Notes on Linear Control Systems: Module IX

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***Abstract*—Stability of feedback systems in frequency domain. Nyquist criteria.**

I. Stability of feedback systems: Nyquist criteria

The stability of a feedback interconnection of $\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$, when the process $\mathbf{P}_2(s)$ on the feedback path is unitary (i.e. $\mathbf{P}_2(s) = 1$) and when the Bode plot of $\mathbf{P}_1(j\omega)$ is available, can be analyzed by means of the so-called *Nyquist criterion*. This corresponds to apply to the process $\mathbf{P}_1$ a control action $-y$, i.e. proportional to its output (Figure 1).

A. The Simplified Nyquist criterion

Consider a process $\mathbf{P}(s)$ and let

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{1}$$

be some its state space realization. The state space representation of the feedback interconnection of $\mathbf{P}(s)$ with unitary feedback is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t),\ \mathbf{u}(t) = \mathbf{v}(t) - \mathbf{y}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{2}$$

i.e.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (A - BC)\mathbf{x}(t) + B\mathbf{v}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{3}$$

(the *feedback or closed-loop* system). Denote by $\mathbf{p}(s)$ the characteristic polynomial of A

$$\mathbf{p}(s) := \det(sI - A) \tag{4}$$

and by $\mathbf{w}(s)$ the characteristic polynomial of the feedback system

$$\mathbf{w}(s) := \det(sI - (A - BC)) = \det(sI - A + BC) \tag{5}$$

Moreover, let $\mathrm{Adj}\{sI - A\}$ denote the adjoint of $sI - A$, i.e. the transpose of the matrix with (i, j)-th element equal to the cofactor of the (i, j)-th element of $sI - A$. Since from matrix algebra

$$\begin{aligned} \mathbf{P}(s) &= \frac{\mathrm{Adj}\{sI - A\}}{\det(sI - A)} = \frac{\mathrm{Adj}\{sI - A\}}{\mathbf{p}(s)}, \\ \mathbf{W}(s) &= \frac{\mathrm{Adj}\{sI - A + BC\}}{\det(sI - A + BC)} = \frac{\mathrm{Adj}\{sI - A + BC\}}{\mathbf{w}(s)} \end{aligned}$$

and assuming there are non zero-pole cancellations in $\mathbf{P}(s)$ and $\mathbf{W}(s)$, we see that the poles of $\mathbf{P}(s)$ and $\mathbf{W}(s)$ are the roots of $\mathbf{p}(s)$ (i.e. the eigenvalues of A) and, respectively, $\mathbf{w}(s)$ (i.e. the eigenvalues of $A - BC$).

Moreover, let $\mathbf{W}(s)$ denote the I/O transfer function of the feedback system. As we know, for feedback interconnections with unitary feedback path

$$\mathbf{W}(s) := \frac{\mathbf{P}(s)}{1 + \mathbf{P}(s)} \tag{6}$$

There is a precise relation between $\mathbf{w}(s)$ and $\mathbf{p}(s)$, on one side, and $P(s)$ on the other.

Proposition *1.1: For all $s \in \mathbb{C}$*

$$\frac{\mathbf{w}(s)}{\mathbf{p}(s)} = 1 + \mathbf{P}(s) \tag{7}$$

Proof. Since (1) is a realization of $\mathbf{P}(s)$ then $\mathbf{P}(s) = C(sI - A)^{-1}B$. Moreover, for any matrices $N(n \times r)$ and $M(r \times n)$

$$\det\{I + MN\} = \det\{I + NM\} \tag{8}$$

where the first identity matrix is $r \times r$ and the second identity matrix is $n \times n$. We have

$$\begin{aligned} &[1 + \mathbf{P}(s)]\mathbf{p}(s) = [1 + C(sI - A)^{-1}B]\det(sI - A) \\ &= \det[1 + C(sI - A)^{-1}B]\det(sI - A) \\ &= \det\{[1 + C(sI - A)^{-1}B](sI - A)\} \\ &= \det\{[1 + BC(sI - A)^{-1}](sI - A)\} \\ &= \det\{sI - A + BC\} = \mathbf{w}(s) \end{aligned}$$

where we used (8) with $M := C(sI - A)^{-1}$ and $N := B$. ◁

Notice that, while ω varies from $-\infty$ to $+\infty$, the end-point of the vector, which points into $1 + \mathbf{P}(j\omega)$ and with application point at the origin of the complex plane $\mathbb{C}$ (denoted by $\overrightarrow{1 + \mathbf{P}(j\omega)}$), describes a certain curve $\gamma(\omega) \in \mathbb{C}$. We want to calculate the total number of times $N(\gamma)$ (positive or negative) the curve $\gamma(\omega)$ encircles the origin of $\mathbb{C}$ while ω varies from $-\infty$ to $+\infty$, conventionally assigning $+1$ each time the encirclement is clockwise and -1 each time the encirclement is clockwise. On the other hand, $N(\gamma)$ is equal to the total phase variation of $\overrightarrow{1 + \mathbf{P}(j\omega)}$ as ω varies from $-\infty$ to $+\infty$ divided by 2π. We associate a positive phase increment to a counterclockwise rotation and negative phase increment to a clockwise rotation. For computing the total phase variation of $\overrightarrow{1 + \mathbf{P}(j\omega)}$ as ω varies from $-\infty$ to $+\infty$, we need the following basic result which is a consequence of simple graphical arguments.

Proposition *1.2: For any given complex number λ with Re$(\lambda) \neq 0$, the phase variation of $\overrightarrow{j\omega - \lambda}$ for ω varying from $-\infty$ to $+\infty$ is $-\pi$ (i.e. one clockwise half tour of $\overrightarrow{j\omega - \lambda}$) if Re$(\lambda) > 0$ and $+\pi$ (i.e. one counterclockwise half tour of $\overrightarrow{j\omega - \lambda}$) if Re$(\lambda) < 0$.*

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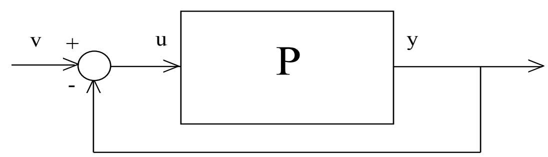


Figure 1. Feedback interconnection with unitary feedback.

We are now in a position to determine the phase variation of $\overrightarrow{\mathbf{p}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$, where $\mathbf{p}(s)$ is any n-degree polynomial without roots on the imaginary axis and, for our convenience, has the form

$$\mathbf{p}(s) \coloneqq \prod_{j=1}^{n} (s - \lambda_j). \tag{9}$$

Let $n^+(\mathbf{p}) \coloneqq$ the number of roots of $\mathbf{p}(s)$ with positive real part, i.e. the cardinality of the set $\{\lambda \in \mathbb{C}^+ : \mathbf{p}(\lambda) = 0\}$. The next result which follows from the fact that

$$\mathrm{Arg}\{(j\omega - \lambda)(j\omega - \mu)\} = \mathrm{Arg}\{j\omega - \lambda\} + \mathrm{Arg}\{j\omega - \mu\}$$

for any complex numbers λ and μ.

Proposition *1.3: For any given n-degree polynomial $\mathbf{p}(s)$ without roots on the imaginary axis, the phase variation of $\overrightarrow{\mathbf{p}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is*

$$-2n^+(\mathbf{p})\pi + n\pi$$

We have also the following consequence of the fact that

$$\mathrm{Arg}\{\frac{j\omega - \lambda}{j\omega - \mu}\} = \mathrm{Arg}\{j\omega - \lambda\} - \mathrm{Arg}\{j\omega - \mu\}$$

for any complex numbers λ and μ.

