Control Systems
27/10/2018

Exercise 1 Denoting $L(s) = G(s)P(s)$, in the Laplace domain the error and output evolutions are described by

$$e(s) = W_e(s)v(s), \quad y(s) = W(s)v(s) + W_d(s)d(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$, $W_e(s) = \frac{1}{1+L(s)}$ and $W_d(s) = \frac{P(s)}{1+L(s)}$. We shall split the controller $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for steady-steady specifications (i.e., $(ii) - (iii)$) and $G_2(s)$ for the remaining ones.

(ii) As $e(s) = W_e(s)v(s)$, one needs $W_e(0) = 0$. As the plant possesses a pole at $s = 0$, the former condition (i.e., $W_e(0) = 0$) is already verified by the plant.

(iii) For $y(t) \equiv 0$ under constant disturbances, one integrator (that is a pole in zero) is needed right before the entering point of the disturbance so to guarantee $W_d(0) = 0$. As the integrating action of the plant is located after the disturbance action, a further integrator needs to be included through the innter controller $G_1(s)$; accordingly, one sets

$$G_1(s) = \frac{1}{s}.$$

(iv) As the innter control action $G_1(s)$ has dimension n_1, the outer feedback loop is constrained to be of at most one-dimensional. Consequently, for maximizing the cross-over frequency ω_t^* while making the phase margin $m_\varphi^* \geq 50^o$, $G_2(s)$ is composed of at most one anticipative action plus one proportional term; namely, it is of the form

$$G_2(s) = kG_a(s), \quad G_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}.$$

By inspecting the Bode plots (Figure 1) of

$$L_1(s) = G_1(s)P(s) = \frac{1}{s^2(s+1)} \tag{1}$$

one notes that it is clear that $\angle L_1(j\omega) \in [-180^o, -270^o]$. As a consequence, let us set the attenuating function so to induce the largest phase contribution; namely, one sets $\omega_n = \omega_t^* \tau_a = 4$ rad/s with $m_a = 16$ in such to increase the phase (at the working frequency ω_t^*) of $\approx 61.9275^o$ with a magnitude contribution of ≈ 12.0412 dB. At this point, for making $m_\varphi^* \geq 50^o$ with such an action, ω_t^* needs to be chosen so that the following inequality is satisfied

$$180^o + \angle L_1(j\omega_t^*) + 61.9275^o \geq 50^o \implies \angle L_1(j\omega_t^*) \geq -191.9275^o.$$

Also, for guaranteeing that the closed-loop system is asymptotically stable it is necessary to guarantee that the extended Nyquist plot of $L(s) = G_2(s)L_1(s)$ makes no encirclements of $-1 + j0$. Since $\angle L_1(j\omega) \in [-180^o, -270^o]$ and we do want to maximize ω_t^*, it is convenient to set ω_t^* as the frequency corresponding to $\angle L_1(j\omega_t^*) \geq -191.9275^o$

