A Unifying Framework for the Semiglobal Stabilization of Nonlinear Uncertain Systems via Measurement Feedback

Stefano Battilotti, *Senior Member, IEEE*

***Abstract*—We study the problem of semiglobally stabilizing an uncertain nonlinear system consisting of a *linear nominal* system perturbed by either nonlinearities or model uncertainties. Our approach relies on well-known $\mathcal{H}_\infty$ linear control tools and allows one to recover and improve, in the unifying framework of a semiglobal separation result, existing results on the semiglobal stabilization via *output* feedback. In particular, we discuss the case of uncorrupted outputs, input and output nonlinearities, or model uncertainties, which may include, for example, practical situations such as backlash, hysteresis, and saturations. The key feature of our design procedure is given by the choice of two continuous functions: the first one is instrumental in constructing a stabilizing controller; the second one arises in the candidate Lyapunov function for the closed-loop system. Relying on our main theorem, we give general tools for achieving large regions of attraction via bounded measurement feedback for a wide class of nonlinear uncertain interconnected systems.**

***Index Terms*—Measurement feedback, robustness, semiglobal stabilization, uncertain systems.**

I. Introduction

The problem of asymptotically stabilizing a nonlinear system with large regions of attraction and partial state feedback has recently gained a renewed interest [13], [8], [9], [26]–[28], [24], [25], [16], [17], [19], [20]. Esfandiari and Khalil [13], [8], [9] were the first to introduce *input saturations* and *high-gain observers* to achieve large regions of attraction. A few years later, Teel and Praly proved a general result for achieving large region of attraction, based on the "complete uniform observability" property [24]: semiglobal stabilization via *state feedback* plus *complete uniform observability* imply semiglobal stabilization via *dynamic output feedback*. Complete uniform observability requires, in particular, that the state can be written as a function Ψ of the output, the input, and their higher order derivatives. A *high-gain* observer is designed to reconstruct the higher order derivatives of y, while *input saturations* are used to avoid peaking [14]. A key point of this design procedure is the availability of a *first-order dynamical model of the output and its higher order derivatives*. Even if the output is corrupted by noise, this key assumption allows one to act as if it were not. Moreover, it is unrealistic to assume the availability of a dynamical model of the output whenever this is affected by noise, since derivatives of noise are involved.

The question of what can be done if either Ψ contains some unknown parameter or a dynamical model of the output and its higher order derivatives is not available, because of the presence of model uncertainties, or the observer gain cannot be taken arbitrarily large, because of physical constraints or robustness requirements, arises quite naturally in the context of robust control. For example, consider a nonlinear system affected by some model uncertainties of which nothing but some *nonlinear bounds* are known. Under which conditions it is asymptotically stabilizable with large regions of attraction whenever only a corrupted measure of the state is available, i.e., through *measurement* feedback? In this paper, we give a general theorem for achieving asymptotic stabilization via measurement feedback and with prescribed regions of attraction (*Theorem III.1*). Our approach relies on simple $\mathcal{H}_\infty$ linear control tools and generalizes to a nonlinear setting a previous result on quadratic stabilization of linear uncertain systems [15]. We believe that the results presented in this paper in the framework of Riccati equations and dissipation inequalities are a necessary step toward constructive procedures for taking into account the effect of both *deterministic* and *stochastic* noise.

Several existing results on the semiglobal stabilization via *output feedback* can be recovered and put in a more general perspective in the framework of our result. In particular, in Section III-B, we discuss the case of "uncorrupted outputs" and sees how high-gain observers arise in this case; in Sections III-C and III-D, we discuss the case of input and output nonlinearities or model uncertainties, referring to practical situations such as backslash and saturations. In Section IV we extend our main result to output regulation, giving a generalization of well-known results in the perspective of measurement feedback [11]. Finally, relying on *Theorem III.1*, in Section V we answer the following question: given a number of interconnected uncertain systems satisfying the assumptions of *Theorem III.1* and given the corresponding measurement feedback controllers which stabilize each one of these systems, is it possible to design a controller which stabilizes the overall system? We give some general tools for semiglobally stabilizing via *bounded measurement feedback* a significant class of nonlinear uncertain interconnected systems (*Theorem V.1*). Here, "bounded" is meant over some compact set of the state space. Our design procedure ends up with *linear* controllers and *quadratic* Lyapunov functions, which can be used together with any well-established step-by-step design tool such

Manuscript received June 20, 1998; revised August 1, 1999 and November 1, 1999. Recommended by Associate Editor M. Krstic.

The author is with the Dipartimento di Informatica e Sistemistica, Università degli Studi "La Sapienza" di Roma, 00184 Rome, Italy (e-mail: battilotti@dis.uniroma1.it).

Publisher Item Identifier S 0018-9286(01)00034-4.

as backstepping and forwarding ([14]) or with small gain theorems for taking into account the presence of appended stable dynamics ([21]). As a result, we obtain that systems

$$\begin{aligned}\dot{x}_1 &= x_2 + p_{11}(x, u, t)\\ \dot{x}_2 &= x_3 + p_{21}(x, u, t)\\ \vdots &= \vdots\\ \dot{x}_s &= u + p_{s1}(x, u, t)\\ y_j &= x_j + p_{j2}(x, u, t), \qquad j = 1, \ldots, s\end{aligned} \tag{1}$$

with $x_j, y, u \in \mathbb{R}$, and $x = \mathrm{col}(x_1, \ldots, x_s)$, are *semiglobally stabilizable via bounded measurement feedback* as long as $p_{j1}(x, u, t)$ and $p_{j2}(x, u, t)$, $j = 1, \ldots, s$, are *higher order* in $x_{j+1}, \ldots, x_s$ and u, uniformly with respect to $t, x_1, \ldots, x_j$ and u [2] and see [27], [28], [16], [17], [19] for state feedback, while systems of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + p_{11}(x, u, t)u\\ \dot{x}_2 &= x_3 + p_{21}(x, u, t)u\\ \vdots &= \vdots\\ \dot{x}_s &= u + p_{s1}(x, u, t)u\\ y_j &= x_j + p_{j2}(x, u, t)u, \qquad j = 1, \ldots, s\end{aligned} \tag{2}$$

with x_j, y_j, and $u \in \mathbb{R}$ are *semiglobally stabilizable via bounded measurement feedback* as long as $p_{1j}(x, u, t)u$ and $p_{2j}(x, u, t)u$ are *of order less than* $1 + j - s$ *and* $j - s$, respectively, with respect to the "generalized" dilation $\delta_l(x, u) = (l^{1-s}x_1, \ldots, l^{-1}x_{s-1}, x_s, lu)$ and uniformly with respect to t [4] and see [23] for state feedback.

II. Notations

- If $\|v\|$ denotes the 2-norm of any given vector v, by $\|A\|$ we denote the induced 2-norm of any given matrix A and we have $\|A\| = \sqrt{\lambda_{\max}\{A^T A\}}$; by $\|v\|_A$ we denote the A-norm of v, i.e., $\|v\|_A = \sqrt{v^T A v}$.
- By $\mathcal{SP}^n$ ($\mathcal{SSP}^n$) we denote the set of $n \times n$ positive definite (positive semidefinite) symmetric matrices; moreover, by S^+ we denote the pseudoinverse of S; $\mathrm{col}(v_1, \ldots, v_n)$ denotes the column vector with jth component equal to v_j and $\mathrm{row}(v_1, \ldots, v_n)$ denotes the row vector with jth component equal to v_j.
- For any vector-valued function η: $\mathbb{R}^s \to \mathbb{R}^r$, we denote by η_i its ith component; for any matrix M we denote by M_i its ith row.
- A function α: $[0, +\infty) \to [0, +\infty)$ is said to be of class $\mathcal{K}$ (or $\alpha \in \mathcal{K}$) if it is continuous, $\alpha(0) = 0$ and it is strictly increasing.
- A function α: $[0, +\infty) \to [0, +\infty)$ is said to be of class $\mathcal{K}_\infty$ (or $\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and $\lim_{s \to +\infty} \alpha(s) = +\infty$.

III. Regional Stabilization via Measurement Feedback

A. Problem Formulation and Main Result

Let us consider

$$\begin{aligned}\dot{x} &= Ax + B_2 u + B_1 \tilde{\Phi}(u, x, t)\\ y &= C_2 x + C_1(u, x, t)\tilde{\Phi}(u, x, t)\end{aligned} \tag{3}$$

with state vector $x \in \mathbb{R}^n$, input vector $u \in \mathbb{R}^m$, output vector $y \in \mathbb{R}^p$, A, B_j, $j = 1, 2$, and C_2 ($C_1(u, x, t)$) matrices (matrix-valued function) with appropriate dimensions and $\tilde{\Phi}(u, x, t) \in \mathbb{R}^k$ satisfying the following property:

- *for some given C^0 positive semidefinite function s: $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ and for each t, x and u*

$$\tilde{\Phi}(u, x, t) \in \{v \in \mathbb{R}^k: \|v\| \le s(u, x)\}. \tag{4}$$

The vector $\tilde{\Phi}$ captures both (model) uncertainties and exogenous disturbances affecting the *nominal* system $\dot{x} = Ax + B_2 u$, $y = C_2 x$, and we refer to such $\tilde{\Phi}$ as *admissible* uncertainties. All the results of this paper can be straightforwardly extended to the case in which $\tilde{\Phi}$ is C^0 with respect to u and x and *Lebesgue measurable* with respect to t and to the case in which the function s depends explicitly on t (tracking, etc.).

The class of *admissible* feedback laws we consider is characterized as follows:

$$\begin{aligned}u &= k(y, \sigma)\\ \dot{\sigma} &= v(y, \sigma), \qquad \sigma \in \mathbb{R}^q\end{aligned} \tag{5}$$

with C^0 functions k: $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m$ and v: $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$, both vanishing at the origin.

In this section, we are interested in the *regional* stabilization of (3) under the constraint (4) for some given function $s(u, x)$. Let $\mathcal{D}$ be the set of admissible uncertainties $\tilde{\Phi}$: $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ and let $(x(t), \sigma(t))$ denote the trajectories of (3)–(5) at time t stemming from (x_0, σ_0). We say that the system (3) is *uniformly locally asymptotically stabilizable via measurement-feedback with region of attraction containing* $\Omega^e \subset \mathbb{R}^n \times \mathbb{R}^q$ (Ω^e*-ULASM*) if there exists an admissible control law (5) such that along the trajectories of (3)–(5)

1) *(uniform stability)* $\forall \epsilon > 0$ there exists $\delta_\epsilon \in \mathbb{R}^+$ such that

$$\left\| \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} \right\| \le \epsilon \tag{6}$$

for all $(x_0, \sigma_0) \in \{v \in \Omega^e: \|v\| \le \delta_\epsilon\}$, $t \ge 0$, and $\tilde{\Phi} \in \mathcal{D}$;

2) *(uniform boundedness plus attraction)* there exists $M > 0$ such that

$$\left\| \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} \right\| < M \tag{7}$$

for all $(x_0, \sigma_0) \in \Omega^e$, $t \ge 0$, and $\tilde{\Phi} \in \mathcal{D}$, and $\forall \epsilon > 0$ there exists $T_\epsilon > 0$ such that

$$\left\| \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} \right\| < \epsilon \tag{8}$$

for all $(x_0, \sigma_0) \in \Omega^e$, $t \ge T_\epsilon$, and $\tilde{\Phi} \in \mathcal{D}$.

If (3) satisfies 1) and 2) with $u = 0$, we say that (3) is *uniformly locally asymptotically stable with region of attraction containing* Ω^e (Ω^e-ULAS). If (3) satisfies 1) and 2) with $u = 0$ and $\Omega^e = \mathbb{R}^{n+q}$, we say that (3) is *uniformly globally asymptotically stable* (UGAS). If $\Omega^e = \mathbb{R}^{n+q}$, we will say that (3) is *uniformly globally asymptotically stabilizable* (UGASM). If Ω^e contains any *a priori* given compact set of $\mathbb{R}^{n+q}$ we will say that (3) is *uniformly semiglobally asymptotically stabilizable via* *measurement feedback*. Throughout the paper, we will consider compact sets containing the origin of the state space.

One can also take into account external disturbances or reference trajectories, modeled through a dynamical system, and distinguish among regulated and measured outputs by considering

$$\begin{aligned}\dot{x} &= Ax + B_2u + B_1\tilde{\Phi}(u,\, w,\, x,\, t)\\ \dot{w} &= s(w)\\ y &= C_2x + C_1\tilde{\Phi}(u,\, w,\, x,\, t)\\ r &= D_2x + D_1\tilde{\Phi}(u,\, w,\, x,\, t)\end{aligned} \tag{9}$$

with external disturbance vector $w \in \mathbb{R}^s$, regulated output vector $r \in \mathbb{R}^r$, and measured output vector $y \in \mathbb{R}^p$. In this case, (4) may not hold for *any* choice of $s(u, x)$, since $\tilde{\Phi}(u, w, x, t)$ may not vanish for $(u, x) = (0, 0)$. This problem may be overcome by trying to move the equilibrium point $(x, \sigma, u) = (0, 0, 0)$ over to $(x, \sigma, u) = (\Pi(w), \Sigma(w), \Gamma(w))$ with C^0 functions $\Pi(\cdot)$, $\Sigma(\cdot)$ and $\Gamma(\cdot)$, all vanishing at the origin and such that $r \equiv 0$ for all w and t whenever $(x, u) = (\Pi(w), \Gamma(w))$ (i.e., regulation). Correspondingly, in (6) and (7), $x(t)$ and $\sigma(t)$ should be replaced by $x(t) - \Pi(w(t))$ and $\sigma(t) - \Sigma(w(t))$, respectively, with $w_0 \in \mathcal{B}_d$ and $\mathcal{B}_d$ is a given compact set of $\mathbb{R}^s$ such that for each $w_0 \in \mathcal{B}_d$ the trajectory $w(t, w_0)$ of $\dot{w} = s(w)$ satisfies $w(t, w_0) \in \mathcal{B} \supseteq \mathcal{B}_d$ for all $t \geq 0$. In this case, we will say that (9) can be *uniformly locally output regulated via measurement feedback with region of attraction containing* $\Omega^e \subset \mathbb{R}^n \times \mathcal{B}_d \times \mathbb{R}^q$ (Ω^e-ULORM).