Proposition *1.4: For any two given n-degree polynomials $\mathbf{p}_1(s)$ and $\mathbf{p}_2(s)$ without roots on the imaginary axis, the total phase variation of $\overrightarrow{\left(\frac{\mathbf{p}_1(j\omega)}{\mathbf{p}_2(j\omega)}\right)}$ for ω varying from $-\infty$ to $+\infty$ is*

$$2\pi(-n^+(\mathbf{p}_1) + n^+(\mathbf{p}_2))$$

where $n^+(\mathbf{p}_1)$ (resp. $n^+(\mathbf{p}_2)$) is the number of roots of ($\mathbf{p}_1(s)$ (resp. ($\mathbf{p}_2(s)$)) with positive real part.

From (7) we conclude that if $\mathbf{p}(s)$ in (4) and $\mathbf{w}(s)$ in (5) are polynomials without roots on the imaginary axis, the total phase variation of

$$\overrightarrow{\left(\frac{\mathbf{w}(j\omega)}{\mathbf{p}(j\omega)}\right)} = \overrightarrow{1 + \mathbf{P}(j\omega)}$$

for ω varying from $-\infty$ to $+\infty$ is

$$2\pi(-n^+(\mathbf{w}) + n^+(\mathbf{p}))$$

where $n^+(\mathbf{p})$ (resp. $n^+(\mathbf{w})$) is the number of roots of $\mathbf{p}(s)$ (resp. $\mathbf{w}(s)$) with positive real part.

Dividing the total phase variation of $\overrightarrow{1 + \mathbf{P}(j\omega)}$ by 2π we obtain the number of counterclockwise tours (minus the number of clockwise tours) of $\overrightarrow{1 + \mathbf{P}(j\omega)}$ around the origin of $\mathbb{C}$ for ω varying from $-\infty$ to $+\infty$, i.e. $-n^+(\mathbf{w}) + n^+(\mathbf{p})$. Since $\mathbf{w}(s)$ has no roots on the imaginary axis by hypothesis, the polynomial $\mathbf{w}(s)$ is Hurwitz [1] (and therefore the closed-loop system $\mathbf{W}(s)$ is asymptotically stable) if and only if $n^+(\mathbf{w}) = 0$. In view of the above discussion, we can conclude that $\mathbf{w}(s)$ is Hurwitz (and therefore the closed-loop system $\mathbf{W}(s)$ is asymptotically stable) if and only if the number of counterclockwise tours (minus the number of clockwise tours) of $\overrightarrow{1 + \mathbf{P}(j\omega)}$ around the origin of $\mathbb{C}$ for ω varying from $-\infty$ to $+\infty$ is exactly $n^+(\mathbf{p})$.

Proposition *1.5: Assume that $\mathbf{p}(s)$ and $\mathbf{w}(s)$ have no roots on the imaginary axis (or equivalently $\mathbf{P}(s)$ and $\mathbf{W}(s)$ have no poles on the imaginary axis). The closed-loop system $\mathbf{W}(s)$ is asymptotically stable if and only if the number of counterclockwise tours (minus the number of clockwise tours) around the origin on behalf of the vector $\overrightarrow{1 + \mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is $n^+(\mathbf{p})$.*

A couple of remarks which refine the statement of proposition 1.5. First, as mentioned before, the roots of $\mathbf{p}(s)$ are the poles of $\mathbf{P}(s)$, so $n^+(\mathbf{p})$ is also the number of poles of $\mathbf{P}(s)$ with positive real part. Moreover, the polynomial $\mathbf{w}(s)$ has at least one root on the imaginary axis if and only if $\mathbf{w}(j\omega^\circ) = 0$ for at least one $\omega^\circ \in (-\infty, +\infty)$. This means, on account of (7), that

$$0 = \frac{\mathbf{w}(j\omega^\circ)}{\mathbf{p}(j\omega^\circ)} = 1 + \mathbf{P}(j\omega^\circ) \tag{10}$$

or equivalently that $\mathbf{P}(j\omega^\circ) = -1$ or that the end-point of $\overrightarrow{\mathbf{P}(j\omega)}$ passes through the point $-1 + j0$ for at least one $\omega^\circ \in (-\infty, +\infty)$. Therefore

Proposition *1.6: The polynomial $\mathbf{w}(s)$ has at least one root on the imaginary axis if and only if the end-point of $\overrightarrow{\mathbf{P}(j\omega)}$ crosses the point $-1 + j0$ for at least one $\omega^\circ \in (-\infty, +\infty)$. If the end-point $\overrightarrow{\mathbf{P}(j\omega)}$ crosses the point $-1 + j0$ for at least one $\omega^\circ \in (-\infty, +\infty)$ then $\mathbf{w}(s)$ is not Hurwitz (i.e. the closed-loop system $\mathbf{W}(s)$ is not asymptotically stable).*

Also notice that the number of counterclockwise tours (minus the number of clockwise tours) around the origin on behalf of the vector $\overrightarrow{1 + \mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is equal to the number of counterclockwise tours (minus the number of clockwise tours) around $-1 + j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$.

We are ready to formulate the *simplified Nyquist criterion.*

Theorem *1.1:* **(Simplified Nyquist criterion)**. *Assume that*

- *$\mathbf{P}(s)$ has no poles on the imaginary axis*
- *the end-point of $\overrightarrow{\mathbf{P}(j\omega)}$ does not cross the point $-1 + j0$ for any $\omega \in (-\infty, +\infty)$ (or, equivalently, $\mathbf{P}(j\omega) \neq -1 + 0j$ for all $\omega \in (-\infty, +\infty)$).*

[1]A n-th degree polynomial $\mathbf{p}(s)$ is Hurwitz if its roots are all in $\mathbb{C}^-$

The closed-loop system $\mathbf{P}(s)$ is asymptotically stable if and only if the number of counterclockwise tours (minus the clockwise tours) of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ around the point $-1 + j0$ for ω varying from $-\infty$ to $+\infty$ is equal to $n^+(\mathbf{p})$. If the number of counterclockwise tours around the point $-1 + j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is not equal to $n^+(\mathbf{p})$ then $\mathbf{w}(s)$ has at least one root with positive real part and the closed-loop system $\mathbf{W}(s)$ is unstable.

The plot of $\overrightarrow{\mathbf{P}(j\omega)}$ on the complex plane for ω varying from $-\infty$ to $+\infty$ is known as *Nyquist plot*. The Nyquist plot is needed for calculating the number of counterclockwise tours (minus the clockwise tours) of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ around the point $-1 + j0$. This plot can be drawn directly from the Bode plot as discussed in the following examples.

Exercize *1.1:* Consider the system

$$\mathbf{P}(s) = \frac{1}{s+1} \tag{11}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 2. Clearly, $\mathbf{P}(s)$ has no poles on the imaginary axis and $n^+(\mathbf{p}) = 0$. Moreover, it is also clear from the Bode plot of $\mathbf{P}(j\omega)$ that $\mathbf{P}(j\omega)$ does not cross the point $-1 + j0$ (i.e. the point with magnitude 1 and phase $-\pi$) for any $\omega \in (-\infty, +\infty)$. Therefore, by the Simplified Nyquist criterion the closed-loop system is asymptotically stable if and only if the number of counterclockwise (decremented by the number of clockwise) tours around the point $-1 + j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is 0.