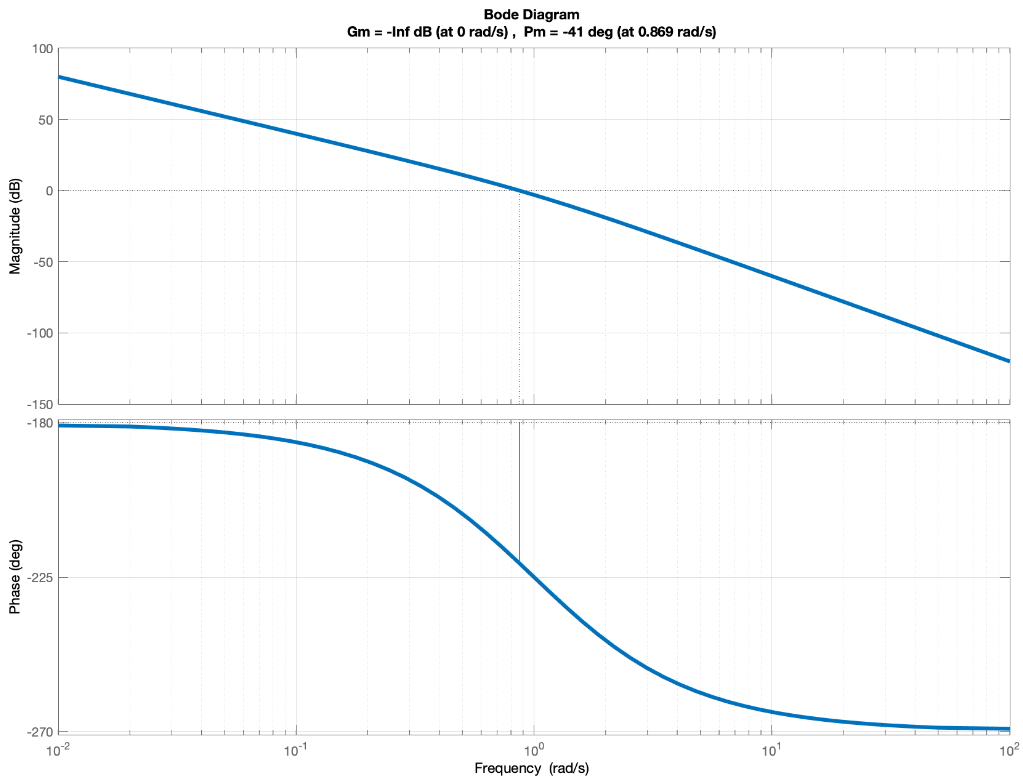

Figure 1: Bode plots of (1)

that is $\omega_t^* \approx 0.2112$ rad/s. As a consequence, one has $\tau_a = \frac{\omega_n}{\omega_t^*} \approx 18.9363$. At this point, the gain $k \in \mathbb{R}$ must be set so to make ω_t^* the new cross-over frequency, that is to guarantee

$$|k|_{dB} + |G_a(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0 \implies |k|_{dB} \approx -38.8639$$

so finally getting $k \approx 0.0113$. Thus, the new open-loop system

$$L(s) = G_2(s)L_1(s) = 0.0113\frac{1 + 18.9363s}{1 + 1.1837s}\frac{1}{s^2(s+1)} \tag{2}$$

satisfies specification (iii) as confirmed by Figure 2.

(i) The Nyquist plot of the open loop system

$$L(s) = kG_a(s)P(s) = 10\frac{1 + 0.7939s}{1 + 0.1323s}\frac{s+1}{s^3} \tag{3}$$

are reported in Figure 3. The number of counter-clockwise encirclements of $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 as the number the open loop poles of L(s) with positive real part. Thus, the system is asymptotically stable in closed loop.

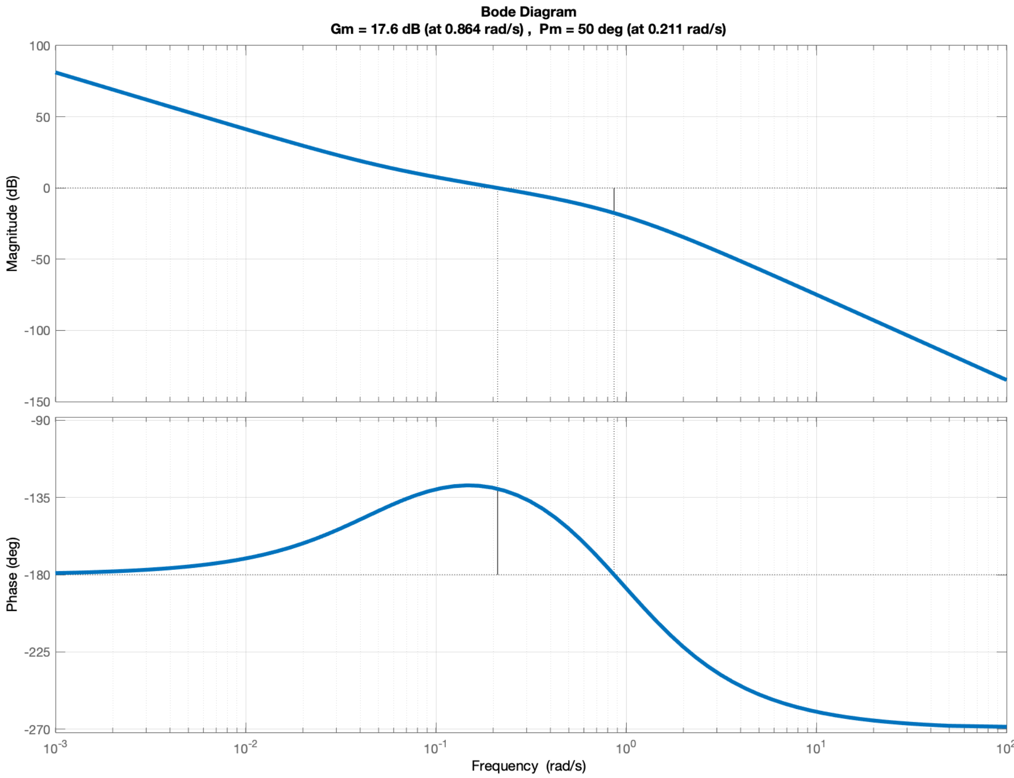

Figure 2: Bode plots of (3)