To understand some key features of our main result, consider the problem of regionally stabilizing (3) under the constraint (4) with

$$s(u, x) = \sqrt{\frac{\|x\|_E^2 + \|u\|_{R_1}^2}{\gamma^2}} \tag{10}$$

for all x *and* u, for some $E \in \mathcal{SSP}^n$, $R_1 \in \mathcal{SP}^m$ and $\gamma > 0$.

Assume the state vector x is available for feedback and the existence of P_{SF}, $Q_{SF} \in \mathcal{SP}^n$ such that

$$A^T P_{SF} + P_{SF}A + P_{SF}\left(\frac{1}{\gamma^2}\, B_1B_1^T - B_2R_1^{-1}B_2^T\right) \cdot P_{SF} + E = -Q_{SF}. \tag{11}$$

Let $V_{SF}(x) = \|x\|_{P_{SF}}^2$ and $F = -R^{-1}B_2^TP_{SF}$. Let us pretend that $\tilde{\Phi}(u, x, t)$ is an "external" disturbance $w \in \mathcal{L}_2[0, \infty)$ affecting the system. Along the trajectories of the system

$$\dot{x} = Ax + B_2u + B_1w \tag{12}$$

by (11) one has

$$\begin{aligned}&\dot{V}_{SF} + \|x\|_E^2 + \|u\|_{R_1}^2 - \gamma^2\|w\|^2\\ &\quad= \|u - Fx\|_{R_1}^2 - \gamma^2\left\|w - \frac{1}{\gamma^2}\, B_1^TP_{SF}x\right\|^2 - \|x\|_{Q_{SF}}^2.\end{aligned} \tag{13}$$

If $u = Fx$, (11) and (13) guarantee that the $\mathcal{L}_2$ gain of the closed-loop system (12) from w to $z = \sqrt{\|x\|_E^2 + \|u\|_{R_1}^2}$ is *less or equal to* γ [5]. We conclude that the admissible controller $u = Fx$ attains for (12) a *guaranteed level of attenuation* (in terms of energy) of the effect of $\tilde{\Phi}$ over the "cost" z.

Since $\|x\|_E^2 + \|u\|_{R_1}^2 \geq \gamma^2\|\tilde{\Phi}(u, x, t)\|^2$ *for all* t, x *and* u, it follows from (13), with w replaced by $\tilde{\Phi}(u, x, t)$, that (3), with $u = Fx$, is UGAS.

When the state vector x is not available for feedback, we should replace x by some *estimate*. To begin with, we illustrate a "dual" problem to the one above. Assume $B_1C_1^T = 0$ and the existence of $R_2 \in \mathcal{SP}^p$ and P_{OI}, $Q_{OI} \in \mathcal{SP}^n$, such that

$$R_2 \geq C_1(x, u, t)C_1^T(x, u, t) \tag{14}$$

for all x, u, and t and

$$A^TP_{OI} + P_{OI}A + P_{OI}B_1B_1^TP_{OI} + \frac{E}{\gamma^2} - C_2^TR_2^{-1}C_2 = -Q_{OI}. \tag{15}$$

Let $V_{OI}(x) = \|x\|_{P_{OI}}^2$. Along the trajectories of the system

$$\begin{aligned}\dot{x} &= Ax - Gy + B_1w\\ y &= C_2x + C_1w\end{aligned} \tag{16}$$

one has

$$\begin{aligned}&\dot{V}_{OI} + \frac{1}{\gamma^2}\|x\|_E^2 - \|w\|^2 = \|(R_2^{-1}C_2 - G^TP_{OI})x\|_{R_2}^2\\ &\qquad - \|w - (B_1 - GC_1)^TP_{OI}x\|^2 - \|x\|_{Q_{OI}}^2.\end{aligned} \tag{17}$$

Note that the *output injection* term Gy in (16) replaces the corresponding control term B_2u in (12). If $G = P_{OI}^{-1}C_2^TR_2^{-1}$, (17) guarantees that the $\mathcal{L}_2$ gain of (16) from any w to $z = \|x\|_E$ is *less or equal to* γ [5].

Assume now that

$$\begin{aligned}P_m &= \gamma^2P_{OI} - P_{SF} > 0\\ Q_m &= \gamma^2Q_{OI} - Q_{SF} > 0.\end{aligned} \tag{18}$$

The first condition is a *coupling* condition between V_{SF} and V_{OI}, while the second one is a *coupling* condition between the *derivatives* of V_{SF} and V_{OI}. Let $e = x - \sigma$ and $W(e) = \|e\|_{P_m}^2$. Let

$$\begin{aligned}u &= F\sigma\\ \dot{\sigma} &= \left(A + \frac{1}{\gamma^2}\, B_1B_1^TP_{SF} - GC_2\right)\sigma + B_2u + Gy\end{aligned} \tag{19}$$

where $G = \gamma^2P_m^{-1}R_2^{-1}C_2^T$, be a candidate admissible controller for (3). Along the trajectories of the closed-loop system (3)–(19)

$$\begin{aligned}\dot{x} &= \left(A + \frac{1}{\gamma^2}\, B_1B_1^T\right)x + B_2u + B_1\tilde{w}\\ \dot{\sigma} &= \left(A + \frac{1}{\gamma^2}\, B_1B_1^T - GC_2\right)\sigma + B_2u + Gy\\ y &= C_2x + C_1\tilde{w}\end{aligned} \tag{20}$$

with $G = \gamma^2P_m^{-1}C_2^TR_2^{-1}$ and $\tilde{w} = w - (1/\gamma^2)B_1^TP_{SF}x$, we have from (11) and (15)

$$\dot{V}_m + \|Fe\|_{R_1}^2 - \gamma^2\|\tilde{w}\|^2 \leq -\|e\|_{Q_m}^2. \tag{21}$$

Summing up (13) and (21), we get

$$\begin{aligned}&\dot{V}_{SF} + \dot{V}_m + \|x\|_E^2 + \|u\|_{R_1}^2 - \gamma^2\|w\|^2\\ &\qquad \leq \|u - Fx\|_{R_1}^2 - \|Fe\|_{R_1}^2 - \|x\|_{Q_{SF}}^2 - \|e\|_{Q_m}^2\end{aligned} \tag{22}$$

which, together with (18), guarantees that the $\mathcal{L}_2$ gain of the closed-loop system (20) from w to

$$z = \sqrt{\|x\|_E^2 + \|u\|_{R_1}^2}$$

is *less or equal to* γ [5]. Since $\|x\|_E^2 + \|u\|_{R_1}^2 \geq \gamma^2 \|\tilde{\Phi}(u, x, t)\|^2$ *for all* t, x, and u, it follows from (22), with w replaced by $\tilde{\Phi}(u, x, t)$, that (3)–(19) is UGAS.

The above procedure extends immediately to the case in which (10) holds only on some *compact set* Ω. While local asymptotic stability is still guaranteed, the region of attraction of (3)–(19) may *shrink as the width of* Ω *increases*. To get into some more detail, pick $c > 0$ and assume that

$$s(u, x) = \sqrt{\frac{\|x\|_{E(c)}^2 + \|u\|_{R_1(c)}^2}{\gamma^2(c)}}$$

whenever $\|u\| \leq \Delta(c)$, for some $\Delta(c) > 0$, and x lives in some compact set $\Omega(c)$, described by the level set of some $P_{SF}(c) \in \mathcal{SP}^n$, i.e., $\Omega(c) = \{v \in \mathbb{R}^n\colon \|v\|_{P_{SF}(c)}^2 \leq c$, where $P_{SF}(c)$ is a solution of (11) for some $Q_{SF}(c) \in \mathcal{SP}^n$. Assume also that one is capable to find $P_{OI}(c) \in \mathcal{SP}^n$ satisfying $P_{OI}(c) > (P_{SF}(c)/\gamma^2(c))$ and (15) for some $Q_{OI}(c) \in \mathcal{SP}^n$ such that $Q_{OI}(c) > (Q_{SF}(c)/\gamma^2(c))$ and $R_2(c) \in \mathcal{SP}^p$ such that $R_2(c) \geq C_1(u, x, t)C_1^T(u, x, t)$ for all $x \in \Omega(c)$, u such that $\|u\| \leq \Delta(c)$ and t. From the discussion above, it follows that (3) is $\Omega^e(c)$-ULASM with admissible controller (19) and $\Omega^e(c) = \{(v_1, v_2) \in \mathbb{R}^{2n}\colon \|v_1\|_{P_{SF}(c)}^2 + \|v_1 - v_2\|_{P_m(c)}^2 \leq c\}$. Unfortunately, since $P_{SF}(c)$ and $P_m(c)$ are themselves functions of c, $\Omega^e(c)$ is not, in general, as wide as we wish and the region of attraction of (3)–(19) may shrink as we attempt to pick c larger and larger. However, this nonlinear phenomenon [7] can be counteracted if an additional *nonlinear* inequality, involving $P_{SF}(c)$, $P_{OI}(c)$, $Q_{SF}(c)$ and $Q_{OI}(c)$ and two C^0 functions $\eta\colon \mathbb{R}^m \to \mathbb{R}^m$ and $\delta\colon \mathbb{R} \to (0, 1-a]$, $a \in (0, 1)$, is satisfied: the first one is instrumental in constructing a stabilizing controller (5), in the sense that $k(y, \sigma)$ is taken as the composition of $\eta(\cdot)$ with the *linear* controller $u = F(c)\sigma$; the second one pops up in the candidate Lyapunov function of the closed-loop system and its choice is critical in the study of the stability of the closed-loop system. This is exactly expressed by the following theorem (see [1] for a preliminary version). For simplicity of computations and resolving formulas, we will assume that $B_1 C_1^T(u, x, t) = 0$ for all u, x and t.

Theorem III.1: Assume $B_1 C_1^T(u, x, t) = 0$ for all u, x and t. Moreover, assume that for each $c > 0$

- *(state feedback)* there exist $P_{SF}(c), Q_{SF}(c) \in \mathcal{SP}^n$, such that
  — for some $\gamma(c) > 0$ and $\Delta(c) \in (0, \infty]$ and for some matrices $E(c) \in \mathcal{SSP}^n$ and $R_1(c) \in \mathcal{SP}^m$ one has

$$\|\tilde{\Phi}(u, x, t)\|^2 \leq \frac{\|x\|_{E(c)}^2 + \|u\|_{R_1(c)}^2}{\gamma^2(c)} \tag{23}$$

  for all t, u such that $\|u\| \leq \Delta(c)$ and $x \in \Omega(c) = \{v \in \mathbb{R}^n\colon \|v\|_{P_{SF}(c)}^2 \leq c\}$;
  — the following Riccati equation is satisfied:

$$\begin{aligned} &A^T P_{SF}(c) + P_{SF}(c)A + P_{SF}(c) \\ &\quad \cdot \left(\frac{1}{\gamma^2(c)} B_1 B_1^T - B_2 R_1^{-1}(c) B_2^T\right) P_{SF}(c) + E(c) \\ &= -Q_{SF}(c) \end{aligned} \tag{24}$$

- *(output injection)* there exist $R_2(c) \in \mathcal{SP}^p$ and $P_{OI}(c)$, $Q_{OI}(c) \in \mathcal{SP}^n$, such that
  — for all $u \in \mathbb{R}^m$ such that $\|u\| \leq \Delta(c)$, $x \in \Omega(c)$, and t

$$R_2(c) \geq C_1(u, x, t)C_1^T(u, x, t) \tag{25}$$

  — the following Riccati inequality is satisfied:

$$\begin{aligned} &A^T P_{OI}(c) + P_{OI}(c)A + P_{OI}(c)B_1 B_1^T P_{OI}(c) + \frac{E(c)}{\gamma^2(c)} \\ &\quad - C_2^T R_2^{-1}(c) C_2 \leq -Q_{OI}(c) \end{aligned} \tag{26}$$

- *(relative speed)* $Q_m(c) = \gamma^2(c)Q_{OI}(c) - Q_{SF}(c) > 0$ and $P_m(c) = \gamma^2(c)P_{OI}(c) - P_{SF}(c) > 0$;
- *(nonlinear coupling)* there exist $a \in (0, 1)$ and C^0 functions $\delta\colon \mathbb{R} \to (0, 1-a]$ and $\eta\colon \mathbb{R}^m \to \mathbb{R}^m$ such that if

$$F(c) = -R_1^{-1}(c)B_2^T P_{SF}(c) \tag{27}$$

$$\varphi(s) = as + \int_0^s \delta(\vartheta)\, d\vartheta \tag{28}$$

  then

$$\begin{aligned} &\|\eta(F(c)(x-e)) - F(c)x\|_{R_1(c)}^2 - \|x\|_{Q_{SF}(c)}^2 \\ &\quad - \left(a + \delta\left(\|e\|_{P_m(c)}^2\right)\right)\left(\|F(c)e\|_{R_1(c)}^2 + \|e\|_{Q_m(c)}^2\right) \end{aligned} \tag{29}$$

  is negative definite for all $x \in \Omega(c)$ and e such that $0 \leq \varphi(\|e\|_{P_m(c)}) \leq c$.