The Nyquist plot can be drawn from the Bode plot as follows. First, from now on we stipulate that the numbers on the real positive axis of the complex plane have phase $0 \pm 2k\pi$, $K = 0, 1, \ldots$, the numbers on the imaginary negative axis have phase $-\frac{\pi}{2} \pm 2k\pi$, the numbers on the real negative axis have phase $-\pi \pm 2k\pi$, the numbers on the imaginary positive axis have phase $-\frac{3\pi}{2} \pm 2k\pi$ and so on. By doing this a clockwise phase variation has negative sign.

Let us draw the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ on the complex plane for ω varying from 0 to $+\infty$. Note from the Bode plot (Figure 2) that

$$\begin{aligned}
&\mathbf{P}(0) = 1, \\
&\lim_{\omega\to+\infty} |\mathbf{P}(j\omega)| = \lim_{\omega\to+\infty} \frac{1}{\sqrt{1+\omega^2}} = 0, \\
&\lim_{\omega\to+\infty} \operatorname{Arg}\{\mathbf{P}(j\omega)\} = \lim_{\omega\to+\infty} \operatorname{Arg}\{\frac{1}{j\omega+1}\} = -\frac{\pi}{2}.
\end{aligned} \tag{12}$$

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ has the direction going from $\mathbf{P}(0) = 1$ to $\lim_{\omega\to+\infty} \mathbf{P}(j\omega) = 0$ (see Figure 3) and approaches the origin with tangent $-\frac{\pi}{2}$ (see the third equation of (12) and Figure 2), remaining in the quadrant which corresponds to phases between $-\frac{\pi}{2} \pm 2k\pi$ and $0 \pm 2k\pi$ (Figure 3). Recall that we stipulated that a positive variation of the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ corresponds to a counterclockwise rotation of $\overrightarrow{\mathbf{P}(j\omega)}$ around $-1+0j$. Note from Figure 2 that the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ decreases monotonically, therefore in Figure 3 the curve does not change its clockwise rotation in going from $\mathbf{P}(0) = 1$ to $\lim_{\omega\to+\infty} \mathbf{P}(j\omega) = 0$.

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0 is obtained directly from the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$. Indeed,

$$\begin{aligned}
\mathbf{P}(-j\omega) &= \frac{1}{-j\omega+1} = \frac{j\omega+1}{\omega^2+1} \\
&= \left(\frac{-j\omega+1}{\omega^2+1}\right)^* = \left(\frac{1}{j\omega+1}\right)^* = \mathbf{P}^*(j\omega)
\end{aligned} \tag{13}$$

Therefore, the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0 is obtained by taking the symmetric curve with respect to the real axis of the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ (see Figure 3).

As it is clear from Figure 3 we obtain a closed curve which goes from $\mathbf{P}(0) = 1$ to the origin, then from the origin back again to $\mathbf{P}(0) = 1$. The number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is 0. Since $n^+(\mathbf{p}) = 0$, by the Simplified Nyquist criterion the closed-loop system is asymptotically stable.

Exercize *1.2:* Consider the system

$$\mathbf{P}(s) = \frac{-s+0.1}{(1+s)(1+0.5s)} \tag{14}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 4. Clearly, $\mathbf{P}(s)$ has no poles on the imaginary axis and $n^+(\mathbf{p}) = 0$. Moreover, it is also clear from the Bode plot of $\mathbf{P}(j\omega)$ that $\mathbf{P}(j\omega)$ does not cross the point $-1 + j0$ for any $\omega \in (-\infty, +\infty)$. Therefore, by the Simplified Nyquist criterion the closed-loop system is asymptotically stable if and only if the number of counterclockwise tours (minus the number of clockwise tours) around the point $-1 + j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is 0.

The Nyquist plot can be drawn from the Bode plot as pointed out in the previous example. Let us draw the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ on the complex plane for ω varying from 0 to $+\infty$. Note that

$$\begin{aligned}
&\mathbf{P}(0) = 0.1, \\
&\lim_{\omega\to+\infty} |\mathbf{P}(j\omega)| = 0, \\
&\lim_{\omega\to+\infty} \operatorname{Arg}\{\mathbf{P}(j\omega)\} = -\frac{3\pi}{2}.
\end{aligned} \tag{15}$$

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ has the direction going from $\mathbf{P}(0) = 0.1$ to $\lim_{\omega\to\pm\infty} \mathbf{P}(j\omega) = 0$ (see Figure 5) and approaches the origin with tangent $-\frac{3\pi}{2}$ (see the third equation of (12) and Figure 4). Note that the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ is contained in the quadrants which corresponds to phases between $-\frac{3\pi}{2} \pm 2k\pi$ and $0 \pm 2k\pi$ (Figure 5). Note also from Figure 4 the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ decreases monotonically, therefore in Figure 5 the curve does not change its clockwise rotation in going from $\mathbf{P}(0) = 0.1$ to $\lim_{\omega\to+\infty} \mathbf{P}(j\omega) = 0$. Moreover, $\overrightarrow{\mathbf{P}(j\omega)}$ crosses the negative real axis for some $\omega^\circ \approx 1.5$rad/sec with $|\mathbf{P}(j\omega^\circ)| < 1$, therefore the curve on the Nyquist plot leaves to its left the point $-1 + 0j$ while crossing the negative real axis (Figure 5). The intersections of the negative real axis can be always calculated from inspection of the Bode plot,

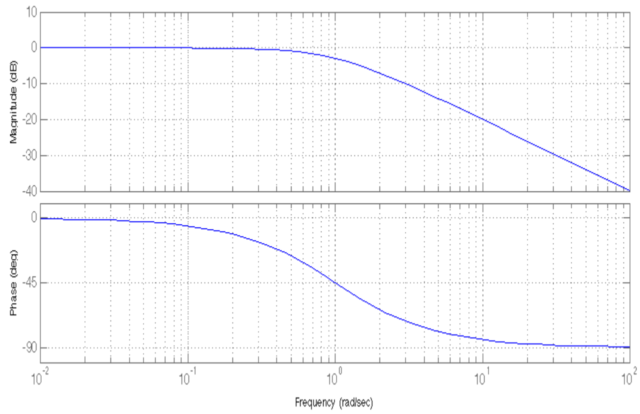


Figure 2. Bode diagrams of to $\mathbf{P}(s) = \frac{1}{s+1}$.

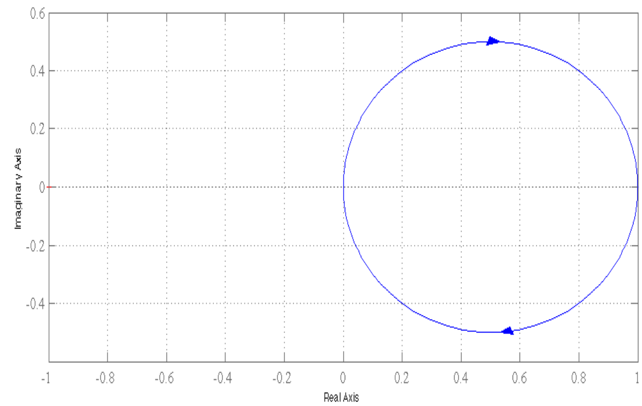


Figure 3. Nyquist plot of $\mathbf{P}(s) = \frac{1}{s+1}$.

