 1, \cdots, m.$$

c) Recalling (14) and (15) and proceeding as above, it can be shown that there exist $n_{wi} \times n_{wi}$ matrices β_{2i} of smooth functions, defined in a neighborhood of x_0^e, such that $\beta_{2i}(0) = I_{n_{wi} \times n_{wi}}$ and R_i^e, $i = 1, \cdots, m$, are invariant under $\tilde{g}_{m+j}^e = g_{m+j}^e \beta_{2j}$, $j = 1, \cdots, m$, if and only if we can solve the following set of partial differential equations

$$L_{X_{ik}^e} \beta_{2i} = \gamma_{ik} \beta_{2i} \qquad i = 1, \cdots, m;\ k = 1, \cdots, s_i \tag{22.1}$$

and

$$L_{X_{jh}^e} \beta_{2i} = 0 \qquad i, j = 1, \cdots, m;\ j \neq i;\ k = 1, \cdots, s_j. \tag{22.2}$$

This equations can be rewritten as

$$L_{X_{jh}^e} \begin{pmatrix} \beta_{2i} \\ \beta_{2,m+1} \end{pmatrix} = \Gamma_{jh} \begin{pmatrix} \beta_{2i} \\ \beta_{2,m+1} \end{pmatrix} \qquad j, i = 1, \cdots, m;\ h = 1, \cdots, s_j \tag{23}$$

where

$$\Gamma_{jh} = \begin{cases} \begin{pmatrix} \gamma_{jh} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

and $\beta_{2,m+1} = 0$ on a submanifold $M^e \ni x_0^e$ complementary to the leaf of R_i^e passing through x_0^e (since from (23) $L_{X_{jh}^e}\beta_{2,m+1} = 0$, then $\beta_{2,m+1} = 0$ in a neighborhood of x_0^e).

Again, using the same arguments as above, we can think of (23) as m separate problems and it can be shown that they can be solved. ●

We denote the closed-loop system resulting from applying the feedback (α^e, β^e) to (7) by

$$\begin{aligned} \dot{x}^e &= \tilde{f}^e(x^e) + \tilde{g}^e(x^e)v^e \\ y_i^e &= h_i^e(x^e) \qquad i = 1, \cdots, m. \end{aligned} \tag{24}$$

For simplicity of notation, set $u^e = v^e$.

Lemma V.3: There exist new local coordinates $z = (z_1^T \cdots z_{m+1}^T)^T$ and a partition $u^e = (u_1^T \cdots u_{m+1}^T)^T$ such that

$$R_i^e = \text{span}\{\partial/\partial z_i\} \qquad i = 1, \cdots, m$$

and (24) is locally expressed as

$$\begin{aligned} \dot{z}_1 &= \tilde{f}_1(z_1, z_{m+1}) + \tilde{g}_{11}(z_1, z_{m+1})u_1 \\ &\quad + \tilde{g}_{1,m+1}(z_1, z_{m+1})u_{m+1} \\ \cdots &\quad \cdots \\ \dot{z}_m &= \tilde{f}_m(z_m, z_{m+1}) + \tilde{g}_{mm}(z_m, z_{m+1})u_m \\ &\quad + \tilde{g}_{m,m+1}(z_m, z_{m+1})u_{m+1} \\ \dot{z}_{m+1} &= \tilde{f}_{m+1}(z_{m+1}) + \tilde{g}_{m+1}(z_{m+1})u_{m+1} \\ y_i^e &= h_i^e(z_i, z_{m+1}) \qquad i = 1, \cdots, m \end{aligned} \tag{25}$$

where $\dim u_i = 1$, $i = 1, \cdots, m$, and $\dim u_{m+1} = n_w$. ◇

Proof: Consider the distributions $E_0^e = \sum_{j=1}^m R_j^e$ and $E_i^e = \sum_{j \neq i} R_j^e$, which from Lemma IV.2 are locally nonsingular and involutive. Since $R_i^e \subset \bigcap_{j \neq i} E_j^e$ and thus

$$E_0^e = \sum_{j=1}^m R_j^e = E_i^e + \bigcap_{j \neq i} E_j^e$$

and since the R_i^e are locally independent, by using similar arguments to those contained in [20, Lemma 4.1], it can be shown that there exists a local coordinate change $z = (z_1^T \cdots z_{m+1}^T)^T$ such that $R_i^e = \text{span}\{\partial/\partial z_i\}$. From the invariance of R_i^e under $\tilde{f}^e$ and $\tilde{g}^e$, since $R_i^e \cap G^e = \text{span}\{\tilde{g}_i^e\}$ and $R_i^e \subset \bigcap_{j \neq i} \ker\{dh_j^e\}$, it follows (25). ●

Note that (25) is noninteractive with respect to $\{u_1, \cdots, u_m\}$ but not necessarily asymptotically stable. Now let $z_0 = z(x_0^e)$ and suppose without loss of generality that $z_0 = 0$.