Under the above assumptions and whenever $\|\eta(F(c)\sigma(t))\| \leq \Delta(c)$ for all $t \geq 0$, (3) is $\Omega^e(c)$-ULASM where

$$\begin{aligned} \Omega^e(c) = \Big\{&(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n\colon \|v_1\|_{P_{SF}(c)}^2 \\ &+\varphi\left(\|v_1 - v_2\|_{P_m(c)}^2\right) \leq c\Big\}. \end{aligned} \tag{30}$$

An admissible stabilizing controller is given by

$$\begin{aligned} u &= \eta(F(c)\sigma) \\ \dot{\sigma} &= H(c)\sigma + B_2\eta(F(c)\sigma) + G(c)y \end{aligned} \tag{31}$$

with

$$\begin{aligned} H(c) &= A + \frac{1}{\gamma^2(c)} B_1 B_1^T P_{SF}(c) - G(c)C_2 \\ G(c) &= \gamma^2(c)P_m^{-1}(c)C_2^T R_2^{-1}(c). \end{aligned} \tag{32}$$

Proof: The proof is based on standard computations of $\mathcal{H}_\infty$ linear control [5]. Throughout the proof, if no ambiguity arises, we omit the argument c. With G and Q_m as above, since $Q_m > 0$, by direct calculations we obtain

$$\begin{aligned} 0 > -Q_m \geq &\left[A + \frac{1}{\gamma^2} B_1 B_1^T P_{SF} - GC_2\right]^T P_m \\ &+ P_m\left[A + \frac{1}{\gamma^2} B_1 B_1^T P_{SF} - GC_2\right] + F^T R_1 F \\ &+ \frac{1}{\gamma^2} P_m (B_1 B_1^T + GR_2 G^T) P_m. \end{aligned} \tag{33}$$

Since $C_1(u, x, t)B_1^T = 0$ for all u, x, and t, the state equations of (3)–(31) can be recast as follows:

$$\begin{aligned} \dot{x} &= \left(A + \frac{1}{\gamma^2} B_1 B_1^T P_{SF}\right)x + B_2\eta(F\sigma) + B_1\tilde{w} \\ \dot{\sigma} &= H\sigma + B_2\eta(F\sigma) + GC_2 x + GC_1(\eta(F\sigma), x, t)\tilde{w} \end{aligned} \tag{34}$$

with $\tilde{w} = \tilde{\Phi}(\eta(F\sigma), x, t) - (1/\gamma^2)B_1^T P_{SF} x$. Let $e = x - \sigma$ and $V_m(e) = \varphi(\|e\|_{P_m}^2)$. Along the trajectories of (34), since

$0 \le a + \delta(s) \le 1$ for all s and using (25) and (33), as long as $(x(t), \sigma(t)) \in \Omega^e$ and $\|\eta(F\sigma(t))\| \le \Delta$ for all $t \ge 0$, one has

$$\begin{aligned}\dot V_m &+ (a + \delta(\|e\|^2_{P_m}))\|Fe\|^2_{R_1} - \gamma^2\|\tilde w\|^2\\ &\le (a+\delta(\|e\|^2_{P_m}))e^T\left[A + \frac{1}{\gamma^2}B_1B_1^TP_{SF} - GC_2\right]^T P_m e\\ &\quad + (a+\delta(\|e\|^2_{P_m}))e^TP_m\left[A + \frac{1}{\gamma^2}B_1B_1^TP_{SF} - GC_2\right]e\\ &\quad + (a+\delta(\|e\|^2_{P_m}))\|Fe\|^2_{R_1} + \frac{(a+\delta(\|e\|^2_{P_m}))^2}{\gamma^2}\\ &\quad \cdot e^TP_m(B_1B_1^T + GR_2G^T)P_m e\\ &\le -(a+\delta(\|e\|^2_{P_m}))\|e\|^2_{Q_m}. \end{aligned} \tag{35}$$

Let $V_{SF}(x) = \|x\|^2_{P_{SF}}$. Along the trajectories of (3), one has for all u

$$\begin{aligned}\dot V_{SF} + \|x\|^2_E + \|u\|^2_{R_1} - \gamma^2\|\tilde\Phi(u, x, t)\|^2\\ = \|u - Fx\|^2_{R_1} - \gamma^2\|\tilde w\|^2 - \|x\|^2_{Q_{SF}}.\end{aligned} \tag{36}$$

From (23), (29), and (33)–(36) we conclude that, as long as $(x(t), \sigma(t)) \in \Omega^e$ and whenever $\|\eta(F\sigma(t))\| \le \Delta$ for all $t \ge 0$,

$$\begin{aligned}\dot V_{SF} + \dot V_m \le \|\eta(F\sigma) - Fx\|^2_{R_1} - \|x\|^2_{Q_{SF}} - \left[\|Fe\|^2_{R_1} + \|e\|^2_{Q_m}\right]\\ \cdot (a + \delta(\|e\|^2_{P_m})) - \|x\|^2_E - \|\eta(F\sigma)\|^2_{R_1}\\ + \gamma^2\|\tilde\Phi(\eta(F\sigma), x, t)\|^2\end{aligned} \tag{37}$$

is negative definite. From (37) and since $V_{SF} + V_m$ is proper and positive definite, it follows by standard results on Lyapunov stability (see, for example, [12]) that, whenever $\|\eta(F\sigma(t))\| \le \Delta$ for all $t \ge 0$, (3)–(31) is Ω^e-ULAS. □

Remark III.1: (Candidate Lyapunov Function): The value $a = 0$ is allowed in *Theorem III.1* as long as $\int_0^s \delta(\vartheta)\,d\vartheta \in \mathcal{K}_\infty$. Moreover, we want to spend few words on the choice of η and δ. These functions must be such to satisfy the coupling condition (29). The possible choices of η are in the existing literature reduced, to some extent, to a *saturation* function (see Section III-B) or a *linear* function (see Section III-C). In our context, the flexibility in the choice of the function η is emphasized together with its intimate connections to the possible choices of the candidate Lyapunov functions (depending on φ and, thus, on δ) which are compatible with the satisfaction of (29). In this sense, while η is a parameter design, δ is not, but it is intimately connected to the choice of η through (29). The relation between η and δ through (29) is essential to understand the solution to semiglobal stabilization problems different from the ones considered in the literature and in the framework of a unifying approach (see Section III-D for example).

Remark III.2: (Input Saturations): The design parameter $\Delta(c)$ allows us to take into account input saturations (see Section III-C). This limiting constraint of the input can be implemented either in the sense of bounding the function $F(c)\sigma$ on the compact set $\Omega^e(c)$ (see Section III-C) or through the function η itself (see Section III-B).

Remark III.3: (Quadratic Stability): Theorem III.1 states that *regional* stabilization can be achieved if one is able to solve: a *semiglobal state feedback* stabilization problem, with controller $u = F(c)x$ and region of attraction being the points inside some level set $\mathcal{L}(c)$ of a Lyapunov function V_{SF}, an *output injection* stabilization problem, assuming the state x lives in $\mathcal{L}(c)$, a *linear* coupling condition which guarantees the observer error converge to zero "sufficiently" faster than the state x does, whenever (3) is plugged with $u = F(c)x$, and a *nonlinear* coupling condition which accounts for peaking and the width of the region of attraction. If (23) and (25) hold for *all* x, u, and t, *Theorem III.1* recovers a well-known result on *quadratic* stabilization of uncertain systems ([15]), since (29) is always satisfied with $a = 0$, $\delta(s) = 1$ and $\eta(s) = s$ for all s (see also the discussion at the beginning of this section).

Remark III.4: (Dissipation Inequalities): From the proof of *Theorem III.1* it also follows that the conclusions of *Theorem III.1* still hold if (24) is replaced directly by the *dissipation inequality* (36) and, moreover, (26) together with the relative speed constraint are replaced by the existence of matrices $P_m(c), Q_m(c) \in \mathcal{SP}^n$ such that

$$\begin{aligned}&\left[A + \frac{1}{\gamma^2}B_1B_1^TP_{SF}(c)\right]^T P_m(c)\\ &+ P_m(c)\left[A + \frac{1}{\gamma^2(c)}B_1B_1^TP_{SF}(c)\right]\\ &+ F^T(c)R_1(c)F(c) + \frac{1}{\gamma^2(c)}P_m(c)B_1B_1^TP_m\\ &- \gamma^2(c)C_2^TR_2^{-1}(c)C_2 \le -Q_m(c)\end{aligned} \tag{38}$$

[or, alternatively, by the dissipation inequality (35)]. In some cases, (38) may be easier to solve than (26) together with the relative speed condition (see [3]).

Remark III.5: (Relative Speed): The condition $Q_m(c) > 0$ can be interpreted as follows: the observer error goes to zero *faster* (but not *arbitrarily*) than the state of (3), with $u = F(c)x$.

B. *Uncorrupted Outputs Revisited*

See [8], [9], [13], [16], [17], [19], [20], [24], and [25]. If $C_1(u, x, t) = 0$ for all u, x and t, for each $c > 0$ the matrix $R_2(c)$ [see (25)] can be taken *any* positive definite matrix. If (A, B_1, C_2) is *left invertible with no zero dynamics* ([18]), (25)–(29) can be met as follows.

Assume $p = m = 1$ and that for each $c > 0$ there exist $\gamma(c)$, $\Delta(c)$, $R_1(c)$, $E(c)$, $P_{SF}(c)$ and $Q_{SF}(c)$ satisfying (23) and (24) (throughout this section we will omit the argument c). Under the above assumptions, one can find C^0 functions $Q_{OI}^{(\cdot)}$, $P_{OI}^{(\cdot)}$ and $R_2^{(\cdot)}$: $((\|E\|/\gamma^2), \infty) \to \mathbb{R}^+$ and $\epsilon_{OI}^{(\cdot)}$: $((\|E\|/\gamma^2), \infty) \to \mathbb{R}^+$ with the following properties.

1) The following Riccati equation is satisfied:

$$\begin{aligned}A^TP_{OI}^{(l)} + P_{OI}^{(l)}A + P_{OI}^{(l)}B_1B_1^TP_{OI}^{(l)}\\ + \frac{E}{\gamma^2} - C_2^T(R_2^{(l)})^{-1}C_2 \le -Q_{OI}^{(l)}\end{aligned} \tag{39}$$

for all $l \in ((\|E\|/\gamma^2), \infty)$.

2) The entries of $P_{OI}^{(l)}$ are polynomial functions of l and

$$Q_{OI}^{(l)} = \epsilon_{OI}^{(l)}P_{OI}^{(l)} \tag{40}$$

$$\lim_{l\to\infty} \frac{\epsilon_{OI}^{(l)}}{l} = \infty \tag{41}$$

$$\lim_{l\to\infty} R_2^{(l)} = 0. \tag{42}$$

3) $P_m^{(l)} = \gamma^2P_{OI}^{(l)} - P_{SF} > 0$ and $Q_m^{(l)} = \gamma^2Q_{OI}^{(l)} - Q_{SF} > 0$ for all $l \in ((\|E\|/\gamma^2), \infty)$.

Let

$$\begin{aligned} u_{\max,i} &= \max_{x\in\Omega}\{|F_{2i}x|\} \\ \eta_i(s_i) &= s_i \min\left\{1, \frac{u_{\max,i}}{|s_i|}\right\} \\ \eta(s_1, \dots, s_m) &= \mathrm{col}(\eta_1(s_1), \dots, \eta_m(s_m)) \\ \delta(s) &= \frac{1}{l(1+s)}, \qquad \text{if } s \geq 0. \end{aligned} \tag{43}$$

Using (40)–(42), since $\eta_i(Fx) = F_{2i}x$ for all $i = 1, \dots, m$ and $x \in \Omega$, one can find $l^* \in ((\|E\|/\gamma^2), \infty)$ such that (29) is negative definite for all $l \in [l^*, \infty)$, $x \in \Omega$ and e such that $0 \leq \|e\|^2_{P_m} \leq c$ (we leave the proof to the reader).

Note that with our choices the region of attraction of the closed-loop system (3)–(31) is given by

$$\Omega^e = \left\{(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n\colon \|v_1\|^2_{P_{SF}} + \frac{1}{l}\ln\left(1 + \|v_1 - v_2\|^2_{P_m^{(l)}}\right) \leq c\right\}. \tag{44}$$

Since

$$\lim_{s\to\infty} \frac{\ln s^k}{s} = 0 \qquad \forall\, k \geq 1$$

it follows that, for each pair of compact sets $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$,

$$\lim_{l\to\infty} \sup_{\substack{x\in\mathcal{S}\\ \sigma\in\mathcal{W}}} \left\{\frac{1}{l}\ln\left(1 + \|x - \sigma\|^2_{P_m^{(l)}}\right)\right\} = 0.$$

We conclude that for each pair of compact sets $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$ one can pick $l \in ((\|E\|^2/\gamma^2), +\infty)$ sufficiently *large* in such a way that Ω^e contains $\mathcal{S} \times \mathcal{W}$. By slightly modifying the above procedure, one can recover also the fact that *semiglobal stabilizability via state feedback [in the sense of (24) or, equivalently, (36)] plus complete uniform observability implies semiglobal stabilization via output feedback*. Indeed, under these assumptions, the output can be taken as a *state* so that one can assume $C_1(u, x, t) = 0$ for all x, u and t and use similar arguments to those above. However, as far as we consider a *linear* nominal systems as we do in *Theorem III.1*, Teel and Praly's result remains still more general than ours as long as a dynamical model of the output and its derivatives is available and the state x can be written as a function of y, u and their higher order derivatives. On the other hand, considering a linear nominal system has revealed itself still satisfactory to recover many classical results on the semiglobal stabilization of nonlinear systems (see following sections).

C. Input Uncertainties

See [16], [17], [19], [20], and [26]. Let us consider the system

$$\begin{aligned} \dot{x} &= Ax + B_2u + B_1\tilde{\Phi}(u, x, t) \\ y &= C_2x + C_1(u, t)\tilde{\Phi}(u, x, t) \end{aligned} \tag{45}$$

with $B_1C_1^T(u, t) = 0$ for all u and t. We make the following assumptions.

(H1) The pairs (A, B_1) and (A, B_2) are stabilizable and the eigenvalues of A have nonpositive real part.

(H2) The pair (C_2, A) is detectable.

(H3) There exist continuous functions $Q_{SF}^{(\cdot)}, P_{SF}^{(\cdot)}\colon (0,1] \to \mathcal{SP}^n$ and $\gamma, \Delta > 0$ such that

1) $\|\tilde{\Phi}(u, x, t)\|^2 \leq (1/\gamma^2)\|u\|^2_{R_1}$ for some $R_1 \in \mathcal{SP}^m$, for all t, x and u such that $\|u\| \leq \Delta$;

2) there exists $R_2 \in \mathcal{SP}^p$ such that

$$R_2 \geq C_1(u, t)C_1^T(u, t)$$

for all t and u such that $\|u\| \leq \Delta$;

3) the following Riccati equation is satisfied:

$$A^TP_{SF}^{(l)} + P_{SF}^{(l)}A + P_{SF}^{(l)} \cdot \left(\frac{B_1B_1^T}{\gamma^2} - B_2R_1^{-1}B_2^T\right)P_{SF}^{(l)} = -Q_{SF}^{(l)} \tag{46}$$

for all $l \in (0, 1]$;

4) $\lim_{l\to 0} Q_{SF}^{(l)} = 0$ and $\lim_{l\to 0} P_{SF}^{(l)} = 0$.

Assumptions (**H**1)–(**H**3) are exactly the same invoked in lemma 3.1 of [26]. The interest in the class of systems (45) relies on the possibility of taking into account *input saturations*.

