met with $u_i = \dots$

$\triangle$

$x=0$ is GES

2.4 LINEAR TIME-VARYING SYSTEMS AND LINEARIZATION

We consider systems

$$\dot{x} = A(t)\,x, \quad x(t_0) = x_0$$

The solution $x(t)$ can be expressed as

$$x(t) = \phi(t,t_0)\,x_0$$

where $\phi(t,t_0)$ is the TRANSITION MATRIX.

$\phi(t,t_0)$ has the semigroup property:

$$\phi(t,t_0) = \phi(t,\tau)\,\phi(\tau,t_0) \quad \forall\, t \geq \tau \geq t_0$$

and inverse given by

$$\phi^{-1}(t, t_0) = \phi(t_0, t)$$

Consequently, it satisfies

$$\boxed{\frac{d}{dt_0}\phi(t, t_0)} = \frac{d}{dt_0}\phi^{-1}(t_0, t) =$$

$$= -\phi^{-1}(t_0, t)\underbrace{A(t_0)\phi(t_0, t)}_{=\frac{d}{dt_0}\phi(t_0, t)}\phi^{-1}(t_0, t)$$

$$= \boxed{-\phi(t, t_0)A(t_0)}$$

(since $\frac{d}{dt_0}\phi^{-1}(t_0, t) = -\phi^{-1}(t_0, t)\frac{d}{dt_0}\phi(t_0, t)\phi^{-1}(t_0, t)$)

and $\boxed{\frac{d}{dt}\phi(t, t_0)} = -\phi(t, t_0)\frac{d}{dt}\phi^{-1}(t, t_0)\phi(t_0, t)$

$$= -\phi(t, t_0)\phi(t_0, t)A(t)\phi(t_0, t)$$

$$= \boxed{A(t)\phi(t, t_0)}$$

<u>Theorem 2.4.1.</u> $x=0$ is (G)UAS for $\dot{x}=A(t)x$ if and only if

$$\|\Phi(t,t_0)\| \leq k e^{-\gamma(t-t_0)} \quad \forall\, t \geq t_0 \geq 0,$$

for some $k, \gamma > 0$.

<u>Proof.</u> Global UAS and UAS are equivalent because $x(t)=\Phi(t,t_0)x_0$, $\forall\, t \geq t_0$. Let's prove ($\Rightarrow$). We have

$$\|x(t)\| \leq \|\Phi(t,t_0)\| \|x_0\| \leq k\|x_0\| e^{-\gamma(t-t_0)}$$

which prove UAS.

We next prove ($\Leftarrow$). Assume $x=0$ is (G)UAS. There exists $\beta \in \mathcal{KL}$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) \quad \forall\, t \geq t_0, \ \forall\, x_0 \in \mathbb{R}^n$$

But

$$\|\Phi(t,t_0)\| = \max_{\|x\|=1} \|\Phi(t,t_0)x\| \leq$$

$$\leq \max_{\|x\|=1} \beta(\|x\|, t-t_0) \leq \beta(1, t-t_0)$$

Since $\beta(1, s) \to 0$ as $s \to \infty$

there exists $T > 0$ such that

$$\beta(1, T) \leq \frac{1}{e}$$

For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each.

Using the property of $\phi(t, t_0)$:

$$\phi(t, t_0) = \phi(t, t_0 + (N-1)T) \cdot \phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \phi(t_0 + T, t_0)$$

Hence

$$\|\phi(t, t_0)\| \leq \|\phi(t, t_0 + (N-1)T)\| \cdot \prod_{k=1}^{N-1} \|\phi(t_0 + kT, t_0 + (k-1)T)\| \leq$$

$$\leq \beta(1, 0) \prod_{k=1}^{N-1} \frac{1}{e} = e\beta(1,0)e^{-N}$$

$$\le e\,\beta(1,0)\,e^{-(t-t_0)/T}$$

(since $t \le t_0 + NT$)

$$\stackrel{\Delta}{=} k e^{-\gamma(t-t_0)}$$

with $k \triangleq e\beta(1,0)$ and $\gamma \triangleq \frac{1}{T}$

Remark (Eigenvalues criterion is not useful!)

Consider

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix}$$

For each t the eigenvalues of $A(t)$ are $-0.25 \pm 0.25\sqrt{7}\,j$ (independent of t!)

However, the origin $x=0$ is unstable.