Lemma V.4: The linear approximation of (25) in a neighborhood of x_0^e is controllable. ◇

Proof: Having set $A = (\partial f/\partial x)(0)$ and $b_i = g_i(0)$, from Lemma III.1.b) and (8) it follows

$$\begin{aligned} \text{span}\{(\partial/\partial x_i)_0 &: i = 1, \cdots, m+1\} \\ &= P_0(0) = \sum_{i=1}^m R_i^*(0) \\ &= \text{span}\{b_i, \cdots, A^{s_i-1}b_i: i = 1, \cdots, m\}, \end{aligned}$$

i.e., the linear approximation of (4) in a neighborhood of x_0 is controllable. But this means that also the linear approximation of (7) in a neighborhood of x_0^e is controllable. Since the feedback (α^e, β^e) is regular, our thesis follows. ●

Lemma V.5: $R_i^e(0)$ is a controllability subspace for the linear approximation of (25) in a neighborhood of x_0^e. ◇

Proof: This is essentially proved in [3]. Here we give a very simple proof. Since from our assumptions

$$R_i^* = \text{span}\{g_i, \cdots, ad_f^{s_i-1}g_i\}$$

then by construction (see Lemma V.2) it must be also

$$R_i^e = \text{span}\{\tilde{g}_i^e, \cdots, ad_{\tilde{f}^e}^{s_i-1}\tilde{g}_i^e\}.$$

This exactly implies our thesis. Note that $R_i^e(0)$ is the controllability subspace associated with u_i in the linear approximation of (25). ●

We have now all we need to stabilize (25) without destroying noninteraction with respect to $\{u_1, \cdots, u_m\}$.

Lemma V.6: There exists a linear state feedback of the form

$$u_i = F_i z_i + v_i \qquad i = 1, \cdots, m, \tag{26.1}$$

$$u_{m+1} = F_{m+1} z_{m+1} \tag{26.2}$$

where v_i are the new inputs, such that the closed-loop system resulting from (25) is locally asymptotically stable and noninteractive. ◇

Proof: Consider the linear approximation of (25) in a neighborhood of x_0

$$\begin{aligned} \dot{z}_i &= A_{ii}z_i + A_{i,m+1}z_{m+1} + b_{ii}u_i + b_{i,m+1}u_{m+1} \\ \dot{z}_{m+1} &= A_{m+1,m+1}z_{m+1} + b_{m+1,m+1}u_{m+1}. \end{aligned}$$

From Lemma V.5 the pairs (A_{ii}, b_{ii}), $i = 1, \cdots, m$, are clearly controllable. But also the pair $(A_{m+1,m+1}, b_{m+1,m+1})$ is controllable, since from Lemma V.4 the overall system is. This means that the system (25) can be locally exponentially stabilized by means of a suitable feedback of the form (26); the resulting closed-loop system is also noninteractive, since this feedback is decentralized. ●

Proof of Theorem V.1: Apply sequentially Lemmas V.2, V.4, and V.6. Finally, set

$$\begin{pmatrix} \alpha(x^e) \\ \delta(x^e) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1(x^e) \\ \vdots \\ \alpha_m(x^e) \end{pmatrix} + g^e\beta^e(x^e)\,\text{diag}(F_1, \cdots, F_{m+1})\,z(x^e) \tag{27.1}$$

where $\text{diag}(\cdot)$ denotes a diagonal block matrix and $z(x^e)$ is the vector function z expressed in x^e coordinates, and

$$\begin{pmatrix} \beta(x^e) \\ \gamma(x^e) \end{pmatrix} = \beta^e(x^e)\begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{m\times m} \\ * \end{pmatrix} \quad (27.2)$$

●

VI. Some Remarks About the Dimension of the Dynamic Extension

In this section, we show how to reduce n_w. A first attempt is given by picking

$$n_w = \sum_{i=1}^{m} s_i \quad (28)$$

which is actually less or equal than $\sum_{i=1}^{m}(n_i + n_0)$ (i.e., the dimension of our previous dynamic extension). To this purpose, we modify the construction of R_i^e in the following way. First, note that

$$\operatorname{span}\{\partial/\partial x_i\} \subset R_i^* + \operatorname{span}\{\partial/\partial x_{m+1}\}. \quad (29)$$

Recall now the form (5) and set $X_i = (X_{i1} \cdots X_{is_i})$. It follows from (29) that from the matrix

$$\overline{X}_i = \left.\begin{pmatrix} Y_{i1} & \cdots & Y_{is_i} \\ Z_{i1} & \cdots & Z_{is_i} \end{pmatrix}\right\} n_i + n_0 \qquad i = 1,\cdots, m$$

which has full rank for all x in a neighborhood of x_0, it is always possible to choose a nonsingular submatrix

$$X_i^* = \left.\begin{pmatrix} Y_{i1} & \cdots & Y_{is_i} \\ * & \cdots & * \end{pmatrix}\right\} s_i \qquad i = 1,\cdots, m.$$

For simplicity, we can suppose that X_i^* consists of the first s_i rows of $\overline{X}_i$. Call x_{m+1}^i the first $s_i - n_i$ components of x_{m+1} and $\overline{x}_{m+1}^i$ the other components. Moreover, choose $\dim w_i = s_i$ and $\dim \lambda_i = n_i$. With this notation, set

$$X_i^e(x^e) = \begin{pmatrix} X_i(x) \\ 0 \\ \vdots \\ X_i^*(x_i, \mu_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1,\cdots, m$$

where

$$X_i^*(x_i, \mu_i) = X_i^*(x)\,|_{\,x_j = 0 \text{ for } j\neq i;\ x_{m+1}^i = \mu_i;\ \overline{x}_{m+1}^i = 0} \qquad i = 1,\cdots, m$$

and define

$$R_i^e = \operatorname{span}\{X_i^e\} \qquad i = 1,\cdots, m.$$

Involutivity can be proven as in Lemma IV.1 if the following additional assumption is made.