We want to show that *under assumptions (H1)–(H3) semiglobal stabilization via linear measurement feedback together with bounded control can be achieved* (we remark that here "bounded control" is meant over some *compact set of the state space*, so that we do not use a truly nonlinear saturation function as in [28]). For, by (**H**2) there exist $Q_{OI}, P_{OI} \in \mathcal{SP}^n$ such that the following Riccati inequality is satisfied:

$$A^TP_{OI} + P_{OI}A + P_{OI}B_1B_1^TP_{OI} - C_2^TR_2^{-1}C_2 \leq -Q_{OI}. \tag{47}$$

Let $l^* \in (0, 1]$ be such that $P_m^{(l)} = \gamma^2P_{OI} - P_{SF}^{(l)} > 0$ for all $l \in (0, l^*]$. This choice is always feasible by (**H**3)-4).

Fix compact sets $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$ and let $c > 0$ be such that (see also [26])

$$c \geq \sup_{\substack{x\in\mathcal{S},\, \sigma\in\mathcal{W}\\ l\in(0,\, l^*]}} \left\{\|x\|^2_{P_{SF}^{(l)}} + \|x - \sigma\|^2_{P_m^{(l)}}\right\}.$$

Choose $a = 0$, $\eta(s) = s$ and $\delta(s) = 1$. Pick $l \in (0, l^*]$ such that

1) if $F^{(l)} = -R_1^{-1}B_2^TP_{SF}^{(l)}$, one has $\|F^{(l)}\sigma\| \leq \Delta$ for all σ such that

$$(x, \sigma) \in \Omega^e(c) = \left\{(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n\colon \|v_1\|^2_{P_{SF}^{(l)}} + \|v_1 - v_2\|^2_{P_m^{(l)}} \leq c\right\}.$$

2) $Q_m^{(l)} = \gamma^2Q_{OI} - Q_{SF}^{(l)} > 0$.

By applying *Theorem III.1*, one concludes that the region of attraction Ω^e contains at least $\mathcal{S} \times \mathcal{W}$. Moreover, since $a = 0$, $\eta(s) = s$ and $\delta(s) = 1$, the resulting controller is *linear* and, by choosing l^* correspondingly, it is *bounded over* Ω^e *by any a priori given positive number*.

D. Output Uncertainties

Let us consider (3) with $B_1 = 0$. Accounting for a term $C_1(u, x, t)\tilde{\Phi}(u, x, t)$ in the output allows us to consider important stabilization cases. As an example, *output saturations* can be modeled in the above form: if $y = \mathrm{sat}(C_2x)$, where sat $(s) = s\min\{1, (1/|s|)\}$, one can take $\tilde{\Phi}(u, x, t) = \mathrm{sat}(C_2x) - C_2x$ and $C_1(u, x, t) = 1$. Even more important nonlinear phenomena characterizing the output of a system can be modeled in the form above. For example, *backlash* (with slope equal to 1 and dead zone $[-1, 1]$) can be modeled with

$$\tilde{\Phi}(u, x, t) = \begin{cases} x_1, & \text{if } |x_1| \leq 1 \\ \operatorname{sgn}(x_1), & \text{else} \end{cases} \tag{48}$$

and $C_1(u, x, t) = 1$. We believe that the results below are a first step toward a characterization of output nonlinearities in terms of *robust control*, alternatively to the classical *describing function*. To state a general result, we assume the following.

(H4) (A, B_2, C_2) is in prime canonical form [22].

(H5) For any C^0 function $c^{(\cdot)}$: $(0, 1] \to \mathbb{R}^+$ such that $\lim_{l\to 0} c^{(l)} = \infty$ there exist C^0 functions $\gamma^{(\cdot)}$, $R_2^{(\cdot)}$: $(0, 1] \to \mathbb{R}^+$ such that

$$\|\tilde{\Phi}(u, x, t)\|^2 \leq \frac{\|C_2 x\|^2}{(\gamma^{(l)})^2} \tag{49}$$

$$\|C_1(u, x, t)C_1^T(u, x, t)\| \leq R_2^{(l)} \tag{50}$$

$$M \geq (\gamma^{(l)})^2 (R_2^{(l)})^{-1} - 1 > 0 \tag{51}$$

for some $M > 0$, for all $l \in (0, 1]$, t, u and x such that $\|C_2 x\|^2 \leq c^{(l)}$ and, in addition,

$$\lim_{l\to 0} \left(c^{(l)}\right)^{1/2} \left(\left(\gamma^{(l)}\right)^2 \left(R_2^{(l)}\right)^{-1} - 1\right)^{((n-1)/(2n-1))} = +\infty. \tag{52}$$

Condition (**H**4) assumes one of the simplest structures that can be thought of. More general structures can be considered, but this will be detailed elsewhere (for the case $B_1 \neq 0$, see [3]).

Condition (52) is a *growth* restriction on the product $(\gamma^{(l)})^2 (R_2^{(l)})^{-1}$ as a function of $c^{(l)}$, which, in turn, parameterizes the width of the compact set included in the region of attraction of the closed-loop system. In the case $y = \operatorname{sat}(C_2 x)$ one can take $\tilde{\Phi}(u, x, t) = \operatorname{sat}(C_2 x) - C_2 x$, $C_1(u, x, t) = 1$, $R_2^{(l)} = 1$, and $\gamma^{(l)} = (\sqrt{c^{(l)}}/(\sqrt{c^{(l)}} - 1))$, with $\lim_{l\to 0} \gamma^{(l)} = 1$, so that $\|\tilde{\Phi}(u, x, t)\|^2 \leq (\|C_2 x\|^2/(\gamma^{(l)})^2)$ for all $x \in \mathbb{R}^n$ such that $\|C_2 x\|^2 \leq c^{(l)}$. Moreover, $(\gamma^{(l)})^2 (R_2^{(l)})^{-1} - 1 = ((2\sqrt{c^{(l)}} - 1)/(\sqrt{c^{(l)}} - 1)^2)$ so that (52) is satisfied without loss of generality (w.l.o.g). We can assume that $\sqrt{c^{(l)}} > 1/2$ for all $l \in (0, 1]$). On the other hand, in the case of backlash, (52) is satisfied with $M > 0$, $\sqrt{M+1} \geq \gamma^{(l)} > 1$ and $R_2^{(l)} = 1$, as long as $|C_2 x| \geq 1$.

We want to show that, if $B_1 = 0$ and under (**H**4) and (**H**5), as a consequence of *Theorem III.1* (3) is *semiglobally stabilizable via measurement feedback*. For simplicity, we will assume $m = p = 1$ and by (**H**4) we have

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From (**H**5) it follows that, whatever the C^0 function $R_1^{(\cdot)}$: $(0, 1] \to \mathbb{R}^+$ is, one has by (50)

$$\|\tilde{\Phi}(u, x, t)\|^2 \leq \frac{\|C_2 x\|^2 + \|u\|^2_{R_1^{(l)}}}{(\gamma^{(l)})^2} \tag{53}$$

for all $t, l \in (0, 1]$, u, and x such that $\|C_2 x\|^2 \leq c^{(l)}$. Let $R_1^{(\cdot)}$ be such that $\lim_{l\to 0} R_1^{(l)} = 0$. We claim that there exist $l^* \in (0, 1]$ and C^0 functions $P_{SF}^{(\cdot)}$, $Q_{SF}^{(\cdot)}$, $P_m^{(\cdot)}$, $Q_m^{(\cdot)}$: $(0, 1] \to \mathcal{SP}^n$ such that

$$\begin{aligned} &A^T P_{SF}^{(l)} + P_{SF}^{(l)} A - P_{SF}^{(l)} B_2 (R_1^{(l)})^{-1} \\ &\quad \cdot B_2^T P_{SF}^{(l)} + C_2^T C_2 = -Q_{SF}^{(l)} \end{aligned} \tag{54}$$

$$\begin{aligned} &A^T P_m^{(l)} + P_m^{(l)} A + P_{SF}^{(l)} B_2 (R_1^{(l)})^{-1} \\ &\quad \cdot B_2^T P_{SF}^{(l)} - (\gamma^{(l)})^2 R_2^{-1} C_2^T C_2 \leq -Q_m^{(l)} \end{aligned} \tag{55}$$

for all $l \in (0, l^*]$. Indeed, define $h_1 = (1/4n)$ and $h_2 = (1/2n)$. Let $\overline{P}_{SF,0}$ be the unique (stabilizing) positive definite symmetric solution of

$$\overline{P}_{SF,0} A + A^T \overline{P}_{SF,0} - \overline{P}_{SF,0} B_2 B_2^T \overline{P}_{SF,0} + C_2^T C_2 = 0. \tag{56}$$

For each fixed C^0 (monotonically decreasing) function $\overline{Q}_{SF}^{(\cdot)}$: $(0, 1] \to \mathcal{SP}^n$ such that $\lim_{l\to 0} \overline{Q}_{SF}^{(l)} = 0$ pick C^0 function $\overline{P}_{SF}^{(\cdot)}$: $(0, 1] \to \mathcal{SP}^n$ such that

$$\overline{P}_{SF}^{(l)} A + A^T \overline{P}_{SF}^{(l)} - \overline{P}_{SF}^{(l)} B_2 B_2^T \overline{P}_{SF}^{(l)} + C_2^T C_2 = -\overline{Q}_{SF}^{(l)} \tag{57}$$

for all $l \in (0, 1]$ and

$$\lim_{l\to 0} \overline{P}_{SF}^{(l)} = \overline{P}_{SF,0}. \tag{58}$$

(see [29]). Define

$$P_{SF}^{(l)} = Z^{(l)} \overline{P}_{SF}^{(l)} Z^{(l)} \tag{59}$$

$$Q_{SF}^{(l)} = Z^{(l)} \overline{Q}_{SF}^{(l)} Z^{(l)} (R_1^{(l)})^{-h_2} \tag{60}$$

where $Z^{(l)} = (R_1^{(l)})^{h_1} \operatorname{diag}\{1, (R_1^{(l)})^{h_2}, \ldots, (R_1^{(l)})^{h_2(n-1)}\}$. By construction, $P_{SF}^{(l)}$ and $Q_{SF}^{(l)}$ solve (54).

Let $c^{(l)} = c(R_1^{(l)})^{-2h_1}$, for some $c > 0$ (independent of l but dependent on the compact set $\mathcal{S} \times \mathcal{W}$ to be included in the region of attraction) to be specified later. Define $k^{(l)} = ((\gamma^{(l)})^2 (R_2^{(l)})^{-1} - 1)^{(1/(2n-1))}$ and pick

$$P_0^{(l)} = W^{(l)} \overline{P}_0 W^{(l)} \tag{61}$$

where $W^{(l)} = \operatorname{diag}\{(k^{(l)})^{n-1}, \ldots, 1\}$, $\overline{P}_0$ is any positive definite symmetric solution of

$$\overline{P}_0 A + A^T \overline{P}_0 - C_2^T C_2 = -\overline{Q}_0$$

for some $\overline{Q}_0 \in \mathcal{SP}^n$. Using (51), (52), and (58) and the fact that $2h_1 = h_2$, pick $\overline{Q}_{SF}^{(l)}$ such that for some $l^* \in (0, 1]$

$$Q_{SF}^{(l)} \leq k^{(l)} \frac{W^{(l)} \overline{Q}_0 W^{(l)}}{2} \tag{62}$$

$$P_{SF}^{(l)} < P_0^{(l)} \tag{63}$$

for all $l \in (0, l^*]$. Define

$$P_m^{(l)} = P_0^{(l)} - P_{SF}^{(l)} \tag{64}$$

$$Q_m^{(l)} = k^{(l)} \frac{W^{(l)} \overline{Q}_0 W^{(l)}}{2}. \tag{65}$$

Finally, fix compact sets $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$ and pick $c > 0$ such that

$$c \geq \sup_{\substack{x\in\mathcal{S},\, \sigma\in\mathcal{W}\\ l\in(0,\,1]}} \left\{ \|x\|^2_{P^{(l)}_{SF}} + \|x-\sigma\|^2_{P^{(l)}_m} \right\} \tag{66}$$

and $\|C_2 x\|^2 \leq c^{(l)}$ for all $l \in (0,\, l^*]$ whenever $\|x\|^2_{P^{(l)}_{SF}} \leq c$ (note that $c < \infty$)). By construction, $P^{(l)}_m$ and $Q^{(l)}_m$ solve (55).

Finally, pick $\eta(s) = s$, $\delta(s) = 1$ and $a = 0$. By *Remark III.4* and (54) and (55), it follows that, if $B_1 = 0$ and under (**H**4) and (**H**5), (3) is semiglobally stabilizable via measurement feedback.

IV. Regional Output Regulation

When achieving regional output *regulation*, two additional difficulties should be faced with respect to regional *stabilization*: 1) the control u should inject a compensation term $\Gamma(w)$ in (9) since the equilibrium point $(x,\, \sigma) = (0,\, 0)$ of (9) is moved by the external disturbances w into $(\Pi(w),\, \Sigma(w))$ and 2) the problem of estimating $\Gamma(w)$. Once these additional problems are correctly addressed in the framework of our main result, the output regulation problem of (9) with some region of attraction can be seen exactly as a problem of rendering Ω^e-ULAS a suitable system, derived from (9). For these reasons, the result of this section is not "different" from the previous ones but arises naturally in the unifying context of the main theorem of our paper and, as such, all the remarks of the previous and next sections can be repeated here.