Exercise 2 As by assigning all poles in -1one is also making the closed-loop system asymptotically stable, specification (i) is satisfied as a byproduct of (ii). Accordingly, we shall set $G(s) = G_2(s) = G_1(s)$.

As the center of the asymptotes of the root locus associated to $P(s)$ is positive, it is straightforward to state that a static feedback $G(s) = k$ cannot stabilize the system assigning all poles with negative real part. Thus, for stabilizing the system with all poles with negative real part the controller needs to possess at least one pole for moving the center of the asymptotes and increase the relative degree to $r = 2$ (that is $G(s) = k\frac{1}{s+p}$, $p > 0$).

Also, note that because the plant $P(s)$ possesses two zeros at -1 (the desired location for the closed-loop poles), it might be convenient to cancel them under feedback in such a way that a finite (not high) gain is admissible for fulfilling the specification. Moreover, by drawing the root locus of $\frac{1}{(s+1)^2}$ it is evident that $\bar{G}(s)$ needs to be set in such a way to create a singularity at $s = -1$. Thus, we assume the feedback of the form

$$G(s) = \frac{as^3 + bs^2 + cs + d}{(s+1)^2(es + f)}$$

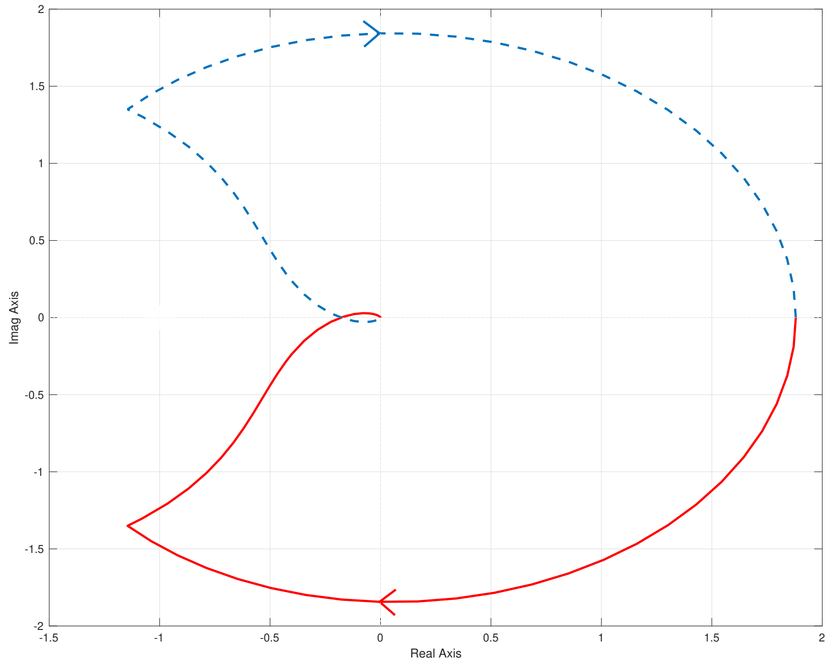

Figure 3: Nyquist plot of (3)

and thus

$$L(s) = \frac{as^3 + bs^2 + cs + d}{(s-1)(s-2)(s-4)(es+f)}.$$

Denoting $p(s)$ as the denominator of the closed-loop input-output transfer function $W(s) = \frac{L(s)}{1+L(s)}$

$$p(s) = (s-1)(s-2)(s-4)(es+f) + as^3 + bs^2 + cs + d$$

one now proceeds by looking at $a, b, c, d, e, f \in \mathbb{R}$ such that

$$p(s) = (s+1)^4$$

that is

$$es^4 + (a - 7e + f)s^3 + (b + 14e - 7f)s^2 + (c - 8e + 14f)s + d - 8f = s^4 + 4s^3 + 6s^2 + 4s + 1$$