Assumption 6:

$$\frac{\partial X_i}{\partial \overline{x}_{m+1}^i} = 0 \qquad i = 1,\cdots, m.$$

It is possible to give weaker versions of Assumption 6, but we do not do this here. Also, local independence of R_i^e can be proven as in Lemma IV.1. Further reduction can be achieved by taking

$$n_w = \sum_{i=2}^{m} \dim\left(R_i \cap \left(\sum_{j<i} R_j\right)\right). \quad (30)$$

Suppose the following.

Assumption 7.1: The distributions $R_i^* \cap (\sum_{j<i} R_j^*)$, $i = 1,\cdots, m$, are nonsingular in a neighborhood of x_0. Moreover, setting $\dim(R_i^* \cap (\sum_{j<i}R_j^*)) = \varrho_i$ for $i = 2,\cdots, m$, it is always possible to find vector fields $\{T_{il}\colon l = 1,\cdots, \varrho_i\}$, $i = 1,\cdots, m$, in the set (2) such that $R_i^* \cap (\sum_{j<i}R_j^*) = \operatorname{span}\{T_{il}\colon l = 1,\cdots,\varrho_i\}$ for $i = 2,\cdots, m$.

From Assumption 7.1 and Remark 1, it is possible to find vector fields $\{X_{ik}\colon i = 1,\cdots, m;\ k = 1,\cdots, s_i\}$ in the set (2) such that locally

$$R_i^* = \operatorname{span}\{X_{ik}\colon k = 1,\cdots, s_i\} \qquad i = 1,\cdots, m \quad (31)$$

and

$$R_i^* \cap \left(\sum_{j<i} R_j^*\right) = \operatorname{span}\{X_{ik}\colon k = 1,\cdots, \varrho_i\} \qquad i = 2,\cdots, m. \quad (32)$$

Note that $X_{ik} \in R_i^* \cap (\sum_{j<i}R_j^*) \subset \operatorname{span}\{\partial/\partial x_{m+1}\}$ for $i = 2,\cdots, m$ and $k = 1,\cdots, \varrho_i$ and that all the vector fields $\{X_{ik}\}$ have the form (5). This, with (32), implies that from the matrix

$$Z_i = (Z_{i1} \cdots Z_{is_i})\}n_0 \qquad i = 2,\cdots, m$$

we can choose a $\varrho_i \times s_i$ submatrix Z_i^* such that it has full rank for all x in a neighborhood of x_0 and the first ϱ_i columns are independent. For simplicity, we can suppose that Z_i^* consists of the first ϱ_i rows of Z_i. As above, call x_{m+1}^i the first ϱ_i components of x_{m+1} and $\overline{x}_{m+1}^i$ the other components. Moreover, choose $\dim w_1 = 0$, $\dim w_i = \dim \mu_i = \varrho_i$ for $i = 2,\cdots, m$. With this notation, set

$$X_1^e(x^e) = \begin{pmatrix} X_1(x) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad X_i^e(x^e) = \begin{pmatrix} X_i(x) \\ 0 \\ \vdots \\ Z_i^*(x_i, \mu_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 2,\cdots, m$$

where

$$Z_i^*(x_i, \mu_i) = Z_i(x)\,|_{\,x_j = 0 \text{ for } j\neq i;\ x_{m+1}^i = \mu_i;\ \overline{x}_{m+1}^i = 0}$$

and define

$$R_i^e = \operatorname{span}\{X_i^e\} \qquad i = 1,\cdots, m.$$

The involutivity of R_i^e can be shown as in Lemma IV.1, if also the following assumption (similar to Assumption 6) is made

Assumption 7.2:

$$\frac{\partial X_i}{\partial \bar{x}^i_{m+1}} = 0 \qquad i = 2,\cdots,m.$$

On the other hand, local independence of the distributions R_i^e can be proved in the following way. It suffices to show that the matrix

$$X^e = \left(X_{11}^e \cdots X_{1s_1}^e \cdots X_{m1}^e \cdots X_{ms_m}\right)$$

has full rank at $x_0^e = 0$. Suppose that this is not true. Thus, there exist some scalars $c_{ik} \neq 0$ such that

$$\sum_{i=1}^{m}\sum_{k=1}^{s_i} c_{ik} X_{ik}(0) = 0 \tag{33}$$

($X_{ik}(0)$ is the vector assigned to 0 by the vector field X_{ik}) and

$$\sum_{k=1}^{s_i} c_{ik} Z_{ik}^*(0) = 0 \qquad i = 2,\cdots,m \tag{34}$$

where Z_{ik}^* is the kth column of Z_i^*. Let i denote the greatest integer such that $c_{ik} \neq 0$. Then, from (33) it follows that

$$-\sum_{j<i}\sum_{k=1}^{s_j} c_{jk} X_{jk}(0) = \sum_{k=1}^{s_i} c_{ik} X_{ik}(0), \tag{35}$$

i.e., there exists $0 \neq v = \sum_{k=1}^{s_i} c_{ik} X_{ik}(0) \in R_i^*(0) \cap (\sum_{j<i} R_j^*(0))$. But from the definition of sum and intersection of smooth distributions (see [23]) we have $R_i^*(0) \cap (\sum_{j<i} R_j^*(0)) = (R_i^* \cap \sum_{j<i} R^*_j)(0)$. These facts, along with (32), implies that $c_{ik} = 0$ for $k > \varrho_i$. But from (34), since the first ϱ_i columns of $Z_i^*(0)$ are independent, it follows that $c_{ik} = 0$ for all $k = 1,\cdots,s_i$ and this is a contradiction.