As to point 1), one simply requires that $(x,\, \sigma,\, u) = (\Pi(w),\, \Sigma(w),\, \Gamma(w))$ is for (9) a set of equilibria, on which regulation is achieved (i.e., $r \equiv 0$). Point 2) can be addressed as follows. The term $\Gamma(w)$ is a "feedforward" term, which moves the equilibrium point $(x,\, \sigma,\, u) = (0,\, 0,\, 0)$ into $(\Pi(w),\, \Sigma(w),\, \Gamma(w))$. If the measure of w is available, this term is exactly known and can be readily included in the controller. However, since in a realistic setting w is not available for feedback, we should try to estimate either w or, directly, $\Gamma(w)$. Bearing in mind our remarks in Section I and the discussion of Section III-B, if a dynamical model of $\Gamma(w)$ and its derivatives is available (except for the case of tracking), then we can try to obtain an estimate of $\Gamma(w)$. A sufficient condition for this to be true is the existence of some $q > 0$ and some *known* locally Lipschitz C^0 function ψ: $\mathbb{R}^q \to \mathbb{R}$ such that

$$L^q_{s(w)}\Gamma(w) = \psi\left(\Gamma(w),\, \Gamma^{(1)}(w),\, \ldots,\, \Gamma^{(q-1)}(w)\right) \tag{67}$$

(see [10]). From a general point of view, this corresponds to the possibility of *immersing* the system

$$\begin{aligned} \dot{w} &= s(w) \\ u &= \Gamma(w) \end{aligned} \tag{68}$$

with output u, into the system

$$\begin{aligned} \dot{\xi} &= J\xi + N\psi(\xi) \\ v &= \xi_1 \end{aligned} \tag{69}$$

with output v, where $\xi = \mathrm{col}(\xi_1,\, \ldots,\, \xi_q)$, $\xi_j = L^{j-1}_s \Gamma(w)$, and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0\cdots & 0 \\ 0 & 0 & 1 & 0\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\ldots & \vdots \\ 0 & 0 & 0 & 0\cdots & 1 \\ 0 & 0 & 0 & 0\cdots & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{70}$$

More generally, we say that (68) can be *immersed* into

$$\begin{aligned} \dot{\xi} &= h(\xi) \\ v &= \beta(\xi) \end{aligned} \tag{71}$$

with $\xi \in \mathbb{R}^q$ and output v, if there exists a locally Lipschitz function τ: $\mathbb{R}^s \to \mathbb{R}^q$ such that

$$\begin{aligned} L_{s(w)}\tau(w) &= h(\tau(w)) \\ \Gamma(w) &= \beta(\tau(w)) \end{aligned} \tag{72}$$

for all $w \in \mathbb{R}^s$ (see [10]). It is clear that, if (67) holds for all $w \in \mathbb{R}^s$, (68) can be immersed into (71) with $\tau(w) = \mathrm{col}(\Gamma(w),\, \ldots,\, \Gamma^{(q-1)}(w))$, $\beta(\xi) = \xi_1$ and $h(\xi) = \mathrm{col}(\xi_2,\, \ldots,\, \xi_q,\, \psi(\xi))$, where $\xi = \mathrm{col}(\xi_1,\, \ldots,\, \xi_q)$. If the functions $L^j_s\Gamma(w)$, $j \geq 1$, are *polynomials with degree not greater than a fixed number* q, the system in which (68) can be immersed is *linear* and *observable* [10].

The above remarks motivate the following result, which is only one of the possible versions one can think of and put in the more general framework of our main theorem (see also *Remark III.4*) the results of [11]. The proof is only sketched, since once the model (69) is included in (9) and the equilibrium point $(x,\, \sigma,\, u) = (0,\, 0,\, 0)$ is shifted through the feedforward term $\Gamma(w)$ into $(\Pi(w),\, \Sigma(w),\, \Gamma(w))$, one can proceed exactly as in the case of *Theorem III.1*.

Theorem IV.1: Assume $B_1 C_1^T = 0$, $B_1^T D_1 = 0$ and let $\mathcal{B} \subset \mathbb{R}^s$ be a given compact set. Moreover, assume that for each $c > 0$

- *(state feedback)* there exist $P_{SF}(c),\, Q_{SF}(c) \in \mathcal{SP}^n$ and (at least) C^1 mappings Π: $\mathbb{R}^s \to \mathbb{R}^n$ and Γ: $\mathbb{R}^s \to \mathbb{R}^m$ such that
  - — for some $\gamma(c) > 0$ and $\Delta(c) \in (0,\, \infty]$ and for some known matrices $E(c) \in \mathcal{SSP}^n$ and $R_1(c) \in \mathcal{SP}^m$ one has

$$\begin{aligned} &\|\tilde{\Phi}(u,\, w,\, x) - \tilde{\Phi}(\Gamma(w),\, w,\, \Pi(w))\|^2 \\ &\quad \leq \frac{1}{\gamma^2(c)} \left[\|x - \Pi(w)\|^2_{E(c)} + \|u - \Gamma(w)\|^2_{R_1(c)} \right] \end{aligned} \tag{73}$$

  for all t, $w \in \mathcal{B}$, $u \in \mathcal{U}(c) = \{v \in \mathbb{R}^m\colon \|v - \Gamma(w)\| \leq \Delta(c);\ w \in \mathcal{B}\}$, $x \in \Omega(c) = \{v \in \mathbb{R}^n\colon \|v - \Pi(w)\|^2_{P_{SF}(c)} \leq c;\ w \in \mathcal{B}\}$;
  - — the following Riccati equation is satisfied:

$$\begin{aligned} AP_{SF}(c) + P_{SF}(c)A^T + P_{SF}(c)\left(\frac{B_1 B_1^T}{\gamma^2(c)} - B_2 R_1^{-1}(c) B_2^T\right) \\ \cdot P_{SF}(c) + E(c) = -Q_{SF}(c) \end{aligned} \tag{74}$$

- *(output regulation)* for all $w \in \mathcal{B}$

$$\frac{\partial \Pi}{\partial w} s(w) = A\Pi(w) + B_2\Gamma(w) + B_1\tilde{\Phi}(\Gamma(w),\, w,\, \Pi(w)) \tag{75}$$

$$0 = C_2\Pi(w) + C_1\tilde{\Phi}(\Gamma(w),\, w,\, \Pi(w)) \tag{76}$$

$$0 = D_2\Pi(w) + D_1\tilde{\Phi}(\Gamma(w),\, w,\, \Pi(w)) \tag{77}$$

and (67) holds for some $q>0$ and for some known locally Lipschitz C^0 function $\psi: \mathbb{R}^q \to \mathbb{R}^m$;

- *(output injection)* there exist $R_2 \in \mathcal{SP}^p$ and $P_m(c), Q_m(c) \in \mathcal{SP}^{n+q}$, such that $R_2 \geq C_1 C_1^T$ and

— the following Riccati inequality is satisfied:

$$\begin{aligned}&\begin{pmatrix} A+\frac{1}{\gamma^2(c)}B_1B_1^TP_{SF}(c) & -B_2M \\ 0 & J\end{pmatrix}^T P_m(c)\\ &+P_m(c)\begin{pmatrix}\left(A+\frac{1}{\gamma^2(c)}B_1B_1^TP_{SF}(c)\right) & -B_2M\\ 0 & J\end{pmatrix}\\ &+\frac{1}{\gamma^2(c)}P_m(c)\begin{pmatrix}B_1B_1^T & 0\\ 0 & NN^T\end{pmatrix}P_m(c)\\ &+\begin{pmatrix}F^T(c)\\ M^T\end{pmatrix}R_1(c)\,(\,F(c) \quad M\,)\\ &-\gamma^2(c)\begin{pmatrix}C_2^T\\ 0\end{pmatrix}R_2^{-1}(c)\,(\,C_2 \quad 0\,)\leq -Q_m(c) \qquad (78)\end{aligned}$$

with $F(c) = -R_1^{-1}(c)B_2^TP_{SF}(c)$ and $M = \mathrm{row}(1, 0, \ldots, 0)$;

- *(nonlinear coupling)* there exist $a \in (0, 1)$ and C^0 functions $\delta: \mathbb{R} \to (0, 1-a]$, $\eta_1: \mathbb{R}^m \to \mathbb{R}^m$, $\eta_2: \mathbb{R} \to \mathbb{R}$ such that, if $\varphi(s) = as + \int_0^s \delta(\vartheta)\,d\vartheta$,

$$\begin{aligned}&\|a_1(x, w, e)\|^2_{R_1(c)} + \gamma^2(c)\|a_2(w, e_2)\|^2 - \|x(t)\|^2_{Q_{SF}(c)}\\ &\quad-(a+\delta(\|e(t)\|^2_{P_m}))(\|K(e_1, e_{21})\|^2_{R_1(c)} + \|e\|^2_{Q_m(c)}) \qquad (79)\end{aligned}$$

is negative definite for all $x \in \Omega(c)$, $w \in \mathcal{B}$ and e such that $0 \leq \varphi(\|e\|_{P_m(c)}) \leq c$, with $e = \mathrm{col}(e_1, e_2)$, $e_2 = \mathrm{col}(e_{21}, \ldots, e_{2q})$ and

$$\begin{aligned}a_1(x, w, e) &= \eta_1(K(x-\Pi(w)-e_1, \Gamma(w)-e_{21}))\\ &\quad - K(x-\Pi(w), \Gamma(w))\\ a_2(w, e_2) &= \eta_2(\psi(\Sigma(w)-e_2)) - \psi(\Sigma(w))\\ \Sigma(w) &= \mathrm{col}(\Gamma(w), L_s\Gamma(w), \ldots, L_s^{q-1}\Gamma(w))\\ K(s, r) &= r + F(c)s. \qquad (80)\end{aligned}$$

Under the above assumptions, (9) is $\Omega^e(c)$-ULORM with

$$\Omega^e(c) = \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^n \times \mathcal{B} \times \mathbb{R}^n \times \mathbb{R}^s: \cdot \|v_1 - \Pi(v_2)\|^2_{P_{SF}(c)} + \varphi\left(\left\|\begin{pmatrix} v_1 - \Pi(v_2) - v_3 \\ \Sigma(v_2) - v_4\end{pmatrix}\right\|^2_{P_m(c)}\right) \leq c\right\} \qquad (81)$$

whenever $\|\eta_1(K(\sigma_1(t), \sigma_{21}(t))) - \Gamma(w(t))\| \leq \Delta(c)$ and $w(t) \in \mathcal{B}$ for all $t \geq 0$, with $\sigma = \mathrm{col}(\sigma_1, \sigma_2)$ and $\sigma_2 = \mathrm{col}(\sigma_{21}, \ldots, \sigma_{2q})$. An admissible stabilizing controller is given by

$$\begin{aligned}u &= \eta_1(K(\sigma_1, \sigma_{21}))\\ \dot\sigma &= H(c)\sigma + \begin{pmatrix}0\\ N\end{pmatrix}\eta_2(\psi(\sigma_2))\\ &\quad + \begin{pmatrix}B_2\\ 0\end{pmatrix}(u-\sigma_{21}) + G(c)y \qquad (82)\end{aligned}$$

with

$$\begin{aligned}H(c) &= \begin{pmatrix} A + \frac{1}{\gamma^2(c)}B_1B_1^TP_{SF}(c) - G_1(c)C_2 & 0\\ -G_2(c)C_2 & J\end{pmatrix}\\ G(c) &= \begin{pmatrix}G_1(c)\\ G_2(c)\end{pmatrix} = \gamma^2(c)P_m^{-1}(c)\begin{pmatrix}C_2^T\\ 0\end{pmatrix}R_2^{-1}(c). \qquad (83)\end{aligned}$$

Proof: Throughout the proof, we omit the argument c. Since $C_1B_1^T = 0$ and by (75) and (76), (9)–(82) can be recast as follows:

$$\begin{aligned}\begin{pmatrix}\dot z\\ \dot\xi\end{pmatrix} &= \begin{pmatrix}A+\frac{1}{\gamma^2}B_1B_1^TP_{SF} & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}z\\ \xi\end{pmatrix} + \begin{pmatrix}B_2\\ 0\end{pmatrix}(u-\xi_1)\\ &\quad+\begin{pmatrix}0\\ N\end{pmatrix}\psi(\xi) + \begin{pmatrix}B_1 & 0\\ 0 & 0\end{pmatrix}\Sigma(u, w, z, \xi, \sigma_2)\\ \dot\sigma &= H\sigma + \begin{pmatrix}0\\ N\end{pmatrix}\psi(\xi) + \begin{pmatrix}B_2\\ 0\end{pmatrix}(u-\sigma_{21}) + GC_2z\\ &\quad+\begin{pmatrix}G_1C_1 & 0\\ G_2C_1 & N\end{pmatrix}\Sigma(u, w, z, \xi, \sigma_2)\\ y &= C_2z + (\,C_1 \quad 0\,)\,\Sigma(u, w, z, \xi, \sigma_2) \qquad (84)\end{aligned}$$

with $z = x - \Pi(w)$, $u = \eta_1(K(\sigma_1, \sigma_{21}))$ and

$$\begin{aligned}&\Sigma(u, w, z, \xi, \sigma_2)\\ &\quad= \Xi(u, w, z, \xi, \sigma_2) - \begin{pmatrix}\frac{1}{\gamma^2}B_1^TP_{SF}z\\ 0\end{pmatrix}\\ &\Xi(u, w, z, \xi, \sigma_2)\\ &\quad= \begin{pmatrix}\tilde\Phi(u, w, z+\Pi(w)) - \tilde\Phi(\xi_1, w, \Pi(w))\\ \eta_2(\psi(\sigma_2)) - \psi(\xi)\end{pmatrix}. \qquad (85)\end{aligned}$$

Let $e_1 = z - \sigma_1$, $e_2 = \xi - \sigma_2$, $e = \mathrm{col}(e_1, e_2)$, and $V_m(e) = \varphi(\|e\|^2_{P_m})$. Along the trajectories of (84), one has

$$\begin{aligned}&\dot V_m + (a+\delta(\|e\|^2_{P_m}))\|K(e_1, e_{21})\|^2_{R_1}\\ &\quad-\gamma^2\|\Sigma(u, w, z\xi, \sigma_2)\|^2\\ &\quad\leq -(a+\delta(\|e\|^2_{P_m(c)}))\|e\|^2_{Q_m}. \qquad (86)\end{aligned}$$

Let $V_{SF}(z) = \|z\|^2_{P_{SF}}$. Along the trajectories of (84), one has for all u

$$\begin{aligned}&\dot V_{SF} + \|z\|^2_E + \|u-\xi_1\|^2_{R_1} + \gamma^2\|a_2(w, e_2)\|^2\\ &\quad-\gamma^2\|\Xi(u, w, z, \xi, \sigma_2)\|^2\\ &\quad= \|u-K(z, \xi_1)\|^2_{R_1} - \|z\|^2_{Q_{SF}} + \gamma^2\|a_2(w, e_2)\|^2\\ &\quad-\gamma^2\|\Sigma(u, w, z, \xi, \sigma_2)\|^2. \qquad (87)\end{aligned}$$

From (73), (86), and (87), we conclude that, whenever $\|\eta_1(K(\sigma_1(t), \sigma_{21}(t))) - \Gamma(w(t))\| \leq \Delta$, and $w(t) \in \mathcal{B}$ for all $t \geq 0$,

$$\begin{aligned}\dot V_{SF} + \dot V_m &\leq \|\eta_1(K(\sigma_1, \sigma_{21})) - K(z, \xi_1)\|^2_{R_1} - \|x\|^2_{Q_{SF}}\\ &\quad-\left[\|K(e_1, e_{21})\|^2_{R_1} + \|e\|^2_{Q_m}\right](a+\delta(\|e\|^2_{P_m}))\\ &\quad+\gamma^2(c)\|a_2(w, e_2)\|^2 \qquad (88)\end{aligned}$$

is negative definite, which together (73), (77), and (79) prove our result. □

Remark IV.1: (Definition of $\mathcal{B}$): The requirement $w(t) \in \mathcal{B}$ for all $t \geq 0$ can be met as long as $\dot w = s(w)$ is Lyapunov stable by choosing the initial conditions w_0 in the set $\mathcal{B}_d$ defined as follows. Let $V(t, w)$ be a C^∞ function such that $V_1(w) \leq V(t, w) \leq V_2(w)$ for all w and t and for some C^∞ positive definite and proper functions $V_1(\cdot)$ and $V_2(\cdot)$ and, in addition, $\dot{V} \leq 0$ along the trajectories of $\dot{w} = s(w)$. Finally, pick $d > 0$ and define $\mathcal{B}_d = \{w_0 \in \mathbb{R}^s\colon V_2(w_0) \leq d\}$ and $\mathcal{B} = \{w_0 \in \mathbb{R}^s\colon V_1(w_0) \leq d\}$. The definition of $\mathcal{B}$ is also crucial when saturating the input u, in the sense that restricting the initial conditions $w_0 \in \mathcal{B}_d$ might be necessary in this case to limit the excursion of the feedforward term $\Gamma(w(t))$.