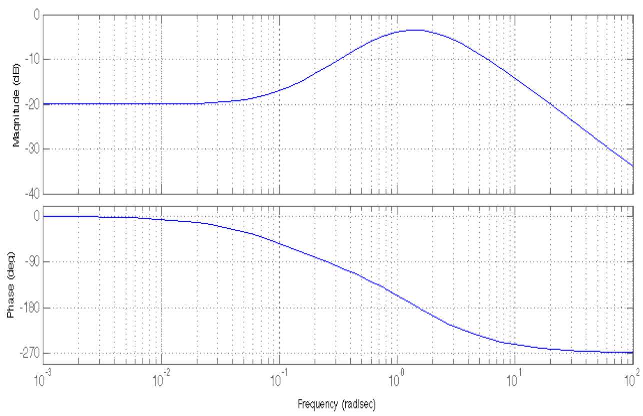


Figure 4. Bode diagrams of to $\mathbf{P}(s) = \frac{-s+0.1}{(1+s)(1+0.5s)}$.

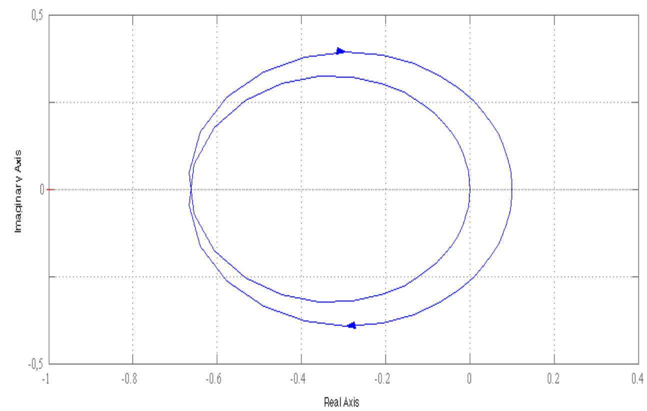


Figure 5. Nyquist plot of $\mathbf{P}(s) = \frac{-s+0.1}{(1+s)(1+0.5s)}$.

seeking for the ω° such that $\text{Arg}\{\mathbf{P}(j\omega^\circ)\} = -\pi$ and then looking at the values of $|\mathbf{P}(j\omega^\circ)|$: if $|\mathbf{P}(j\omega^\circ)| < 1$ the curve on the Nyquist plot leaves to her left the point $-1+0j$ while crossing the negative real axis, if $|\mathbf{P}(j\omega^\circ)| > 1$ the curve on the Nyquist plot leaves to her right the point $-1+0j$ while crossing the negative real axis, otherwise if $|\mathbf{P}(j\omega^\circ)| = 1$ the curve on the Nyquist plot crosses the point $-1+0j$.

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0 is obtained by taking the symmetric curve with respect to the real axis of the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ (see Figure 5).

As it is clear from Figure 5 we obtain a closed curve which goes from $\mathbf{P}(0) = 0.1$ to the origin, then from the origin back again to $\mathbf{P}(0) = 0.1$. The number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is 0. Since $n^+(\mathbf{p}) = 0$, by the Simplified Nyquist criterion the closed-loop system is asymptotically stable.

Exercize *1.3:* Consider the system

$$\mathbf{P}(s) = 20\frac{-s+0.1}{(1+s)(1+0.5s)} \tag{16}$$

which is exactly the previous $\mathbf{P}(s)$ multiplied by 20. The Bode plot of $\mathbf{P}(j\omega)$ are drawn in Figure 6. Note that

$$\begin{aligned} &\mathbf{P}(0) = 2 \\ &\lim_{\omega\to+\infty} |\mathbf{P}(j\omega)| = 0 \\ &\lim_{\omega\to+\infty} \text{Arg}\{\mathbf{P}(j\omega)\} = -\frac{3\pi}{2} \end{aligned} \tag{17}$$

As it is clear from Figure 7 we obtain a closed curve which goes from $\mathbf{P}(0) = 2$ to the origin, then from the origin back again to $\mathbf{P}(0) = 2$. Moreover, $\overrightarrow{\mathbf{P}(j\omega)}$ crosses the negative real axis for some ω° between 1 and 2rad/sec with $|\mathbf{P}(j\omega)| > 1$, therefore the curve on the Nyquist plot leaves to her right the point $-1+0j$ while crossing the negative real axis (Figure 7).

The number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is -2 (= 2 clockwise tours). Since $n^+(\mathbf{p}) = 0$, by the Simplified Nyquist criterion the closed-loop system is unstable. We have seen that by increasing the gain of $\mathbf{P}(s)$ (i.e. the value of $\mathbf{P}(0)$) from 0.1 (Figure 5) to 2 (Figure 7) the closed-loop system has become unstable.

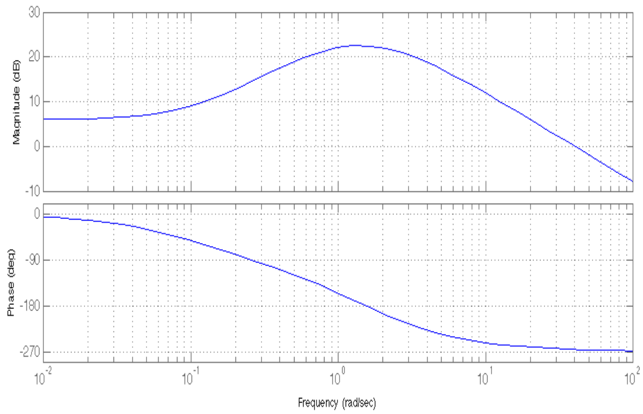


Figure 6. Bode diagrams of $\mathbf{P}(s) = 20\frac{-s+0.1}{(1+s)(1+0.5s)}$.

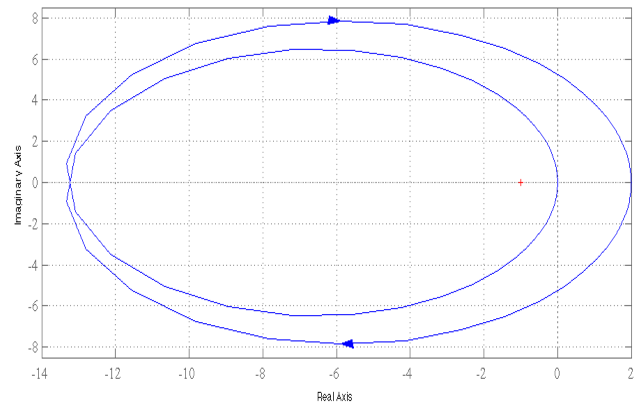


Figure 7. Nyquist plot of $\mathbf{P}(s) = 20\frac{-s+0.1}{(1+s)(1+0.5s)}$.

B. The Extended Nyquist criterion

We have seen that the Simplified Nyquist criterion assumes that $\mathbf{p}(s)$ has no roots on the imaginary axis. If $\mathbf{p}(s)$ has at least one root $z := j\omega^\circ$ on the imaginary axis then the magnitude of $\mathbf{P}(j\omega)$ goes to infinity as $\omega \to \omega^\circ$. This means that the plot of $\overrightarrow{1+\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$ is an open curve and it is not possible to evaluate the number of tours of $-1+0j$ on behalf of $\overrightarrow{1+\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to $+\infty$. However, it is possible to close the curve in a suitable way and then evaluate the number of tours of $-1+0j$ on behalf of $\overrightarrow{\mathbf{P}(j\omega)}$. We want to give an extended version of the Nyquist criterion in this sense, by relaxing the assumption that $\mathbf{p}(s)$ has no roots on the imaginary axis.

One key result from complex analysis, based on Laurent expansions and Cauchy's residuals theorem, is needed for this extension.