The results of Section V are still true if we use as R_i^e either one of the extended distributions constructed above, but we do not show this here. We want to stress the importance of constructing a set of distributions for which Lemma IV.1 is true and such that X_{ik}^* depends only on x_i and μ_i.

It is worth noting that the lower bound (30) is the best possible if we want to construct a set of extended independent distributions R_i^e. As a matter of fact, independently of the value of n_w, it must be

$$n + n_w - \sum_{i=1}^{m} s_i \geq 0$$

so that

$$n_w \geq \sum_{i=1}^{m} s_i - n = \sum_{i=1}^{m} \dim R_i^* - \dim\left(\sum_{i=1}^{m} R_i^*\right) = \sum_{i=2}^{m} \dim\left(R_i^* \cap \left(\sum_{j<i} R_j^*\right)\right).$$

The last equality in the above formula can be proven as in [3], since from Assumption 7.1 the distributions $\sum_{j=1}^{i} R_j^*$ for $i = 1,\cdots,m$ are locally nonsingular. As a matter of fact, suppose by induction that $\sum_{j=1}^{i-1} R_j^*$ is locally nonsingular. Then also $\sum_{j=1}^{i} R_i^* = \sum_{j=1}^{i-1} R_j^* + R_i^*$ must be locally nonsingular, since $R_i^* \cap (\sum_{j<i} R_j^*)$, R_i^*, and $\sum_{j<i} R_j^*$ are.

In Section VIII an example of the above construction will be given.

VII. The $\Delta_{MIX} \neq 0$ Case

As already noted, the requirement $\Delta_{MIX} = 0$ is usually too strong a condition. In this section, we relax this assumption, showing that the asymptotic stability of the Δ_{MIX} dynamics will be sufficient to our purposes. The idea consists of factoring out the Δ_{MIX} part and to apply the results of Sections IV and V, after having checked that the regularity assumptions and (8) still hold for the "quotient" system.

Let us go back to (4). We restate Assumption 2 as follows.

Assumption 2.1: The distributions P_i^*, R_i^*, and P_0 are locally finitely computable and

$$P_0,\, P_i^*,\, P^*,\, P_i^* + \bigcap_{j\neq i} P_j^*,\, R_i^*,\, \sum_{j\neq i} R_i^*,\, \sum_{j=1}^{m} R_j^*,\, \Delta_{MIX} \text{ and } \Delta_{MIX} + R_i^*$$

are nonsingular in a neighborhood of x_0.

Since Δ_{MIX} is nonsingular, by definition it is also invariant under f and g. Moreover, $\Delta_{MIX} \subset P_i^*$ for $i = 1,\cdots,m$ (see also [22]). It can be easily shown that there exist local coordinates $x = (x_1^T \cdots x_m^T x_{m+1,1}^T x_{m+1,2}^T)^T$ such that (4) is locally expressed by

$$\begin{aligned}
\dot{x}_i &= f_i(x_i) + g_{ii}(x_i)u_i \qquad i = 1,\cdots,m\\
\dot{x}_{m+1,1} &= f_{m+1,1}(x_1,\cdots,x_m,x_{m+1,1}) + \sum_{j=1}^{m} g_{m+1,1j}(x_1,\cdots,x_m,x_{m+1,1})u_j\\
\dot{x}_{m+1,2} &= f_{m+1,2}(x) + \sum_{j=1}^{m} g_{m+1,2j}(x)u_j\\
y_i &= h_i(x_i) \qquad i = 1,\cdots,m
\end{aligned} \tag{36}$$

where $P_i^* = \text{span}\{\partial/\partial x_j\colon\ j \neq i\}$ and $\Delta_{MIX} = \text{span}\{\partial/\partial x_{m+1,2}\}$. The Δ_{MIX} dynamics are defined as

$$\dot{x}_{m+1,2} = f_{m+1,2}(0,\cdots,x_{m+1,2}) = f \mid \Delta_{MIX}. \tag{37}$$

Consider now the class of regular dynamic noninteraction feedbacks

$$\begin{aligned}
u &= \alpha(x^e) + \beta(x^e)v\\
\dot{w} &= \delta(x^e) + \gamma(x^e)v.
\end{aligned} \tag{38}$$