Remark IV.2: Among the other things, *Theorem IV.1* differs from *Theorem III.1* in the output regulation constraint which one hand guarantees that $x(t) = \Pi(w(t))$ for all $t \geq T$ whenever $x(T) = \Pi(w(T))$ for some T and, on the other, ensures the possibility of immersing the exosystem into (69). Note that (78) has dimension $n + q$, since in (84) both x and $\Gamma(w)$ (through the dynamical system in which the exosystem is immersed) are estimated by an observer. Moreover, the expression of (82) differs from (31) in the *nonlinear* term $N\eta_2(\psi(\sigma_2))$. Note also that η_1 and η_2 have the same role as η in *Theorem III.1*: in the case of the systems considered in the previous sections, η_1 and η_2 should be chosen as *saturations* (uncorrupted outputs) or *linear* functions (input uncertainties).

Consider as an example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{w} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} w \\ y &= \mathrm{sat}(x_1 + w_1) \end{aligned} \tag{89}$$

with $\omega \in \mathbb{R}$, $x = \mathrm{col}(x_1, x_2)$ and $w = \mathrm{col}(w_1, w_2)$. Let $V(v) = \|v\|^2$ and define $\mathcal{B}_d = \mathcal{B} = \{v \in \mathbb{R}^2\colon \|v\|^2 \leq d\}$ for some given $d > 0$. It can be easily seen that all the assumptions of *Theorem IV.1* are satisfied with $\eta_1(s) = s$, $\eta_2(s) = s$, $\delta(s) = 1$ and $a = 0$ as long as there exist $P_m, Q_m \in \mathcal{SP}^4$ such that

$$\begin{aligned} &P_m \begin{pmatrix} A & -B_2M \\ 0 & J \end{pmatrix} + \begin{pmatrix} A & -B_2M \\ 0 & J \end{pmatrix}^T P_m \\ &+ P_m \begin{pmatrix} 0 & 0 \\ 0 & NN^T \end{pmatrix} P_m + \omega^4 \begin{pmatrix} 0 & 0 \\ 0 & M^TM \end{pmatrix} \leq -Q_m \end{aligned} \tag{90}$$

where $M = \mathrm{row}(1, 0)$ and $N = \mathrm{col}(0, 1)$. Since $(\left(\begin{smallmatrix} A & -B_2M \\ 0 & J \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & NN^T \end{smallmatrix}\right))$ is stabilizable and $\left(\begin{smallmatrix} A & -B_2M \\ 0 & J \end{smallmatrix}\right)$ has no eigenvalues on the right-half complex plane, (90) can be met for sufficiently small ω.

V. Semiglobal Stabilization of Uncertain Nonlinear Systems

Let us consider the interconnected system

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_{12}x_2 + B_{11}\Psi_1(u, x, t) \\ \tilde{y}_1 &= C_{12}x_1 + C_{11}\Psi_1(u, x, t) \\ \vdots &= \vdots \\ \dot{x}_s &= A_sx_s + B_{s2}u + B_{s1}\Psi_s(u, x, t) \\ \tilde{y}_s &= C_{s2}x_s + C_{s1}\Psi_s(u, x, t) \end{aligned} \tag{91}$$

with $x_j \in \mathbb{R}^{n_j}$, $x = \mathrm{col}(x_1, \ldots, x_s)$, $u \in \mathbb{R}^m$, $\tilde{y}_j \in \mathbb{R}^{p_j}$, A_j, B_{ji}, C_{ji} matrices of suitable dimensions such that $B_{j1}C_{j1}^T = 0$ and rank $(B_{j1}) = n_j$ for all j, $\Psi_j(x, u, t) \in \mathbb{R}^{k_j}$ are admissible uncertainties and $\sum_{j=1}^s n_j = n$. In what follows, for later convenience we will denote u alternatively by either $\tilde{x}_{s+1}$ or x_{s+1} and m by n_{s+1}.

Let us consider each system

$$\begin{aligned} \dot{x}_i &= A_ix_i + B_{i2}x_{i+1} + B_{i1}\Psi_i(u, x, t) \\ \tilde{y}_i &= C_{i2}x_1 + C_{i1}\Psi_i(u, x, t) \end{aligned} \tag{92}$$

with output $\tilde{y}_i$ and input x_{i+1} and assume that each one of these systems satisfies the assumptions of *Theorem III.1* with $a = 0$, $\delta(s) = 1$ and $\eta(s) = s$. Correspondingly, let

$$\begin{aligned} x_{i+1} &= F_i\sigma_i \\ \dot{\sigma}_i &= H_i\sigma_i + B_{i2}F_i\sigma_i + G_i\tilde{y}_i, \qquad \sigma_i \in \mathbb{R}^{n_i} \end{aligned} \tag{93}$$

be a measurement feedback controller as in (31) for (92). Pick as new coordinates for (91)

$$\begin{aligned} \zeta_1 &= x_1 \\ \zeta_2 &= x_2 - \tilde{x}_2 \\ \vdots &= \vdots \\ \zeta_s &= x_s - \tilde{x}_s \end{aligned} \tag{94}$$

with $\tilde{x}_j = F_{j-1}\sigma_{j-1}$, and slightly modify the controllers (93) in such a way to define the following candidate measurement feedback controller for the overall system (91)

$$\begin{aligned} \tilde{x}_{i+1} &= F_i\sigma_i \\ \dot{\sigma}_i &= H_i\sigma_i + B_{i2}\tilde{x}_{i+1} + G_iy_i, \qquad i = 1, \ldots, s \end{aligned} \tag{95}$$

where $y_1, \ldots, y_s$ are defined as follows:

$$\begin{aligned} y_1 &= \tilde{y}_1 \\ y_j &= \tilde{y}_j - C_{j2}\tilde{x}_j, \qquad 2 \leq j \leq s. \end{aligned} \tag{96}$$

Note that, contrary to (93), *the jth controller (95) is coupled with the $(j-1)$th one through the injection term $C_{j2}\tilde{x}_j$.*

In the new coordinates, (91)–(95) can be rewritten as follows:

$$\begin{aligned} \dot{\zeta}_1 &= A_1\zeta_1 + B_{12}\tilde{x}_2 + B_{11}\tilde{\Phi}_1(\zeta, \sigma, \tilde{x}, t) \\ \dot{\sigma}_1 &= H_1\sigma_1 + B_{12}\tilde{x}_2 + G_1y_1 \\ y_1 &= C_{12}\zeta_1 + C_{11}\tilde{\Phi}_1(\zeta, \sigma, \tilde{x}, t) \\ \vdots &= \vdots \\ \dot{\zeta}_n &= A_s\zeta_s + B_{s2}\tilde{x}_{s+1} + B_{s1}\tilde{\Phi}_s(\zeta, \sigma, \tilde{x}, t) \\ \dot{\sigma}_s &= H_s\sigma_s + B_{s2}\tilde{x}_{s+1} + G_sy_s \\ y_s &= C_{s2}\zeta_s + \tilde{C}_{s1}\tilde{\Phi}_s(\zeta, \sigma, \tilde{x}, t) \end{aligned} \tag{97}$$

with $\zeta = \mathrm{col}(\zeta_1, \ldots \zeta_s)$, $\sigma = \mathrm{col}(\sigma_1, \ldots, \sigma_s)$, $\tilde{x} = \mathrm{col}(\tilde{x}_2, \ldots, \tilde{x}_{s+1})$, and $\tilde{\Phi}_j(\zeta, \sigma, \tilde{x}, t) \in \mathbb{R}^{k_j}$.

We want to give conditions under which the controller (95) semiglobally stabilizes (97) [and, thus, (91)] and we will do this for a wide class of interconnected system [16], [17], [19], [27], [28]. We want to remark that if $\tilde{y}_1, \ldots, \tilde{y}_n$ were not the *true outputs* of (91) (i.e., not available from sensors), we could estimate $\tilde{y}_1, \ldots, \tilde{y}_n$ and put in (95) these estimates by assuming the following "complete uniform observability" property: for each $i = 1, \ldots, s$ and for some $h, k > 0$

$$\begin{aligned} &C_{i2}x_i + C_{i1}\Psi_i(u, x, t) \\ &\quad = M_i(y, \dot{y}, \ldots, y^{(h)}, u, \dot{u}, \ldots, u^{(k)}, t) \end{aligned} \tag{98}$$

$y^{(h+1)}$

$$= N(C_{12}x_1 + C_{11}\Psi_1(u, x, t), \ldots, C_{s2}x_s + C_{s1}\Psi_s \cdot (u, x, t), u, \dot{u}, \ldots, u^{(k)}, t) \tag{99}$$

where M_i and N are locally Lipschitz functions (uniformly with respect to t) and y is the *true* output of (91). If this is the case, one can apply the arguments of [24] to prove that (91) can be semiglobally stabilized through the output y.

We illustrate the main ideas of this section through an example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 u^2 \\ \tilde{y}_1 &= x_1 + \psi(x_2) \\ \dot{x}_2 &= u \\ \tilde{y}_2 &= x_2 \end{aligned} \tag{100}$$

where ψ is an *unknown* locally Lipschitz function (or bounded by a *known* locally Lipschitz function), vanishing at the origin. Rewrite (100) as follows:

$$\begin{aligned} \dot{x}_1 &= \tilde{x}_2 + B_{11}\tilde{\Psi}_1(u, x, t) \\ \tilde{y}_1 &= x_1 + C_{11}\tilde{\Psi}_1(u, x, t) \\ \dot{x}_2 &= u + B_{21}\tilde{\Psi}_2(u, x, t) \\ \tilde{y}_2 &= x_2 + C_{21}\tilde{\Psi}_2(u, x, t) \end{aligned} \tag{101}$$

where

$$\tilde{\Psi}_1(u, x, t) = \begin{pmatrix} x_2 - \tilde{x}_2 + x_1 u^2 \\ \psi(x_2) \end{pmatrix}$$

$\tilde{\Psi}_2(u, x, t) = 0$, and $B_{j1}C_{j1}^T = 0$, $j = 1, 2$. Note that

$$\begin{aligned} \tilde{\Psi}_1^2(u, x, t) &\le R_{11}\tilde{x}_2^2 + \lambda_{12}(|x_2 - \tilde{x}_2|) + 2(\Delta_2^{(l)})^4 x_1^2 \\ \tilde{\Psi}_2^2(u, x, t) &\le R_{21}u^2 \end{aligned} \tag{102}$$

for some $R_{j1} > 0$, $j = 1, 2$, such that $B_{j1}B_{j1}^T < R_{j1}^{-1}$ and for some $\lambda_{12} \in \mathcal{K}$, whenever $|\tilde{x}_{j+1}| \le \Delta_j^{(l)}$, $j = 1, 2$ (remember that $\tilde{x}_3 = u$), and for some C^0 functions $\Delta_j^{(\cdot)}\colon (0, 1] \to \mathbb{R}^+$ such that $\lim_{l\to 0} \Delta_2^{(l)} = 0$. It is easy to see that, as long as $|\tilde{x}_{j+1}| \le \Delta_j^{(l)}$ for $j = 1, 2$,

$$\begin{aligned} \dot{x}_1 &= \tilde{x}_2 + B_{11}\tilde{\Psi}_1(u, x, t) \\ \tilde{y}_1 &= x_1 + C_{11}\tilde{\Psi}_1(u, x, t) \end{aligned} \tag{103}$$

and, respectively,

$$\begin{aligned} \dot{x}_2 &= u + B_{21}\tilde{\Psi}_2(u, x, t) \\ \tilde{y}_2 &= x_2 + C_{21}\tilde{\Psi}_1(u, x, t) \end{aligned} \tag{104}$$

satisfy the assumptions of *Theorem III.1* with $\eta(s) = s$, $\delta(s) = 1$, $a = 0$ and large regions of attraction (see Section III-C as a reference). In particular, for each pair of C^0 functions $P_{SF,j}^{(\cdot)}\colon (0, 1] \to \mathbb{R}^+$, $j = 1, 2$, satisfying $\lim_{l\to 0} P_{SF,j}^{(l)} = 0$ and such that $(P_{SF,1}^{(l)})^2(R_{11}^{-1} - B_{11}B_{11}^T) > 2(\Delta_2^{(l)})^4$ and $(R_{j1}/4(P_{SF,j}^{(l)})^2) \ge C_{j1}C_{j1}^T$ for all $l \in (0, 1]$, there exist C^0 functions $Q_{SF,j}^{(\cdot)}, Q_{OI,j}^{(\cdot)}, R_{j2}^{(\cdot)}\colon (0, 1] \to \mathbb{R}^+$, $j = 1, 2$, such that $\lim_{l\to 0} Q_{SF,j}^{(l)} = 0$, $(R_{j1}/4(P_{SF,j}^{(l)})^2) \ge R_{j2}^{(l)} \ge C_{j1}C_{j1}^T$, $Q_{OI,j}^{(l)} > Q_{SF,j}^{(l)}$, and

$$\begin{aligned} (P_{SF,j}^{(l)})^2(B_{j1}B_{j1}^T - R_{j1}^{-1}) + E_j^{(l)} &= -Q_{SF,j}^{(l)} \\ (P_{OI,j}^{(l)})^2 B_{j1}B_{j1}^T - (R_{j2}^{(l)})^{-1} + E_j^{(l)} &= -Q_{OI,j}^{(l)} \end{aligned} \tag{105}$$

for $j = 1, 2$, with $E_1^{(l)} = 0$, $E_2^{(l)} = 2(\Delta_2^{(l)})^4$ and $P_{OI}^{(l)} = 2P_{SF,j}^{(l)}$ for all $l \in (0, 1]$.