Indeed,

$$\Phi(t,0) = \begin{pmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{pmatrix}$$

and there are x_0 arbitrarily close to $x=0$ for which the solution $x(t)$ is unbounded

In Example 2.3.1 we have seen that: if we can find a positive definite bounded matrix $P(t)$ satisfying

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \quad (*)$$

for some positive definite $Q(t)$ then $V(t,x) = x^T P(t) x$ is a Lyapunov function for $\dot{x} = A(t)x$.

If $Q(t)$ is also <u>bounded</u>: i.e.

$$0 < c_3 I \leq Q(t) \leq c_4 I, \quad \forall t \geq 0$$

and $A(t)$ continuous and bounded, when $x=0$ is UAS there is indeed a solution $P(t)$ of $(*)$ above.

Theorem 2.4.2 Let $x=0$ be UAS for $\dot{x} = A(t)x$. Assume $A(t)$ continuous and bounded, $Q(t)$ be continuous, bounded and positive definite, symmetric matrix. There is a continuosly differentiable bounded positive definite, symmetric matrix $P(t)$ satisfying $(*)$. The function $V(t,x) = x^T P(t) x$ is a Lyapunov function for $\dot{x} = A(t)x$.

Proof. Define

$$P(t) = \int_t^\infty \Phi^T(\tau,t)\, Q(\tau)\, \Phi(\tau,t)\, d\tau$$

and let $\psi(\tau,t,x)$ be the solution of $\dot{x} = A(t)x$ starting at t from x. We know $\psi(\tau,t,x) = \Phi(\tau,t)x$ so that

$$x^T P(t) x = \int_t^\infty \psi^T(\tau,t,x)\, Q(\tau)\, \psi(\tau,t,x)\, d\tau$$

Since $x=0$ is UAS from theorem 2.3.1

$$x^T P(t)\, x \le \int_t^\infty c_4 \|\phi(\tau,t)\|^2 \|x\|^2 \, d\tau$$

(recall $Q(t) \le c_4 I \quad \forall t \ge 0$)

$$\le \int_t^\infty k^2 e^{-2\gamma(\tau - t)} \, d\tau \cdot c_4 \|x\|^2$$

$$\le \frac{k^2 c_4}{2\gamma} \|x\|^2 \triangleq c_2 \|x\|^2$$

On the other hand

$$\|A(t)\| \le L \quad \forall t \ge 0$$

($A(t)$ is bounded!)

It follows

$$\left| \dot{\overline{\|\psi(\tau,t,x)\|^2}} \right| \le 2L \|\psi(\tau,t,x)\|^2$$

therefore

$$e^{-2L(\tau-t)} \|x\|^2 \le \|\psi(\tau,t,x)\|^2 \le e^{2L(\tau-t)} \|x\|^2$$

Hence

$$x^T P(t) x \geq \int_t^\infty c_3 \, \| \psi(\tau, t, x) \|^2 d\tau$$

(recall $c_3 I \leq Q(t) \ \forall \ t \geq 0$)

$$\geq \int_t^\infty e^{-2L(\tau - t)} d\tau \cdot c_3 \| x \|^2$$

$$= \frac{c_3}{2L} \| x \|^2 \triangleq c_1 \| x \|^2$$

Thus

$$c_1 \| x \|^2 \leq x^T P(t) x \leq c_2 \| x \|^2$$

which proves $P(t)$ being positive definite and bounded. It is also symmetric

$$P(t) = \int_t^\infty \psi^T(\tau, t, x) Q(t) \psi(\tau, t, x) \, d\tau$$

$$= \left(\int_t^\infty \psi^T(\tau, t, x) Q(t) \psi(\tau, t, x) \, d\tau \right)^T$$

$$= P^T(t)$$

and continuosly differentiable

$$\dot{P}(t) = \int_t^\infty \phi^T(\tau,t)\, Q(\tau) \frac{d}{dt}\phi(\tau,t)\, d\tau$$

$$+ \int_t^\infty \frac{\partial}{\partial t}\phi^T(\tau,t)\, Q(\tau)\, \phi(\tau,t)\, d\tau - Q(t)$$

$$= - \int_t^\infty \phi^T(\tau,t)\, Q(\tau)\, \phi(\tau,t)\, d\tau \cdot A(t)$$

(since $\frac{d}{dt}\phi(\tau,t) = -\phi(\tau,t)A(t)$)

$$- A^T(t) \int_t^\infty \phi^T(\tau,t)\, Q(\tau)\, \phi(\tau,t)\, d\tau - Q(t)$$

$$= -P(t)A(t) - A^T(t)P(t) - Q(t)$$

Therefore $V(t,x) = x^T P(t) x$ is a Lyapunov function for $\dot{x} = A(t)x$

EX. (Time-invariant systems) When

$\dot{x} = Ax \implies \phi(\tau,t) = e^{A(\tau - t)}$

and $P = \int_t^\infty e^{A^T(\tau - t)}\, Q\, e^{A(\tau - t)}\, d\tau$

(with constant Q)