Let

$$\bar{f}^e(x^e) = \begin{pmatrix} f(x) + g(x)\alpha(x^e)\\ \delta(x^e)\end{pmatrix}$$

$$\bar{g}^e(x^e) = \begin{pmatrix} g(x)\beta(x^e)\\ \gamma(x^e)\end{pmatrix}.$$

It is worth noting that the dynamic static feedback we use in Theorem V.1 is regular, since, as it can be checked, the relative degree of the extended closed-loop system is the same as that of the open-loop system (4). Suppose also that the distribution Δ^e_{MIX}, defined as Δ_{MIX} but with f and g_i replaced by $\bar{f}^e$ and $\bar{g}^e_i$, is nonsingular at x^e_0. In [22] it is shown that, under the above assumptions, if we define the map

$$\pi: U \times V \to U$$
$$x^e = (x, w) \mapsto x$$

where U and V are neighborhoods, respectively, of x_0 and w_0, and $(\pi*)_{x^e}$ is its differential at x^e, then

$$(\pi*)_{x^e}(\Delta^e_{MIX}(x^e)) = \Delta_{MIX}(x)$$

and

$$\bar{f}^e \mid \Delta^e_{MIX} = \begin{pmatrix} f_{m+1,2}(0,\cdots,x_{m+1,2}) \\ f(x_{m+1,2}, \bar{w}) \end{pmatrix}$$

where $x_{m+1,2}$ and $\bar{w}$ form a coordinate system on the leaf of Δ^e_{MIX} passing through x^e_0. Thus, it is clear that the local asymptotic stability of the Δ_{MIX} dynamics is a necessary condition in order to obtain local stability and noninteraction by means of any regular dynamic feedback.

The main theorem of this section is the following.

Theorem VII.1: Suppose Assumptions 1 and 2.1 hold. The problem of local noninteracting control with stability by means of regular dynamic state feedback (38) is solvable if the Δ_{MIX} dynamics is locally asymptotically stable and

$$R^*_i = \text{span}\{g_i, \cdots, ad_f^{s_i-1} g_i\} \qquad i = 1, \cdots, m \quad (39)$$

in a neighborhood of x_0. ◇

Before proving this, we need a preliminary result. Note that, since from Assumption 2.1 $\Delta_{MIX} + R^*_i$ and Δ_{MIX} are locally nonsingular, then the integer $\hat{s}_i = \dim(\Delta_{MIX} + R^*_i) - \dim \Delta_{MIX}$ is well-defined in a neighborhood of x_0. Consider now the map

$$\sigma: U \to U/\mathscr{F}^{\Delta_{MIX}}$$
$$x \mapsto (x_1, \cdots, x_m, x_{m+1,1})$$

where U is such that (U, x) is a Frobenius chart with coordinate functions x and $\mathscr{F}^{\Delta_{MIX}}$ is the foliation induced in U by Δ_{MIX}, and denote by $(\sigma*)_x$ its differential at x. Let $\hat{P}_0$ and $\hat{P}_i$ be the distributions, defined on $U/\mathscr{F}^{\Delta_{MIX}}$, which assign to each point $\sigma(x)$, respectively, the subspaces $(\sigma*)_x(P_0(x))$ and $(\sigma*)_x(P^*_i(x))$ and $\hat{R}_i$ the distribution defined on $U/\mathscr{F}^{\Delta_{MIX}}$, which assign to each point $\sigma(x)$ the subspace $(\sigma*)_x((\Delta_{MIX} + R^*_i)(x))$ (in general, $\Delta_{MIX} \not\subset R^*_i$). Note that these distributions are locally nonsingular, since P^*_i, P_0, $\Delta_{MIX} + R_i$ and Δ_{MIX} are. Note also that for each θ in the set (2) there exists a vector field θ, defined on $U/\mathscr{F}^{\Delta_{MIX}}$, such that

$$\hat{\theta} \circ \sigma = \sigma * \theta \quad (40)$$

($\circ$ denotes composition) and vector fields $\hat{f}$ and $\hat{g}_j$, $j = 1, \cdots, m$, such that $\hat{f} = \sigma * f$ and $\hat{g}_j = \sigma * g_j$. This is obvious if one thinks that Δ_{MIX} is nonsingular and thus invariant under the vector fields f, g_j, $j = 1, \cdots, m$, and θ, which implies that only the component $x_{m+1,2}$ of f, g_j, $j = 1, \cdots, m$, and θ depend on $x_{m+1,2}$. Moreover, $\dim \hat{R}_i = \hat{s}_i$ in a neighborhood of $\sigma(x_0)$.

Lemma VII.1: Under assumptions of Theorem VII.1 we have

a) $\hat{P} = \sigma * P^*_i = \text{span}\{\partial/\partial x_j : j \notin \{i, (m+1, 2)\}\}$
$= \langle \hat{f}, \hat{g}_1, \cdots, \hat{g}_m \mid \text{span}\{\hat{g}_j : j \neq i\}\rangle \; i = 1, \cdots, m;$

b) $\hat{P}_0 = \sigma * P_0 = \text{span}\{\partial/\partial x_j : j \neq (m+1, 2)\}$
$= \langle \hat{f}, \hat{g}_1, \cdots, \hat{g}_m \mid \text{span}\{\hat{g}_i : i = 1, \cdots, m\}\rangle;$

c) $\hat{R}_i = \sigma*(R^*_i + \Delta_{MIX}) = \text{span}\{\hat{g}_i, \cdots, ad_{\hat{f}}^{\hat{s}_i-1}\hat{g}_i\}$
$= \langle \hat{f}, \hat{g}_1, \cdots, \hat{g}_m \mid \text{span}\{\hat{g}_i\}\rangle \qquad i = 1, \cdots, m.$