Thus, by equating the terms with the same power of s^i $(i = 0, 1, \ldots, 4)$ one gets

$$a = 11, \quad b = -8, \quad c = 12, \quad d = 1, \quad e = 1 \quad \text{and} \quad f = 0$$

and thus

$$G(s) = \frac{11s^3 - 8s^2 + 12s + 1}{s(s+1)^2}$$
$$L(s) = \frac{11s^3 - 8s^2 + 12s + 1}{s(s-1)(s-2)(s-4)} = 11\frac{(s+0.07875)(s^2 - 0.806s + 1.154)}{s(s-1)(s-2)(s-4)}.$$

As the relative degree of $L(s)$ is $r = 4 - 3 = 1$, both the positive and negative locii possess a horizontal asymptote centered at

$$s_0 \approx 1 + 2 + 4 - 0.8060 + 0.07875 = 6.2728.$$

Denote by $\hat{p}(s,k)$ the denominator of the input-output transfer function $W(s) = \frac{kL(s)}{1+kL(s)}$. By construction, the positive locus possess a singularity of multiplicity $\mu = 4$ at $s^* = -1$ corresponding to $k^* = 1$. Also, by computing the solutions to the equalities above

$$\hat{p}(s,k) = s^4 + (11k-7)s^3 + (14-8k)s^2 + (12k-8)s + k$$
$$\frac{\partial \hat{p}(s,k)}{\partial s} = 4s^3 + 3(11k-7)s^2 + 2(14-8k)s + 12k - 8$$

one has that: the positive locus possesses two further singularities (both of multiplicity $\mu = 2$) at $(s^*, k^*) \approx (0.188, 0.345)$ and $(s^*, k^*) \approx (2.92, 0.023)$; the negative locus exhibits one singularity at $(s^*, k^*) \approx (1.35, -0.0274)$.

Finally, values of $k \in \mathbb{R}$ making the Routh table of the closed-loop pole polynomial $\hat{p}(s,k)$ correspond to the points in which the locus intersects the imaginary axis.Thus, by computing the Routh table

r^4	1	$14-8k$	k
r^3	$11k-7$	$12k-8$	
r^2	$\frac{-88k^2+198k-90}{11k-7}$	k	
r^1	$\frac{1177k^3-3234k^2+2713k-720}{88k^2-198k+90}$		
r^0	k		

One has that the positive locus intersects the imaginary axis for $k \approx \{0, 0.6321, 0.669, 1.4465\}$ corresponding to $s^* \approx \{0, \pm i2.98, \pm i0.297, \pm i1.02\}$. The plot of the root locus is reported in Figure 4.

Exercise 3 The system possesses one pseudo-periodical mode associated to the eigenvalues $\lambda = \frac{-k}{2} \pm i\frac{\sqrt{3}}{2}k$. The transfer function associated to the system is given by

$$P(s) = \frac{1}{s^2 + ks + k^2}$$

so that for $u(t) = u_1(t) + u_2(t)$ with $u_1(t) = 1_+$ and $u_2(t) = \sin_+(t)$, the forced output response can be computed in the Laplace domain as

$$y(t) = y_1(t) + y_2(t), \quad y_i(t) = \mathcal{L}^{-1}(P(s)u_i(s))[t], \qquad u_i(s) = \mathcal{L}(u_i(t))[s]$$

for $i = 1, 2$.