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$$= \int_0^\infty e^{A^T s} Q e^{As} ds$$

(P is constant!)

and P is the unique positive definite solution of

$$PA + A^T P = -Q$$

Consider now

$$\dot{x} = f(t, x)$$

with $f : [0, \infty) \times D_r \longrightarrow \mathbb{R}^n$ continuously differentiable and $D_r \triangleq \{ x \in \mathbb{R}^n : \|x\| < r \}$. Suppose $x = 0$ is an equilibrium point for $\dot{x} = f(t,x)$: $f(t,0) = 0$ $\forall t \geq 0$. Suppose also

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq k \quad \forall x \in D_r \quad \forall t \geq 0 \qquad \text{(A1)}$$

$$\left\| \frac{\partial f}{\partial x}(t,x_1) - \frac{\partial f}{\partial x}(t,x_2) \right\| \leq L \| x_1 - x_2 \| \quad \text{A2}$$

$$\forall\, x_1 x_2 \in D_2, \quad \forall\, t \geq 0.$$

We have

$$f(t,x) = \underset{(0)}{f(t,0)} + \left(\int_0^1 \left. \frac{\partial f}{\partial x} \right|_{\theta x} d\theta \right) x$$

Then

$$f(t,x) = \left(\int_0^1 \left. \frac{\partial f}{\partial x} \right|_{\theta x} d\theta \right) x =$$

$$\left. \frac{\partial f}{\partial x} \right|_{x=0} x + \left[\int_0^1 \left(\left. \frac{\partial f}{\partial x} \right|_{\theta x} - \left. \frac{\partial f}{\partial x} \right|_{x=0} \right) d\theta \right] x$$

$$\triangleq A(t)x + g(t,x)$$

But

$$\| g(t,x) \| \leq \int_0^1 \left\| \left. \frac{\partial f}{\partial x} \right|_{\theta x} - \left. \frac{\partial f}{\partial x} \right|_{x=0} \right\| d\theta \cdot \| x \|$$

$$\leq L \int_0^1 \| \theta x \| d\theta \cdot \| x \| \leq L \| x \|^2$$

This means that $f(t,x)$ can be locally approximated by $A(t)x$ around $x = 0$.

<u>Theorem 2.4.3</u> Let $x=0$ be an equilibrium point for $\dot{x}=f(t,x)$ and A1, A2 hold true. Let

$$A(t)=\frac{\partial f}{\partial x}\Big|_{x=0}$$

Then $x=0$ is LES for $\dot{x}=f(t,x)$ if it is ES for $\dot{x}=A(t)x$.

<u>Proof</u>. Since $x=0$ is ES for $\dot{x}=A(t)x$ and $A(t)$ is continuous and bounded, theorem 2.4.2 gives the existence of $P(t)$ such that

$$-\dot{P}(t)=P(t)A(t)+A^T(t)P(t)+Q(t)$$

for continuous, positive definite and symmetric $Q(t)$. Let $V(t,x)\triangleq x^TP(t)x$:

$$\dot{V}(t,x)=x^TP(t)f(t,x)+f^T(t,x)P(t)x$$
$$+x^T\dot{P}(t)x=$$
$$=x^T\left[P(t)A(t)+A^T(t)P(t)+\dot{P}(t)\right]x$$
$$=-x^TQ(t)x+2x^TP(t)g(t,x)$$

$\leq -c_3 \|x\|^2 + 2c_2 L \|x\|^3$ (where $c_3 I \leq Q(t)$ and $P(t) \leq c_2 I$)

$\leq -(c_3 - 2c_2 L\rho)\|x\| \quad \forall \|x\| < \rho$

If, $\rho < \dfrac{c_3}{2c_2 L}$ then $\dot{V}(t,x)$ is negative definite inside $\|x\| < \rho$.

Corollary 2.3.2 (all conditions satisfied for $\|x\| < \rho$) concludes that $x=0$ is LES for $\dot{x} = f(t,x)$