◇

Proof: Let us prove a). Since P^*_i is finitely computable, the sequence

$$S_0 = \text{span}\{g_j : j \neq i\}$$
$$S_k = [f, S_{k-1}] + \sum_{i=1}^{m} [g_i, S_{k-1}] + S_{k-1}$$

converges in a finite number of steps to the distribution P^*_i (see Section II). From Lemma II.1 and since $\sigma*[f, \theta] = [\hat{f}, \sigma*\theta]$ and $\sigma*[g_j, \theta] = [\hat{g}_j, \sigma*\theta]$ for $j = 1, \cdots, m$ and θ in the set (2), it follows that $(\sigma*)(S_k)$ must also converge in a finite number of steps to $\langle \hat{f}, \hat{g}_1, \cdots, \hat{g}_m \mid \text{span}\{\hat{g}_j : j \neq i\}\rangle$ so that a) follows. Similarly, b) follows. To prove c), note first that the well-defined matrix

$$\hat{D}_i = \sigma*(g_i \; \cdots \; ad_f^{s_i-1} g_i) = (\hat{g}_i \; \cdots \; ad_{\hat{f}}^{s_i-1}\hat{g}_i) \qquad i = 1, \cdots, m \quad (41)$$

has rank $\hat{s}_i$ for all $\sigma(x)$ in a neighborhood of $\sigma(x_0)$, since $\dim \hat{R}_i = \hat{s}_i$ and, from (39) and the nonsingularity assumptions, $\text{span}\{\hat{D}_i\} = \hat{R}_i$. Moreover, the first $\hat{s}_i$ columns of $\hat{D}_i$ must be locally independent, since otherwise at $\sigma(x_0) = 0$ we should have

$$\dim(\text{span}\{b_i, \cdots, \hat{A}^{\hat{s}_i-1}\hat{b}_i\} < \hat{s}_i \qquad i = 1, \cdots, m$$

where $\hat{A} = (\partial\hat{f}/\partial\hat{x})(0)$ and $\hat{b}_i = \hat{g}_i(0)$. Since rank $\hat{D}_i = \hat{s}_i$, this would be a contradiction. By dimensionality arguments, since $\text{span}\{\hat{g}_i, \cdots, ad_{\hat{f}}^{\hat{s}_i-1}\hat{g}_i\} \subset \hat{R}_i$, it follows that

$$\text{span}\{\hat{g}_i, \cdots, ad_{\hat{f}}^{\hat{s}_i-1}\hat{g}_i\} = \hat{R}_i \qquad i = 1, \cdots, m. \quad (42)$$

From this, using (2), (3), and (40), it also follows that

$$\begin{aligned} \text{span}\{\theta : \theta = \hat{g}_i \text{ or } \theta = [\hat{g}_{j_h}, [\cdots, [\hat{g}_{j_1}, \hat{g}_i]\cdots]], \\ 1 \le h \le n-1, 0 \le j_k \le m \qquad \text{for } 1 \le k \le h\} \\ = \hat{R}_i = \text{span}\{\hat{g}_i, \cdots, ad_{\hat{f}}^{\hat{s}_i-1}\hat{g}_i\} \qquad i = 1, \cdots, m. \end{aligned}$$

This means that the sequence

$$\hat{S}_0 = \text{span}\{\hat{g}_i\}$$
$$\hat{S}_k = [\hat{f}, \hat{S}_{k-1}] + \sum_{j=1}^{m} [\hat{g}_j, \hat{S}_{k-1}] + \hat{S}_{k-1}$$

converges after a finite number of steps; from Lemma II.1 it follows that

$$\langle \hat{f}, \hat{g}_1, \cdots, \hat{g}_m | \operatorname{span}\{\hat{g}_i\}\rangle = \hat{R}_i = \operatorname{span}\{\hat{g}_i, \cdots, ad_{\hat{f}}^{\hat{s}_i - 1}\hat{g}_i\} \qquad i = 1, \cdots, m. \quad \bullet$$

Proof of Theorem VII.1: Let us consider the system

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_{ii}(x_i)u_i \qquad i = 1, \cdots, m \\ \dot{x}_{m+1,1} &= f_{m+1,1}(x_1, \cdots, x_m, x_{m+1,1}) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1})u_j \\ y_i &= h_i(x_i) \qquad i = 1, \cdots, m. \end{aligned} \tag{43}$$

Since Δ_{MIX} is locally nonsingular, using similar arguments to those used to derive (40), it easily follows that $\hat{\Delta}_{MIX} = 0$, where $\hat{\Delta}_{MIX}$ is defined as Δ_{MIX} but with f and g_j replaced by $\hat{f}$ and $\hat{g}_j$. From Lemma VII.1 and Assumptions 1 and 2.1, Theorem V.1 still holds with (4) replaced by (43) and P_i^*, P_0, and R_i^* replaced by $\hat{P}_i$, $\hat{P}_0$, and $\hat{R}_i$, so that (43) can be rendered locally noninteractive and asymptotically stable by means of a regular dynamic state feedback of the form (27). If we set $\hat{x}^e = (x_1^T \cdots x_m^T x_{m+1,1}^T w^T)^T$, the resulting closed-loop system (in $\hat{x}^e$ coordinates) is given by