Let

$$\begin{aligned} \tilde{x}_{j+1} &= F_j^{(l)}\sigma_j \\ \dot{\sigma}_j &= H_j^{(l)}\sigma_j + B_{j2}\tilde{x}_{j+1} + G_j^{(l)}y_j, \qquad j = 1, 2 \end{aligned} \tag{106}$$

be as in (31) with $\eta(s) = s$, $y_1 = \tilde{y}_1$, and $y_2 = \tilde{y}_2 - \tilde{x}_2$. Change coordinates as $\zeta_1 = x_1$ and $\zeta_2 = x_2 - \tilde{x}_2$ and rewrite (100)–(106) as

$$\begin{aligned} \dot{x}_1 &= \tilde{x}_2 + B_{11}\tilde{\Phi}_1(\zeta, \sigma, \tilde{x}, t) \\ \dot{\sigma}_1 &= H_1^{(l)}\sigma_1 + B_{12}\tilde{x}_2 + G_1^{(l)}y_1 \\ y_1 &= x_1 + C_{11}\tilde{\Phi}_1(\zeta, \sigma, \tilde{x}, t) \\ \dot{\zeta}_2 &= \tilde{x}_3 + B_{21}\tilde{\Phi}_2(\zeta, \sigma, \tilde{x}, t) \\ \dot{\sigma}_2 &= H_2^{(l)}\sigma_2 + B_{22}\tilde{x}_3 + G_2^{(l)}y_2 \\ y_2 &= \zeta_2 + C_{21}\tilde{\Phi}_2(\zeta, \sigma, \tilde{x}, t). \end{aligned} \tag{107}$$

Let $W_j(\zeta_j, \sigma_j) = \|\zeta_j\|^2_{P_{SF,j}^{(l)}} + \|\zeta_j - \sigma_j\|^2_{P_{SF,j}^{(l)}}$, $j = 1, 2$. It follows the existence of C^0 functions $P_{SF,1}^{(\cdot)}, P_{SF,2}^{(\cdot)}, k_2^{(\cdot)}, c_1^{(\cdot)}\colon (0, 1] \to \mathbb{R}^+$ such that for *sufficiently small* $l^\circ \in (0, 1]$ the level set $\{W_1(\zeta_1, \sigma_1) + k_2^{(l)}W_2(\zeta_2, \sigma_2) \le c_1^{(l)}\}$ contains any *a priori* given compact set Ω and $\dot{W}_1(\zeta_1, \sigma_1) + k_2^{(l)}\dot{W}_2(\zeta_2, \sigma_2)$ is negative definite for all $l \in (0, l^\circ]$. This is enough to conclude that the measurement feedback controller (106) semiglobally stabilizes (107) [and, thus, (100)] and $\|u(t)\| = \|F_2^{(l)}\sigma_2(t)\| \le \Delta$ for any given $\Delta > 0$, for all $t \ge 0$ and for all initial conditions in Ω.

We want to generalize the above arguments to (91). For simplicity, assume that rank $(B_{1j}) = n_j$ for all $j \ge 1$ and that (91) is already in the following form [see (101)]

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_{12}\tilde{x}_2 + B_{11}\tilde{\Psi}_1(u, x, t) \\ \tilde{y}_1 &= C_{12}x_1 + C_{11}\tilde{\Psi}_1(u, x, t) \\ \vdots &= \vdots \\ \dot{x}_s &= A_sx_s + B_{s2}\tilde{x}_{s+1} + B_{s1}\tilde{\Psi}_s(u, x, t) \\ \tilde{y}_s &= C_{s2}x_s + C_{s1}\tilde{\Psi}_s(u, x, t). \end{aligned} \tag{108}$$

Moreover, assume that each system (92) satisfies the assumptions of *Theorem III.1* with $a = 0$, $\delta(s) = 1$ and $\eta(s) = s$ and with large regions of attraction. In particular, assume that

(H6) there exist $\gamma_j > 0$, $R_{j1} \in \mathcal{SP}^{n_{j+1}}$ and C^0 functions $P_{SF,j}^{(\cdot)}, Q_{SF,j}^{(\cdot)}, P_{OI,j}^{(\cdot)}, Q_{OI,j}^{(\cdot)}\colon (0, 1] \to \mathcal{SP}^{n_j}$, $\Delta_j^{(\cdot)}\colon (0, 1] \to \mathbb{R}^+$, $E_j^{(\cdot)}\colon (0, 1] \to \mathcal{SSP}^j$ and $R_{j2}^{(\cdot)}\colon (0, 1] \to \mathcal{SP}^{p_j}$, $j = 1, \ldots, s$, satisfying (24)–(29) with $A_j, B_{j2}, B_{j1}, C_{j2}$ and C_{j1}, $a = 0$, $\eta(s) = s$ and $\delta(s) = 1$, and such that $\lim_{l\to 0} P_{SF,j}^{(l)} = 0$ and $P \ge P_{OI,j}^{(l)} \ge (2P_{SF,j}^{(l)}/\gamma_j^2)$ for some $P \in \mathcal{SP}^n$, $j = 1, \ldots, s$ and for all $l \in (0, 1]$.

While $P_{OI,j}^{(l)} \ge (2P_{SF,j}^{(l)}/\gamma_j^2)$ is made for guaranteeing the relative speed constraint, $P \ge P_{OI,j}^{(l)}$ requires the boundedness of $P_{OI,j}^{(l)}$, but they both can be replaced in many different ways.

We restrict the structure of the admissible uncertainties $\tilde{\Psi}_j(u, x, t)$, $j = 1, \ldots, s$, as follows. Let $\zeta_j = x_j - \tilde{x}_j$,

$j = 1, \ldots, s$, with $\tilde{x}_1 = 0$, $\tilde{x}_{j+1} = F_j^{(l)} \sigma_j$, $j = 1, \ldots, s$, and $F_j^{(l)} = -R_{j1}^{-1} P_{SF,j}^{(l)}$.

(H7) There exist C^0 matrix-valued functions of appropriate dimensions $E_{ji}(\cdot, \cdot)$, $1 \le j \le s$, $j \le i \le s+1$, $\tilde{E}_{ji}(\cdot, \cdot)$, $1 \le j \le s$, $j-1 \le i \le s+1$, $N_{ji}(\cdot, \cdot)$, $1 \le j \le s$, $j \le i \le s$, $\tilde{N}_{ji}(\cdot, \cdot)$, $1 \le j \le s$, $j-1 \le i \le s$, such that N_{ji} and $\tilde{N}_{ji}$, $i \ge j+1$, are uniformly bounded with respect to $x_1, \ldots, x_j, \tilde{x}_2, \ldots, \tilde{x}_j$ and

$$\begin{aligned} &\|\tilde{\Psi}_j(u, x, t)\|^2 \\ &\quad \le \frac{1}{\gamma_j^2} \left[\sum_{j \le i \le s} \|\zeta_i\|^2_{N_{ji}(x, \tilde{x})} + \sum_{j \le i \le s+1} \|\tilde{x}_i\|^2_{E_{ji}(x, \tilde{x})} \right] \end{aligned} \tag{109}$$

for $j = 1, \ldots, s$ and

$$\begin{aligned} &\left\| B_{j1}^{+} R_{j-1,1}^{-1} B_{j-1,2}^{T} P_{SF,j-1}^{(l)} G_{j-1}^{(l)} C_{j-1,1} \tilde{\Psi}_{j-1}(u, x, t) \right\|^2 \\ &\quad \le \frac{1}{\gamma_{j-1}^2} \left[\sum_{j-1 \le i \le s} \|\zeta_i\|^2_{\tilde{N}_{ji}(x, \tilde{x})} \right. \\ &\qquad \left. + \sum_{j-1 \le i \le s+1} \|\tilde{x}_i\|^2_{\tilde{E}_{ji}(x, \tilde{x})} \right] \end{aligned} \tag{110}$$

for $j = 2, \ldots, s$ and for all $l \in (0, 1]$.

Before getting to the main result of this section, we need some definitions. Let

$$\begin{aligned} P_{mj}^{(l)} &= \gamma_j^2 P_{OI,j}^{(l)} - P_{SF,j}^{(l)} \\ V_{SF,j}(v_1) &= \|v_1\|^2_{P_{SF,j}^{(l)}} \\ W_j(v_1, v_2) &= V_{SF,j}(v_1) + \|v_1 - v_2\|^2_{P_{mj}^{(l)}} \end{aligned} \tag{111}$$

for $j = 1, \ldots, s$. For each pair of compact sets $\tilde{\mathcal{S}}_s, \tilde{\mathcal{W}}_s \subset \mathbb{R}^{n_s}$ define the following C^0 functions:

$$c_s^{(l)} = 2^{s-1} \max_{\substack{v_1 \in \tilde{\mathcal{S}}_s \\ v_2 \in \tilde{\mathcal{W}}_s}} \{W_s(v_1, v_2)\} \tag{112}$$

$$\tilde{\Delta}_{s+1}^{(l)} = 2\|R_{s1}^{-1} B_{s2}^T \sqrt{P_{SF,s}^{(l)}}\| \sqrt{c_s^{(l)}} \tag{113}$$

$$\tilde{\Delta}_s^{(l)} = \max_{V_{SF,s}(\zeta_s) \le c_s^{(l)}} \{\|\zeta_s\|\} \tag{114}$$

$$k_s^{(l)} = 4 \max_{\substack{0 \le V_{SF,s}(\zeta_s) \le c_s^{(l)} \\ \|\tilde{x}_{j+1}\| \le \tilde{\Delta}_{j+1}^{(l)},\, j = s-1,\, s}} \left\{ \frac{\|\zeta_s\|^2_{\Gamma_{s-1,s}(x, \tilde{x})}}{\|\zeta_s\|^2_{Q_{SF,s}^{(l)}}} \right\} \tag{115}$$

$$\vdots = \vdots$$

$$c_i^{(l)} = c_{i+1}^{(l)} k_{i+1}^{(l)} \tag{116}$$

$$\tilde{\Delta}_i^{(l)} = \max_{V_{SF,i}(\zeta_i) \le c_i^{(l)}} \{\|\zeta_i\|\} \tag{117}$$

$$\begin{aligned} k_i^{(l)} = 2^{s-i+2} &\max_{\substack{0 \le V_{SF,h}(\zeta_h) \le c_h^{(l)},\, h = i, \ldots, s \\ \|\tilde{x}_{j+1}\| \le \tilde{\Delta}_{j+1}^{(l)},\, j = i-1, \ldots, s}} \\ &\cdot \left\{ \frac{\|\zeta_j\|^2_{\Gamma_{i-1,j}(x, \tilde{x})}}{k_{i+1}^{(l)} \cdots k_j^{(l)} \|\zeta_j\|^2_{Q_{SF,j}^{(l)}}},\ j = i, \ldots, s \right\} \end{aligned} \tag{118}$$

$$\vdots = \vdots$$

$$c_1^{(l)} = c_2^{(l)} k_2^{(l)} \tag{119}$$

with

$$\Gamma_{1i}(x, \tilde{x}) = N_{1i}(x, \tilde{x}) \tag{120}$$

for $i = 2, \ldots, s$ and

$$\Gamma_{ji}(x, \tilde{x}) = \frac{4\gamma_j^2}{\gamma_{j-1}^2} \tilde{N}_{ji}(x, \tilde{x}) + 2N_{ji}(x, \tilde{x}) \tag{121}$$

for all $j = 2, \ldots, s$ and $i = j+1, \ldots, s$. Finally, let

$$\begin{aligned} D_j^{(l)} &= B_{j+1,1}^{+} \left[A_{j+1} R_{j1}^{-1} B_{j2}^T P_{SF,j}^{(l)} + R_{j1}^{-1} B_{j2}^T \right. \\ &\qquad \left. \cdot \left(A_j^T P_{SF,j}^{(s)} + Q_{SF,j}^{(l)} + P_{SF,j}^{(s)} G_j^{(l)} C_{j2} \right) \right] \\ S_j^{(l)} &= B_{j+1,1}^{+} \left[A_{j+1} R_{j1}^{-1} B_{j2}^T P_{SF,j}^{(l)} + R_{j1}^{-1} B_{j2}^T \right. \\ &\qquad \left. \cdot \left(A_j^T P_{SF,j}^{(s)} + Q_{SF,j}^{(l)} \right) \right]. \end{aligned} \tag{122}$$

The numbers $c_1^{(l)}, \ldots, c_s^{(l)}$ are instrumental in constructing the number $c^{(l)}$, which parameterizes, together with $P_{SF,j}^{(l)}$ and $P_{mj}^{(l)}$, $j = 1, \ldots, s$, the width of the region of attraction, and a Lyapunov function $\tilde{W}_1(\zeta, \sigma)$ for the closed-loop system. In particular,

$$\begin{aligned} \tilde{W}_1(\zeta, \sigma) = W_1(\zeta_1, \sigma_1) + k_2^{(l)} &\\ \cdot \Big(W_2(\zeta_2, \sigma_2) + k_3^{(l)} \Big(\cdots \Big(&W_{s-1}(\zeta_{s-1}, \sigma_{s-1}) \\ &+ k_s^{(l)} W_s(\zeta_s, \sigma_s) \Big) \cdots \Big) \Big) \end{aligned}$$

and $c^{(l)} = c_1^{(l)}$. The region of attraction of the closed-loop system contains at least the level set $\tilde{W}_1(\zeta, \sigma) \le c_1^{(l)}$. On the other hand, $\tilde{\Delta}_j^{(l)}$, $j = 1, \ldots, s$, quantifies the worst excursion of $\|\zeta_j(t)\|$ over the compact set $\Omega^{(l)}$ (depending on $P_{SF,j}^{(l)}$, $P_{mj}^{(l)}$ and $c_j^{(l)}$, $j = 1, \ldots, s$) which captures the trajectories of the closed-loop system, while $\tilde{\Delta}_{s+1}^{(l)}$ gives an upper limit for $\|\tilde{x}_{s+1}(t)\|$ over $\Omega^{(l)}$. Note that the definition of $\tilde{\Delta}_j^{(l)}$, $j = 1, \ldots, s$, implicitly requires that the excursion of $\|\tilde{x}_{j+1}(t)\|$ along the trajectories of the closed-loop system be less or equal to $\tilde{\Delta}_{j+1}^{(l)}$.