Proposition *1.7: Let Γ be a closed curve in some domain $D \subset \mathbb{C}$ and $\mathbf{f}(s)$ an olomorphic function on D, except for a finite number of points. The number of counterclockwise tours around the origin on behalf of the vector $\overrightarrow{\mathbf{f}(s)}$ for $s \in \Gamma$ varying clockwisely is $-n_z(\mathbf{f})+n_p(\mathbf{f})$, where $n_z(\mathbf{f})$ (resp. $n_p(\mathbf{f})$) is the number of zeroes (resp. poles) of $\mathbf{f}(s)$ inside Γ counted with their multiplicity.*

Assume that $\mathbf{p}(s)$ has some roots $z_1, \dots, z_r$ on the imaginary axis. Define a curve $\Gamma_{R,\rho} \subset \mathbb{C}$ as the concatenation of the following curves:

(i) a half-circumference in the right-half complex plane with radius R and centered at the origin

(ii) r half-circumferences in the right-half complex plane, each with radius ρ and centered at the point $z_i := j\omega_i$ of the imaginary axis (denote by $z_{i,\rho}^- := j\omega_{i,\rho}^-$ the initial point of each half-circumference and by $z_{i,\rho}^+ := j\omega_{i,\rho}^+$ its final point, both on the imaginary axis)

(ii) the segments on the imaginary axis joining the r half-circumferences

Let s vary over the curve $\Gamma_{R,\rho}$ clockwisely (when $R \to \infty$ and $\rho \to 0$ this corresponds to a variation of s over the imaginary axis from $-j\infty$ to $+j\infty$).

The function $1+\mathbf{P}(s)$ is an olomorphic function on $\mathbb{C}$, except for a finite number of points (its poles). By virtue of proposition 1.1 and proposition 1.7 and since $\Gamma_{R,\rho}$ encircles only the zeroes of $\mathbf{w}(s)$ and $\mathbf{p}(s)$ with positive real part, the number of counterclockwise tours around the origin on behalf of the vector $\overrightarrow{1+\mathbf{P}(s)}$ for $s \in \Gamma_{R,\rho}$ varying clockwisely is $-n_{R,\rho}^+(\mathbf{w})+n_{R,\rho}^+(\mathbf{p})$, where $n_{R,\rho}^+(\mathbf{w})$ (resp. $n_{R,\rho}^+(\mathbf{p})$) denotes the number of zeroes of $\mathbf{w}(s)$ (resp. $\mathbf{p}(s)$) with positive real part inside $\Gamma_{R,\rho}$ counted with their multiplicity.

Notice that as s goes from the point $z_{i,\rho}^- := j\omega_{i,\rho}^-$ to the point $z_{i,\rho}^+ := j\omega_{i,\rho}^+$ along each half-circumference in the right-half complex plane, with radius ρ and centered at the point $z_i = j\omega_i$, the vector $\overrightarrow{1+\mathbf{P}(s)}$ goes from the point $1+\mathbf{P}(j\omega_{i,\rho}^-)$ to the point $1+\mathbf{P}(j\omega_{i,\rho}^+)$ subject to a phase decrement of $r_i\pi$ and, therefore, to r_i clockwise half tours around the origin. Indeed, if $s = z_i + \rho e^{j\theta}$ with θ varying from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ (i.e. s varies from $z_{i,\rho}^-$ to $z_{i,\rho}^+$ along each half-circumference in the right-half complex plane, with radius ρ and centered at the point z_i) then for $\rho \approx 0$

$$\begin{aligned} &\mathrm{Arg}\{\mathbf{P}(z_{i,\rho}^+)\} - \mathrm{Arg}\{\mathbf{P}(z_{i,\rho}^-)\} \\ &= \mathrm{Arg}\{\mathbf{P}(z_i + \rho e^{j\frac{\pi}{2}})\} - \mathrm{Arg}\{\mathbf{P}(z_i + \rho e^{-j\frac{\pi}{2}})\} \\ &= \mathrm{Arg}\{\frac{1}{e^{j\frac{\pi}{2}r_i}}\} - \mathrm{Arg}\{\frac{1}{e^{-j\frac{\pi}{2}r_i}}\} = -r_i\pi \end{aligned} \tag{18}$$

since for $\rho \approx 0$ the only term in $\mathbf{P}(s)$ which contribute to the phase variation is $\frac{1}{(s-z_i)^{r_i}}$.

As $R \to \infty$ and $\rho \to 0$ then $n_{R,\rho}^+(\mathbf{w}) \to n^+(\mathbf{w})$, $n_{R,\rho}^+(\mathbf{p}) \to n^+(\mathbf{p})$ and the curve $\Gamma_{R,\rho}$ tends to coincide with the imaginary axis.

Definition *1.1: The curve of $\mathbf{P}(j\omega)$ in $\mathbb{C}$ modified in such a way that, for each pole $z_i = j\omega_i$ of $\mathbf{P}(s)$ on the imaginary axis and with multiplicity r_i, we close the curve of $\mathbf{P}(j\omega)$ from $\mathbf{P}(j\omega_i^-)$ to $\mathbf{P}(j\omega_i^+)$ with r_i clockwise half tours (at infinity) as the point $s = j\omega$ on the imaginary axis crosses the point z_i from $z_i^- = j\omega_i^-$ to $z_i^+ = j\omega_i^+$ is called Nyquist plot of $\mathbf{P}(j\omega)$.*

On account of the above discussion, the extension of the Nyquist criterion when $\mathbf{p}(s)$ has at least one root on the imaginary axis is the same as its simplified version (theorem 1.1) as long as we substitute the polar plot with the Nyquist plot. We obtain the following *extended Nyquist criterion.*

Theorem *1.2:* **(Extended Nyquist criterion)**. *Assume that the end-point of $\overrightarrow{\mathbf{P}(j\omega)}$ does not cross the point $-1+j0$ for any $\omega \in (-\infty,+\infty)$ (or, equivalently, $\mathbf{P}(j\omega) \neq -1+j0$ for all $\omega \in (-\infty,+\infty)$). The polynomial $\mathbf{w}(s)$ is Hurwitz (i.e. the closed-loop system is asymptotically stable) if and only if the number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is $n^+(\mathbf{p})$. If the number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ is not equal to $n^+(\mathbf{p})$ then $\mathbf{w}(s)$ has at least one root with positive real part (i.e. the closed-loop system is unstable).* The Nyquist plot is needed for calculating the number of counterclockwise tours around the point $-1+j0$ on behalf of the vector $\overrightarrow{\mathbf{P}(j\omega)}$. This plot can be drawn directly from the Bode plot as discussed in the following examples.

Exercize *1.4:* Consider the system

$$\mathbf{P}(s) = \frac{1}{s(s+1)} \tag{19}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 8. Clearly, $\mathbf{P}(s)$ has a pole on the imaginary axis (more precisely, at 0) and $n^+(\mathbf{p}) = 0$. Moreover, it is also clear from the Bode plot of $\mathbf{P}(j\omega)$ that $\mathbf{P}(j\omega)$ does not cross the point $-1+j0$ (i.e. the point with magnitude 1 and phase $-\pi$) for any $\omega \in (-\infty,+\infty)$. Therefore, by the Extended Nyquist criterion the closed-loop system is asymptotically stable if and only if the number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is $n^+(\mathbf{p}) = 0$.

The polar plot (Figure 9) can be drawn from the Bode plot as follows. First, as usual we stipulate that the numbers on the real negative axis of the complex plane has phase $0 \pm 2k\pi$, $K = 0, 1, \ldots$, the numbers on the positive imaginary axis have phase $-\frac{\pi}{2} \pm 2k\pi$, the numbers on the negative real axis have phase $-\pi \pm 2k\pi$, the numbers on the positive imaginary axis have phase $-\frac{3\pi}{2} \pm 2k\pi$ and so on.