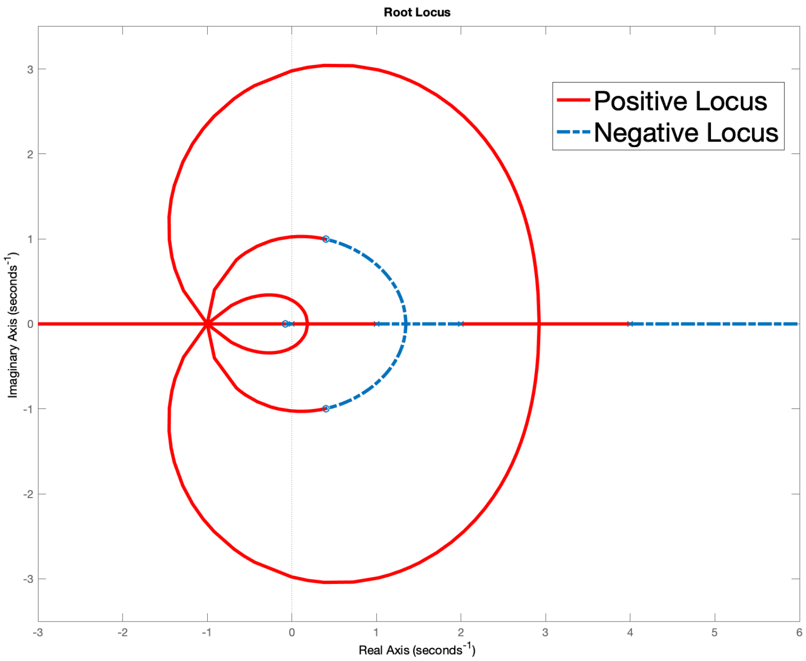

Figure 4: Root Locus of $L(s) = \frac{11s^3-8s^2+12s+1}{s(s-1)(s-2)(s-4)}$.

Accordingly, by computing

$$\begin{aligned}
y_1(s) = P(s)u_1(s) =& \frac{1}{s^2+ks+k^2}\frac{1}{s} = \frac{1}{k^2}\frac{1}{s} - \frac{1}{k^2}\frac{s+k}{s^2+ks+k^2} \\
=& \frac{1}{k^2}\frac{1}{s} - \frac{1}{k^2}\frac{s+\frac{k}{2}}{(s+\frac{k}{2})^2+\frac{3}{4}k^2} - \frac{1}{\sqrt{3}k^2}\frac{\frac{\sqrt{3}}{2}k}{(s+\frac{k}{2})^2+\frac{3}{4}k^2} \\
y_2(s) = P(s)u_2(s) =& \frac{1}{s^2+ks+k^2}\frac{1}{s^2+1} \\
=& \frac{As+B}{s^2+1} + \frac{C(s+\frac{k}{2})+D}{s^2+ks+k^2}
\end{aligned}$$

with

$$A = -\frac{k}{k^4-k^2+1}, \quad B = \frac{k^2-1}{k^4-k^2+1}, \quad C = \frac{k}{k^4-k^2+1}, \quad D = -\frac{k^2-2}{2(k^4-k^2+1)}$$

one gets

$$\begin{aligned}
y_1(t) =& \frac{1}{k^2}_+ - \frac{1}{k^2}\Big(e^{-\frac{k}{2}t}\cos\big(\frac{\sqrt{3}}{2}kt\big)\Big)_+ - \frac{1}{\sqrt{3}k^2}\Big(e^{-\frac{k}{2}t}\sin\big(\frac{\sqrt{3}}{2}kt\big)\Big)_+ \\
y_2(t) =& A(\sin t)_+ + B(\cos t)_+ + C\Big(e^{-\frac{k}{2}t}\cos\big(\frac{\sqrt{3}}{2}kt\big)\Big)_+ + \frac{2\sqrt{3}}{3}D\Big(e^{-\frac{k}{2}t}\sin\big(\frac{\sqrt{3}}{2}kt\big)\Big)_+.
\end{aligned}$$

If the system is asymptotically stable (that is if, and only if, $k > 0$), the steady-state response is given by

$$y_{ss}(t) = \frac{1}{k^2} + A\sin t + B\cos t.$$

The corresponding transient response gets the form

$$y_{\text{tran}}(t) = y(t) - y_{ss}(t) = (-\frac{1}{k^2} + C)e^{-\frac{k}{2}t}\cos\big(\frac{\sqrt{3}}{2}kt\big) + \frac{\sqrt{3}}{3}\big(-\frac{1}{k^2} + 2D\big)e^{-\frac{k}{2}t}\sin\big(\frac{\sqrt{3}}{2}kt\big).$$

As a requirement, the transient response must be bounded, for a suitable $k > 0$, as follows

$$|y_{\text{tran}}(t)| \leq 0.05, \quad \forall t \geq T_a = 10^{-2}.$$

Accordingly, denoting $R := (|-\frac{1}{k^2} + C| + |\frac{\sqrt{3}}{3}\big(-\frac{1}{k^2} + 2D\big)|)$, one has that

$$|y_{\text{tran}}(t)| \leq Re^{-\frac{k}{2}t}$$

so that k must verify

$$e^{\frac{k}{2}T_a} \geq \frac{20}{R} \implies k \geq \frac{2}{T_a}\ln 20R.$$

The above condition can be qualitatively checked for temptative values of $k \in \mathbb{R}$. As an example, the requirement is fulfilled for $k = 10$.