The main result of this section is the following. The proof follows essentially the lines of [2] and will be omitted.

Theorem V.1: Let $\Delta > 0$, $\mathcal{S}_j, \mathcal{W}_j, \tilde{\mathcal{S}}_j, \tilde{\mathcal{W}}_j \subset \mathbb{R}^{n_j}$, $j = 1, \ldots, s$, be given compact sets with $\mathcal{S}_j \subset \tilde{\mathcal{S}}_j$ and $\mathcal{W}_j \subset \tilde{\mathcal{W}}_j$ and assume (**H**6)–(**H**7). Assume, in addition,

$$\lim_{l \to 0} c_j^{(l)} k_j^{(l)} = \infty, \qquad j = 2, \ldots, s \tag{123}$$

and the existence of $l^* \in (0, 1]$ such that

- for all $l \in (0, l^*]$ and $j = 1, \ldots, s-1$

$$2 \left\| R_{j1}^{-1} B_{j2}^T \sqrt{P_{SF,j}^{(l)}} \right\| \sqrt{c_j^{(l)}} \le \Delta_j^{(l)} \le \tilde{\Delta}_{j+1}^{(l)} \tag{124}$$

$$(D_j^{(l)})^T D_j^{(l)} \le \frac{Q_{mj}^{(l)}}{16\gamma_{j+1}^2 k_{j+1}^{(l)}} \tag{125}$$

$$\left(S_j^{(l)}\right)^T S_j^{(l)} + \frac{1}{2\gamma_j^2} \tilde{N}_{j+1,j}(x, \tilde{x}) \le \frac{Q_{SF,j}^{(l)}}{32\gamma_{j+1}^2 k_{j+1}^{(l)}} \tag{126}$$

- for all $l \in (0, l^*]$ and whenever $V_{SF,i}(\zeta_i) \leq c_i^{(l)}$ and $\|\tilde{x}_{i+1}\| \leq 2\|R_{i1}^{-1}B_{i2}^T\sqrt{P_{SF,i}^{(l)}}\|\sqrt{c_i^{(l)}}$, $i = 1, \ldots, s$,

$$E_{jh}(x, \tilde{x}) + \frac{2\gamma_j^2}{\gamma_{j-1}^2}\tilde{E}_{jh}(x, \tilde{x}) \leq \begin{cases} \dfrac{k_{j+1}^{(l)}\cdots k_{h-1}^{(l)}R_{h-1,1}}{2^{n-j+2}}, & \text{if } 2 \leq j \leq h-2,\ 4 \leq h \leq s+1 \\ \dfrac{R_{h-1,1}}{8}, & \text{if } j = h-1,\ 3 \leq h \leq s+1 \end{cases} \tag{127}$$

$$E_{1h}(x, \tilde{x}) \leq \begin{cases} \dfrac{k_2^{(l)}\cdots k_{h-1}^{(l)}R_{h-1,1}}{2^{s+1}}, & \text{if } 3 \leq h \leq s+1 \\ \dfrac{R_{11}}{8}, & \text{if } h = 2 \end{cases} \tag{128}$$

$$\tilde{E}_{j,j-1}(x, \tilde{x}) \leq \frac{\gamma_{j-1}^2 R_{j-2,1}}{32\gamma_j^2 k_{j-1}^{(l)}k_j^{(l)}}, \qquad j = 3, \ldots, s \tag{129}$$

$$E_{jj}(x, \tilde{x}) + \frac{2\gamma_j^2}{\gamma_{j-1}^2}\tilde{E}_{jj}(x, \tilde{x}) \leq \frac{R_{j-1,1}}{8k_j^{(l)}}, \qquad j = 2, \ldots, s \tag{130}$$

$$N_{jj}(x, \tilde{x}) + \frac{2\gamma_j^2}{\gamma_{j-1}^2}\tilde{N}_{jj}(x, \tilde{x}) \leq \frac{Q_{SF,j}^{(l)}}{8}, \qquad j = 2, \ldots, s \tag{131}$$

then there exist $l^\circ \in (0, 1]$ such that for all $l \in (0, l^\circ]$, with

$$\begin{aligned} \tilde{x}_{i+1} &= F_i^{(l)}\sigma_i \\ \dot{\sigma}_i &= H_i^{(l)}\sigma_i + B_{i2}\tilde{x}_{i+1} + G_i^{(l)}y_i, \qquad i = 1, \ldots, s \end{aligned} \tag{132}$$

where $y_1, \ldots, y_s$ are defined as in (96) and $H_i^{(l)}$ and $G_i^{(l)}$ are defined for each (92) as in (31) with $\eta(s) = s$, the closed-loop system (108)–(132) is $(\mathcal{S}_1 \times \mathcal{W}_1 \times \cdots \times \mathcal{S}_s \times \mathcal{W}_s)$-ULAS and $\|\tilde{x}_{s+1}(t)\| \leq \Delta$ for all $t \geq 0$ and for all initial conditions in $\mathcal{S}_1 \times \mathcal{W}_1 \times \cdots \times \mathcal{S}_s \times \mathcal{W}_s$. $\diamond$

While (123) allows us to include arbitrarily large compact sets in the region of attraction of the close-loop system, (124) guarantees that the excursion of $\|\tilde{x}_{j+1}(t)\|$, $j = 1, \ldots, s-1$, along the trajectories of the closed-loop system is always less or equal to that of $\|\zeta_{j+1}(t)\|$. On the other hand, the fact that $P_{SF,s}^{(l)} \to 0$ as $l \to 0$ and $P_{OI,s}^{(l)}$ (and, thus, $c_s^{(l)}$) is bounded is instrumental in rendering the amplitude of $\|\tilde{x}_{s+1}(t)\|$ less or equal to Δ. The remaining conditions of *Theorem V.1* are needed to render the derivative of $\tilde{W}_1(\zeta, \sigma)$ negative definite along the trajectories of the closed-loop system.

Two significant applications of *Theorem V.1* are worthwhile being discussed. We will assume that $n_i = 1$ for all $i = 1, \ldots, s$ (i.e., one-dimensional blocks).

Case A (Feedforward Structures): Assume that $\tilde{E}_{j,j-1} = 0$, $\tilde{N}_{j,j-1} = 0$, $E_{jj} = 0$, $N_{jj} = 0$, E_{jh}, N_{jh}, $\tilde{E}_{jh}$ and $\tilde{N}_{jh}$, $h \geq j+1$, depend only on $x_{j+1}, \ldots, x_s$ and $\tilde{x}_{j+1}, \ldots, \tilde{x}_{s+1}$ and $\tilde{E}_{jj}$ and $\tilde{N}_{jj}$ depend only on $x_j, \ldots, x_s$ and $\tilde{x}_j, \ldots, \tilde{x}_{s+1}$. This is indeed the case of (1), where $p_{j1}(x, u, t)$ and $p_{j2}(x, u, t)$, $j = 1, \ldots, s$, are *higher order* in $x_{j+1}, \ldots, x_s, u$, uniformly with respect to t and $x_1, \ldots, x_j$ [28]. It is easy to see that (124)–(131) amount to a condition of the form $P_{SF,j}^{(l)} \leq \varphi_j(P_{SF,j+1}^{(l)}) + \xi_j^{(l)}$, $j = 1, \ldots, s-1$, where $l \in (0, 1]$ and $\varphi_j(\cdot)$ and $\xi_j^{(\cdot)}$ are C^0 functions with $\varphi_j(0) = 0$ (see [2] for details).

Case B (Homogeneous Structure): Assume that $N_{j,j+1}$ and $\tilde{N}_{j,j+1}$ constant, $N_{jh} = 0$ and $\tilde{N}_{jh} = 0$ for all $h \neq s+1$, $E_{jh} = 0$ and $\tilde{E}_{jh} = 0$ for all $h \neq s+1$. This is the case of (2), where $p_{1j}(x, u, t)u$ and $p_{2j}(x, u, t)u$ are *of order less than* $1 + j - s$ *and* $j - s$, respectively, with respect to the "generalized" dilation $\delta_l(x, u) = (l^{1-s}x_1, \ldots, l^{-1}x_{s-1}, x_s, lu)$ and uniformly with respect to t [23]. It is easy to see that (124)–(131) amount to a condition of the form $P_{SF,j}^{(l)} = \varphi_j P_{SF,j+1}^{(l)}$, $j = 1, \ldots, s-1$, where $l \in (0, 1]$ and $\varphi_j > 0$ (see [4] for details).

References

[1] S. Battilotti, "A general theorem on the semiglobal stabilization of nonlinear uncertain systems via measurement feedback," in *Proc. IFAC Nonlinear Control Systems Design Symp.*, 1998.

[2] ——, "Semiglobal stabilization of uncertain block-feedforward systems via measurement feedback," in *Proc. IFAC Nonlinear Control Systems Design Symp.*, 1998.

[3] ——, "Semiglobal stabilization of triangular systems via measurement feedback," in *Proc. IEEE Conf. Decision and Control*.

[4] ——, "Generalized dilations and the stabilization of uncertain systems via measurement feedback," *Syst. Control Lett.*, to be published.

[5] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, 1989.

[6] H. J. Sussmann and P. V. Kokotovic, "The peaking phenomenon and the global stabilization of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 424–439, 1991.

[7] P. Kokotovic and R. Marino, "On vanishing stability regions in nonlinear systems with high gain feedback," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 967–970, 1986.

[8] F. Esfandiari and H. K. Khalil, "Output feedback stabilization of fully linearizable systems," *Int. J. Control*, vol. 56, pp. 1007–1037, 1992.

[9] H. K. Khalil and F. Esfandiari, "Semiglobal stabilization of a class of nonlinear system using output feedback," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1412–1415, 1993.

[10] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin, Germany: Springer-Verlag, 1996.

[11] ——, "Semiglobal robust regulation of nonlinear systems," in *Proc. Colloq. Automatic Control, Lecture Notes in Control and Information Sciences*, vol. 215, 1997, pp. 27–55.

[12] H. K. Khalil, *Nonlinear Systems*, 2nd ed. New York: MacMillan, 1996.

[13] ——, "Robust servomechanism output feedback controllers for feedback linearizable systems," *Automatica*, vol. 30, pp. 1587–1599, 1994.

[14] R. Sepulchre, M. Jankovic, and P. Kokotovic, *Constructive Nonlinear Control*. Berlin, Germany: Springer-Verlag, 1996.

[15] P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ theory," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 356–361, 1989.

[16] Z. Lin and A. Saberi, "Semiglobal exponential stabilization of linear systems subject to input saturations via linear feedbacks," *Syst. Control Lett.*, vol. 21, pp. 225–239, 1993.

[17] ——, "A semiglobal low-and-high gain design technique for linear systems with input saturation-stabilization and disturbance rejection," *Int. J. Robust Nonlinear Control*, vol. 5, pp. 381–398, 1995.

[18] A. Saberi, Z. Lin, and A. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 368–378, 1996.

[19] Z. Lin and A. Saberi, "Semiglobal exponential stabilization of linear systems subject to input saturations via linear feedbacks," *Syst. Control Lett.*, vol. 21, pp. 225–239, 1993.

[20] ——, "Robust semiglobal stabilization of minimum ohase input output linearizable systems via partial state and output feedback," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1029–1042, 1996.

[21] Z. P. Jiang, A. R. Teel, and L. Praly, “Small-gain theorem for ISS systems and applications,” *Math. Control, Signals Syst.*, vol. 7, pp. 95–120, 1994.
[22] S. Morse, “Structural invariants of linear multivariable systems,” *SIAM J. Control Optim.*, vol. 11, pp. 446–465, 1973.
[23] R. Sepulchre, “Slow peaking and low-gain designs for global sstabilization of nonlinear systems,” in *Proc. 36th IEEE Conf. Decision and Control*, San Diego, CA, Dec. 1997.
[24] A. R. Teel and L. Praly, “Tools for semiglobal stabilization by partial state and output feedback,” *SIAM J. Control Optim.*, vol. 33, pp. 1443–1488, 1995.
[25] ——, “Global stabilizability and observability imply semiglobal stabilizability by output feedback,” *Syst. Control Lett.*, vol. 22, pp. 313–325, 1994.
[26] A. R. Teel, “Semiglobal stabilizability of linear null controllable systems with input nonlinearities,” *IEEE Trans. Automat. Contr.*, vol. 40, pp. 96–100, 1995.
[27] ——, “A nonlinear small gain theorem for the analysis of control systems with saturations,” *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1256–1270, 1996.
[28] ——, “Using saturation to stabilize a class of single input partially linear composite systems,” in *Proc. IFAC Nonlinear Control Systems Design Symp.*, 1992, pp. 232–237.
[29] H. L. Trentelman, “Families of linear quadratic problems: Continuity properties,” *IEEE Trans. Automat. Contr.*, vol. 32, pp. 323–329, 1987.

Stefano Battilotti (S’89–M’95–SM’00) was born in Rome, Italy, in 1962. He received the degree in electrical engineering and the Ph.D. degree from the from the Università degli Studi “La Sapienza” di Roma, Rome, Italy, in 1987 and 1992, respectively.

Since 1994, he has been with the Dipartimento di Informatica e Sistemistica, Università degli Studi “La Sapienza” di Roma, where he is currently an Associate Professor. His research interests are focused on nonlinear deterministic and stochastic systems, with particular attention to disturbance and input–output decoupling with internal stability and robust control via measurement feedback.