We draw on the complex plane the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ (i.e. for $\omega \to 0$ from the left) to $+\infty$. Note that from the Bode plot (Figure 8) and since $\mathbf{P}(-j\omega) = \mathbf{P}^*(j\omega)$

$$\begin{aligned}
&\lim_{\omega\to 0^+} |\mathbf{P}(j\omega)| = \lim_{\omega\to 0^+} \frac{1}{\omega\sqrt{1+\omega^2}} = +\infty,\\
&\lim_{\omega\to 0^+} \mathrm{Arg}\{\mathbf{P}(j\omega)\} = \lim_{\omega\to 0^+} \mathrm{Arg}\{\frac{1}{j\omega(j\omega+1)}\} = -\frac{\pi}{2},\\
&\lim_{\omega\to +\infty} |\mathbf{P}(j\omega)| = \lim_{\omega\to +\infty} \frac{1}{\omega\sqrt{1+\omega^2}} = 0,\\
&\lim_{\omega\to +\infty} \mathrm{Arg}\{\mathbf{P}(j\omega)\} = \lim_{\omega\to +\infty} \mathrm{Arg}\{\frac{1}{j\omega(j\omega+1)}\} = -\pi.
\end{aligned} \tag{20}$$

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ to $+\infty$ stems from $\mathbf{P}(0^+)$ and approaches the origin with asymptotic phase $-\pi$ (see the fourth equation of (12) and Figure 8), remaining in the quadrants which correspond to phases between $\frac{\pi}{2} \pm 2k\pi$ and $-\pi \pm 2k\pi$ (Figure 9). Recall that we stipulated that a positive variation of the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ corresponds to a counterclockwise rotation on behalf of $\overrightarrow{\mathbf{P}(j\omega)}$. Note from Figure 8 that the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ to $+\infty$ decreases monotonically, therefore in Figure 9 the curve does not change its clockwise rotation in going from $\mathbf{P}(0^+)$ to the origin.

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0^- is obtained directly from the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ to $+\infty$. Indeed,

$$\begin{aligned}
W(-j\omega) &= \frac{1}{-j\omega(-j\omega+1)}\\
&= \frac{-j(j\omega+1)}{\omega(\omega^2+1)} = \Big(\frac{\omega-j}{\omega(\omega^2+1)}\Big)^*\\
&= \Big(\frac{1}{j\omega(j\omega+1)}\Big)^* = W^*(j\omega)
\end{aligned} \tag{21}$$

Therefore, the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0 is obtained by drawing the symmetric curve with respect to the real axis of the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ (see Figure 9).

We obtain the Nyquist plot from Figure 9 by closing the curve from $\mathbf{P}(0^-)$ to $\mathbf{P}(0^+)$ with a clockwise half tour around 0 (Figure 10). The number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is 0. Since $n^+(\mathbf{p}_1) = 0$, by the Extended Nyquist criterion the closed-loop system is asymptotically stable.

Exercize *1.5:* Consider the system

$$\mathbf{P}(s) = \frac{(s+1)^2}{(s-1)(s+2)(s+3)s} \tag{22}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 11. Clearly, $\mathbf{P}(s)$ has a pole on the imaginary axis and $n^+(\mathbf{p}) = 1$. Moreover, it is also clear from the Bode plot of $\mathbf{P}(j\omega)$ that $\mathbf{P}(j\omega)$ does not cross the point $-1+j0$ (i.e. the point with magnitude 1 and phase $-\pi$) for any $\omega \in (-\infty,+\infty)$. Therefore, by the Extended Nyquist criterion the closed-loop system is asymptotically stable if and only if the number of counterclockwise tours (minus the number of clockwise tours) around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is $n^+(\mathbf{p}) = 1$.

The polar plot (Figure 12) can be drawn from the Bode plot as follows. As usual we stipulate that the numbers on the positive real axis of the complex plane have phase $0 \pm 2k\pi$, $K = 0, 1, \ldots$, the numbers on the negative imaginary axis have phase $-\frac{\pi}{2} \pm 2k\pi$, the numbers on the negative real axis have phase $-\pi \pm 2k\pi$, the numbers on the positive imaginary axis have phase $-\frac{3\pi}{2} \pm 2k\pi$ and so on.

We draw on the complex plane the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ (i.e. for $\omega \to 0$ from the left) to $+\infty$. Note that from the Bode plot (Figure 11)

$$\begin{aligned}
&\lim_{\omega\to 0^+} |\mathbf{P}(j\omega)| = +\infty\\
&\lim_{\omega\to 0^+} \mathrm{Arg}\{\mathbf{P}(j\omega)\} = -\frac{3\pi}{2}\\
&\lim_{\omega\to +\infty} |\mathbf{P}(j\omega)| = 0\\
&\lim_{\omega\to +\infty} \mathrm{Arg}\{\mathbf{P}(j\omega)\} = -\pi
\end{aligned} \tag{23}$$

The curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ to $+\infty$ has the direction going from $\mathbf{P}(0^+)$ to $\lim_{\omega\to+\infty} \mathbf{P}(j\omega) = 0$

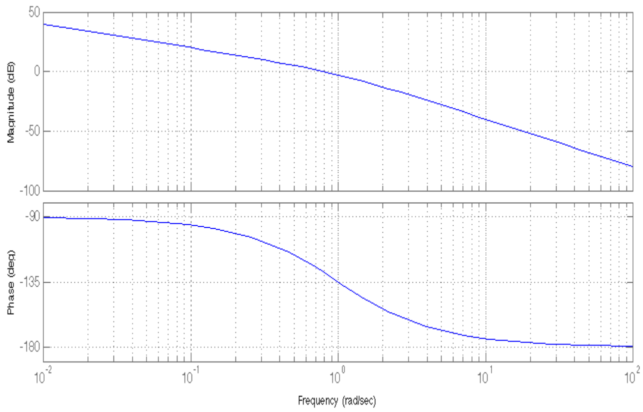


Figure 8. Bode diagrams of to $\mathbf{P}(s) = \frac{1}{s(s+1)}$.

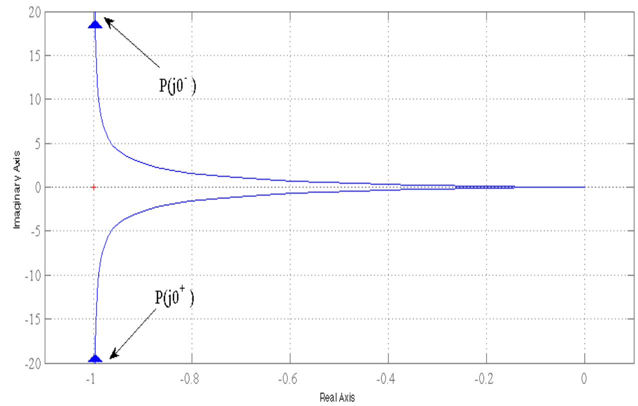


Figure 9. Polar plot of $\mathbf{P}(s) = \frac{1}{s(s+1)}$.

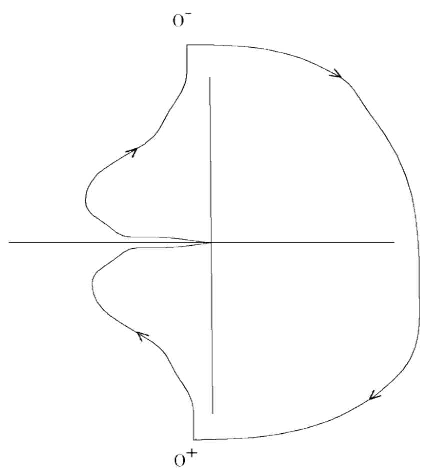


Figure 10. Nyquist plot of $\mathbf{P}(s) = \frac{1}{s(s+1)}$.