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_{ii}(x_i)F_i z_i(\hat{x}^e) + g_{ii}(x_i)v_i \quad i = 1, \cdots, m \\ \dot{x}_{m+1,1} &= f_{m+1,1}(x_1, \cdots, x_m, x_{m+1,1}) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1})F_j z_j(\hat{x}^e) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1})v_j \\ \dot{w} &= \delta(\hat{x}^e) + \gamma(\hat{x}^e)v \\ y_i &= h_i(x_i) \qquad i = 1, \cdots, m \end{aligned}$$

where the $z_i(\hat{x}^e)$ are the functions z_i (see Lemma V.3) expressed in $\hat{x}^e$ coordinates. Since the Δ_{MIX} dynamics is locally asymptotically stable and the outputs do not depend on $x_{m+1,2}$, it follows from standard stability results for triangular systems ([23, Appendix]) that also the composite system

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_{ii}(x_i)F_i z_i(\hat{x}^e) + g_{ii}(x_i)v_i \\ \dot{x}_{m+1,1} &= f_{m+1,1}(x_1, \cdots, x_m, x_{m+1,1}) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1})F_j z_j(\hat{x}^e) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1})v_j \\ \dot{w} &= \delta(\hat{x}^e) + \gamma(\hat{x}^e)v \\ \dot{x}_{m+1,2} &= f_{m+1,2}(x_1, \cdots, x_m, x_{m+1,1}, x_{m+1,2}) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1}, x_{m+1,2}) \\ &\quad \cdot F_j z_j(\hat{x}^e) \\ &\quad + \sum_{j=1}^{m} g_{m+1,1j}(x_1, \cdots, x_m, x_{m+1,1}, x_{m+1,2})v_j \\ y_i &= h_i(x_i) \qquad i = 1, \cdots, m \end{aligned}$$

is locally noninteractive and asymptotically stable. $\bullet$

VIII. Example

Let us consider the simple decoupled system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= \sin x_1 + \sin x_2 + x_3 \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned} \tag{44}$$

It can be seen that

$$\begin{aligned} R_1^* &= \operatorname{span}\{\partial/\partial x_1, \partial/\partial x_3\} \\ R_2^* &= \operatorname{span}\{\partial/\partial x_2, \partial/\partial x_3\} \\ P_1^* &= R_2^* \\ P_2^* &= R_1^* \\ P^* &= \operatorname{span}\{\partial/\partial x_3\}. \end{aligned}$$

The P^* dynamics is given by

$$\dot{x}_3 = x_3$$

which is clearly unstable. Thus no regular state feedback can help us obtain noninteraction and stability.

However, (44) is strongly accessible at $x_0 = 0$ and it has relative degree $\{1, 1\}$ at x_0. Moreover

$$R_i^* = \operatorname{span}\{g_i, [f, g_i]\} \qquad i = 1, \cdots, m$$

and $\Delta_{MIX} = 0$. Thus, our result applies. The distributions R_i^e are given by

$$R_1^e = \operatorname{span}\{g_1^e, [f, g_1]^e\} = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\cos x_1 \\ 1 & 0 \\ 0 & -\cos x_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$$

$$R_2^e = \operatorname{span}\{g_2^e, [f, g_2]^e\} = \operatorname{span}\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -\cos x_2 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -\cos x_2 \end{pmatrix}\right\}.$$

It is easy to check that a suitable feedback (α^e, β^e) is given by

$$\alpha^e = \begin{pmatrix} 0 & 0 & 0 & 0 & (\sin x_1 + \mu_1) & 0 & (\sin x_2 + \mu_2) \end{pmatrix}^T$$

$$\beta^e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

The linear coordinate change

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= \mu_1 \\ z_3 &= x_2 \\ z_4 &= \mu_2 \\ z_5 &= \lambda_1 - x_1 \\ z_6 &= \lambda_2 - x_2 \\ z_7 &= x_3 - \mu_1 - \mu_2 \end{aligned}$$

puts the closed-loop system, resulting from applying (45) to the extended system (7), in the form (25). This system can be rendered locally exponentially stable and noninteractive by means of a suitable feedback of the form (26).

From the results of Section VI, it follows that, since locally $R_1^* \cap R_2^* = \text{span}\{[f, g_2]\}$ and the vector fields $\{X_{ik}: i = 1,2;\ k = 1,2\}$ do not depend at all on x_3, we could have taken $n_w = \dim(R_1^* \cap R_2^*) = \dim \mu_2 = 1$ as well. The R_i^e would have been given in this case by

$$R_1^e = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\cos x_1 \\ 0 & 0 \end{pmatrix}\right\}$$

$$R_2^e = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -\cos x_2 \\ 0 & -\cos x_2 \end{pmatrix}\right\}$$

and

$$\alpha^e = \begin{pmatrix} 0 & 0 & 0 & (\sin x_2 + \mu_2) \end{pmatrix}^T$$

$$\beta^e = I_{3\times 3}.$$

IX. Conclusions

No result so far has considered the problem of computing a dynamic state feedback which ensures both local asymptotic stability and noninteracting control. We give a sufficient condition to solve this problem. For simplicity, we treat the class of systems (1) which can be rendered locally noninteractive by means of regular static state feedback. Our condition, which turns out to be useful when the P^* dynamics are unstable, requires the Δ_{MIX} dynamics to be locally asymptotically stable and some rank conditions to be satisfied, under the assumption that the overall system is strongly accessible at x_0. If we consider the class of regular dynamic noninteraction feedbacks, the requirement of Δ_{MIX} to be locally asymptotically stable is also necessary in order to obtain local stability and noninteraction [22], so that our assumption seems to be very natural. On the other hand, the rank condition (8) seems still too strong to be also necessary.