(see Figure 12) and approaches the origin with asymptotic phase $-\pi$ (see the fourth equation of (23) and Figure 11), remaining in the quadrants which correspond to phases between $-\frac{\pi}{2} \pm 2k\pi$ and $-\frac{3\pi}{2} \pm 2k\pi$ (Figure 12). Note from Figure 11 that the phase of $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0^+ to $+\infty$ increases and finally decreases monotonically, therefore in Figure 12 the curve changes its rotation (from counterclockwise to clockwise) in going from $\mathbf{P}(0^+)$ to $\lim_{\omega \to +\infty} \mathbf{P}(j\omega) = 0$.

As usual, the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from $-\infty$ to 0 is obtained by taking the symmetric curve with respect to the real axis of the curve described by $\overrightarrow{\mathbf{P}(j\omega)}$ for ω varying from 0 to $+\infty$ (see Figure 12).

We obtain the Nyquist plot from Figure 12 by closing the curve from $\mathbf{P}(0^-)$ to $\mathbf{P}(0^+)$ with a clockwise half tour around 0. The number of counterclockwise tours (minus the number of clockwise tours) around the point $-1 + j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is -1. Since $n^+(\mathbf{p}) = 1$, by the Extended Nyquist criterion the closed-loop system is unstable.

II. Proportional and derivative feedback control

As we have already noticed, from the point of view of stability the feedback interconnection with unitary feedback corresponds to apply to the process a control action $-y$, i.e. proportional to its output (proportional feedback). If the interconnection with unitary feedback is not asymptotically stable, it is convenient to refer to more general proportional feedback control actions like $-Ky$, where K is a real parameter. In this case the Nyquist criterion can be applied to $K\mathbf{P}(s)$ in place of $\mathbf{P}(s)$, by obtaining a qualitative analysis of the stability of the feedback interconnection as a function of the parameter K. Notice that increasing $K > 0$ in the product $K\mathbf{P}(j\omega)$ corresponds to a radial expansion of the Nyquist plot of $\mathbf{P}(j\omega)$ while decreasing $K > 0$ corresponds to its radial contraction. On the other hand, decreasing $K < 0$ in the product $K\mathbf{P}(j\omega)$ corresponds to a radial expansion of the Nyquist plot of $\mathbf{P}(j\omega)$ with a clockwise rotation of π, while increasing $K < 0$ corresponds to a radial contraction of the Nyquist plot of $\mathbf{P}(j\omega)$ subject to a clockwise rotation by $180°$. Therefore, by varying K we modulate the radial extension of the Nyquist plot of $\mathbf{P}(j\omega)$ and, therefore, the relative position between the point $-1 + 0j$ and the intersections of the Nyquist plot of $\mathbf{P}(j\omega)$ with the real negative axis. More simply, one can think of keeping unaltered the Nyquist plot of $\mathbf{P}(j\omega)$ (therefore, the intersections of the Nyquist plot of $\mathbf{P}(j\omega)$ with the real negative axis) and vary the relative position of the point $-1 + 0j$ with respect to the intersections of the Nyquist plot of $\mathbf{P}(j\omega)$ with the real negative axis. In particular, letting K vary from 0 to ∞ amounts to letting the $-1 + 0j$ vary on the negative real axis from $-\infty$ to 0 (the Nyquist plot of $\mathbf{P}(j\omega)$ is left unaltered) and letting K vary from $-\infty$ to 0 amounts to letting the $-1 + 0j$ vary on the positive real axis from 0 to $+\infty$ (the Nyquist plot of $\mathbf{P}(j\omega)$ is left unaltered). For each different position of the point $-1 + 0j$ on the real axis, we apply each time the Nyquist criterion. Clearly, the number of counterclockwise tours of $-1 + 0j$ will vary according to the relative position of the point $-1 + 0j$ and the Nyquist plot of $\mathbf{P}(j\omega)$. This allows a qualitative discussion of the stability of the feedback interconnection according to the values of K.

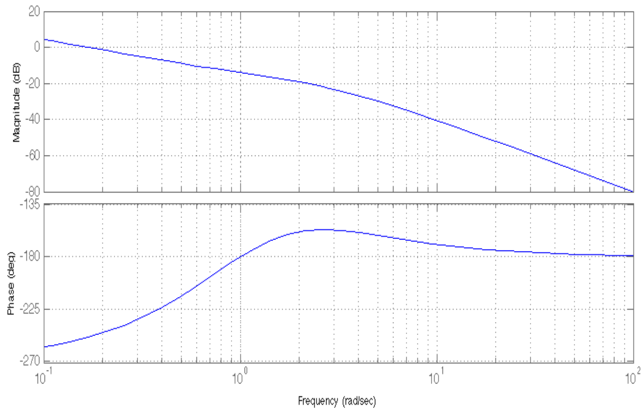


Figure 11. Bode diagrams of to $\mathbf{P}(s) = \frac{(s+1)^2}{(s-1)(s+2)(s+3)s}$.

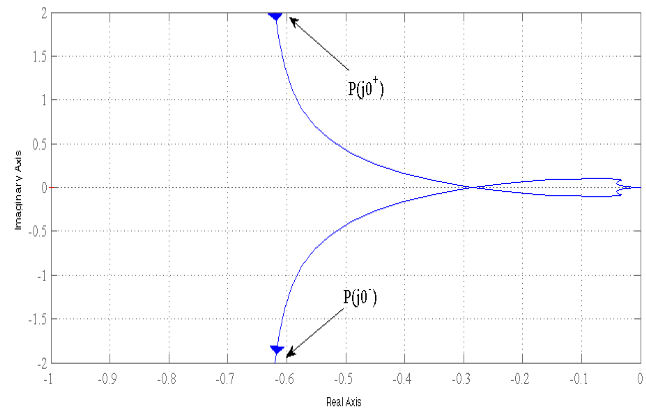


Figure 12. Polar plot of $\mathbf{P}(s) = \frac{(s+1)^2}{(s-1)(s+2)(s+3)s}$.

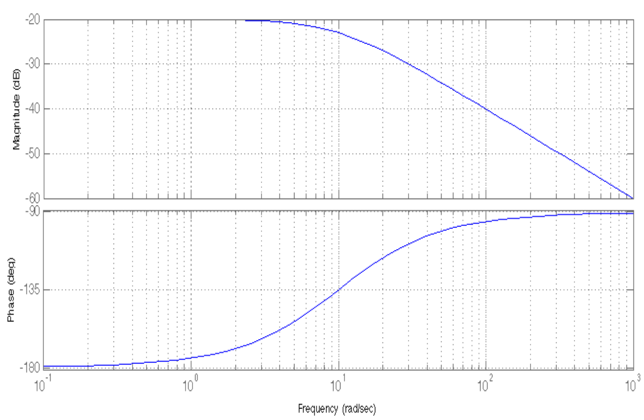


Figure 13. Bode diagrams of to $\mathbf{P}(s) = \frac{1}{s-10}$.