From our result, it also follows that for the class of square right invertible systems (in the sense of Fliess [14], [15], [16]) we can first apply the dynamic extension algorithm [13] (see also [23]) in order to get some relative degree and then, after having applied a noninteracting regular static state feedback, check if our conditions are satisfied on the system thus obtained.

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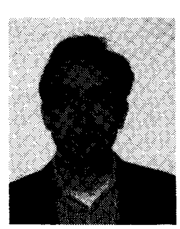
References

[1] G. Gilbert, "The decoupling of multivariable systems by state feedback," *SIAM J. Optimiz. Contr.*, vol. 7, pp. 50-63, 1969.
[2] H. J. Sussmann and V. Jurdjevic, "Controllability of nonlinear systems," *J. Differential Equation*, vol. 12, pp. 95-116, 1972.
[3] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd ed., New York: Springer-Verlag, 1979.
[4] W. A. Porter, "Diagonalization and inverse for nonlinear systems," *Int. J. Contr.*, vol. 10, pp. 67-76, 1970.
[5] S. N. Singh and W. J. Rugh, "Decoupling in a class of nonlinear systems by state variable feedback," *Trans. ASME*, vol. 94, pp. 323-329.
[6] E. Freund, "The structure of decoupled nonlinear systems," *Int. J. Contr.*, vol. 21, pp. 651-659, 1975.
[7] S. Singh, "Decoupling of invertible nonlinear systems with state-feedback and precompensation," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 1237-1239, 1980.
[8] A. Isidori, A. J. Krener, C. Gori Giorgi, and S. Monaco, "Nonlinear decoupling via feedback: A differential geometric approach," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 331-345, 1981.
[9] —, "Locally (f, g)-invariant distributions," *Syst. Contr. Lett.*, vol. 1, pp. 12-15, 1981.
[10] W. Respondek, "On decomposition of nonlinear control systems," *Syst. Contr. Lett.*, vol. 1, pp. 301-308, 1982.
[11] D. Claude, "Decoupling of nonlinear systems," *Syst. Contr. Lett.*, vol. 1, pp. 242-248, 1982.
[12] H. Nijmeijer and J. M. Schumacher, "Zeroes at infinity for affine nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 566-573, 1985.
[13] J. Descusse and C. H. Moog, "Decoupling with dynamic compensation for strong invertible affine systems," *Int. J. Contr.*, vol. 42, pp. 1387-1398, 1986.
[14] M. Fliess, "A new approach to the noninteracting control problem in nonlinear system theory," in *Proc. 23rd Allerton Conf.*, Monticello, IL, 1985, pp. 123-129.
[15] —, "A new approach to the structure at infinity of nonlinear systems," *Syst. Contr. Lett.*, vol. 7, pp. 419-421, 1986.
[16] —, "A note on the invertibility of nonlinear input-output differential systems," *Syst. Contr. Lett.*, vol. 8, pp. 146-151, 1986.
[17] H. Nijmeijer and J. M. Schumacher, "The regular local noninteracting control problem for nonlinear control systems," *SIAM J. Contr. Optimiz.*, vol. 24, pp. 1232-1245, 1986.
[18] H. Nijmeijer and W. Respondek, "Decoupling via dynamic compensation for nonlinear control systems," in *Proc. 25th IEEE Conf. Decision Contr.*, 1986, pp. 192-197.
[19] J. Descusse and C. H. Moog, "Dynamic decoupling for right-invertible nonlinear systems," *Syst. Contr. Lett.*, vol. 8, pp. 345-349, 1987.
[20] A. Isidori and J. W. Grizzle, "Fixed modes and nonlinear noninteracting control with stability," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 907-914, 1988.
[21] J. W. Grizzle and A. Isidori, "Block noninteracting control with stability via static state feedback," *Math. Contr. Signals Syst.*, pp. 315-342, 1989.
[22] K. Wagner, "Nonlinear noninteraction with stability by dynamic state-feedback," *SIAM J. Contr. Optimiz.*, to be published.
[23] A. Isidori, *Nonlinear Control Systems*, 2nd ed., New York: Springer-Verlag, 1989.
[24] H. Nijmeijer, "Feedback decomposition of nonlinear control," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 861-862, 1983.



Stefano Battilotti was born in Rome, Italy, on August 21, 1962. He received the degree in electrical engineering from the University of Rome, Italy, in 1987 and is currently working towards the Ph.D. degree at the same university.

He was a Visiting Student at the University of Maryland, College Park. His research interests are in nonlinear geometric control and stability theory.