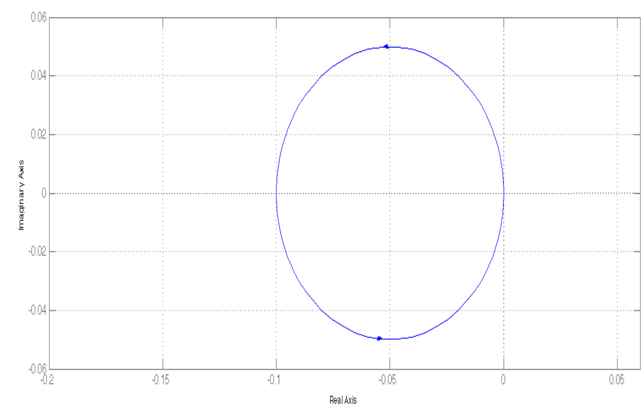


Figure 14. Nyquist plot of $\mathbf{P}(s) = \frac{1}{s-10}$.

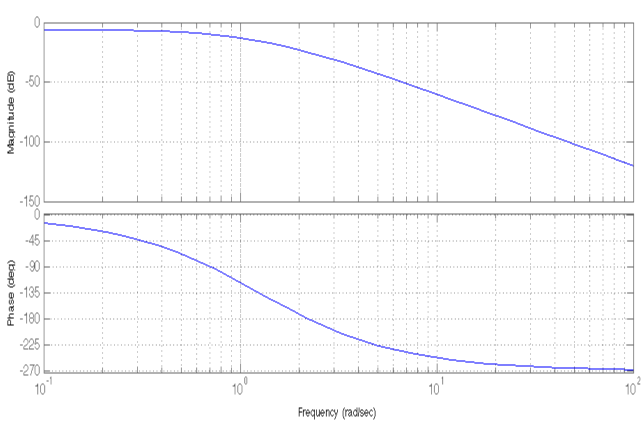


Figure 15. Bode diagrams of to $\mathbf{P}(s) = \frac{1}{(s+1)^2(s+2)}$.

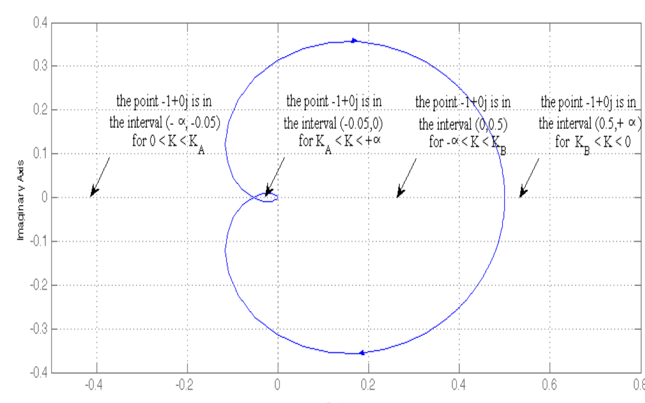


Figure 16. Polar plot of $\mathbf{P}(s) = \frac{1}{(s+1)^2(s+2)}$.

Exercize *2.1:* (*High gain feedback control*). Consider the system

$$\mathbf{P}(s) = \frac{1}{s-10} \tag{24}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 13 and the Nyquist plot in Figure 14. The number of counterclockwise tours around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is 0. Since $n^{+}(\mathbf{p}) = 1$, by the Simplified Nyquist criterion the closed-loop system is unstable, therefore, P cannot be stabilized with a unitary proportional feedback control action.

If we use a proportional feedback control action $u = -Ky$, with real K, in order to investigate to what extent stabiliity of the closed-loop system can be achieved by increasing $|K|$ we apply the Nyquist criterion to

$$\mathbf{P}(s) = \frac{K}{s-10} \tag{25}$$

in place of (24). To this aim, let us go back to the Nyquist plot of $\mathbf{P}$ in Figure 14. Notice that the plot crosses the negative real axis at the point -0.1. If we imagine to let the point $-1+0j$ vary on the negative real axis from $-\infty$ to 0, we change the relative position of the point $-1+0j$ and the crossing point -0.1. Note also that there is a value of K, say $K := K_A$ such that the point $-1+0j$ and the crossing point -0.1 will be coincident. This corresponds to the case in which for some value of $K = K_A$ the Nyquist plot of $K\mathbf{P}$ crosses the point $-1+0j$. Therefore, for this value of K the closed-loop system is not asymptotically stable. For $K \in (0, K_A)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its left, while for $K \in (K_A, +\infty)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its right. In the first situation, we see from Figure 17 that the number of counterclockwise tours of $-1+0j$ is 0 while in the second situation the number of counterclockwise tours around $-1+0j$ is 1. Since $n^{+}(\mathbf{p}) = 1$, for $K \in (0, K_A)$ the closed-loop system is unstable, while for $K \in (K_A, +\infty)$ the closed-loop system is asymptotically stable. Therefore, by increasing the amplitude K (beyond K_A) of the proportional feedback control action we stabilize the process. On the other hand, for $K < 0$ the point $-1+0j$ is positioned on the positive real axis and the number of counterclockwise tours around $-1+0j$ is -1, therefore the closed-loop system is not asymptotically stable. Note that the closed-loop system with proportional control action $u = -Ky$ is

$$W(s) = \frac{K\mathbf{P}(s)}{1+K\mathbf{P}(s)} = \frac{K}{s-10+K} \tag{26}$$

The pole of the closed-loop system is $s = 10 - K$ which is negative for $K > 10$. Therefore, $K_A = 10$. The number K_A represents the critical value of K for which the closed-loop system change its stability behaviour depending from the values of K.

Exercize *2.2:* (*Low gain feedback control*). Consider the system

$$\mathbf{P}(s) = \frac{1}{(s+1)^2(s+2)} \tag{27}$$

The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 15 and the Nyquist plot in Figure 16. The number of counterclockwise tours around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is 0. Since $n^{+}(\mathbf{p}) = 0$, by the Simplified Nyquist criterion the closed-loop system is asymptotically stable and P can be stabilized with a unitary proportional feedback control action. However, if we consider the amplified process $20\mathbf{P}(j\omega)$, we discover from the Nyquist plot and the Simplified Nyquist criterion that P cannot be stabilized with a unitary proportional feedback control action. This means that the stability of the closed-loop system is not compatible with increasing values of the proportional control action.

If we use a proportional feedback control action $u = -Ky$, with real K, in order to investigate to what extent stability of the closed-loop system can be maintained by increasing $|K|$ we apply the Nyquist criterion to $K\mathbf{P}(j\omega)$. We will see that it is not possible to increase $|K|$ without losing the property of the closed-plop stability. To this aim, let us go back to the Nyquist plot of $\mathbf{P}$ in Figure 16. Notice that the plot crosses the real axis at the points -0.05 and 0.5. If we imagine to let the point $-1+0j$ vary on the real axis from $-\infty$ to $+\infty$, we change the relative position of the point $-1+0j$ and the crossing points -0.05 and 0.5. Note also that there are two values of K, say $K_A > 0$ and $K_B < 0$, such that the point $-1+0j$ and the crossing point -0.05 (resp. 0.5) will be coincident. This corresponds to the case in which for some values of K the Nyquist plot of $K\mathbf{P}$ crosses the point $-1+0j$. Therefore, for these values of K the closed-loop system is not asymptotically stable. For $K \in (0, K_A)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its left, while for $K \in (K_A, +\infty)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its right. In the first situation, we see from Figure 16 that the number of counterclockwise tours of $-1+0j$ is 0 while in the second situation the number of counterclockwise tours of $-1+0j$ is -1. Since $n^{+}(\mathbf{p}) = 0$, for $K \in (0, K_A)$ the closed-loop system is asymptotically stable, while for $K \in (K_A, +\infty)$ the closed-loop system is unstable. Therefore, by increasing the amplitude K (below K_A) of the proportional feedback control action we stabilize the process. For $K \in (-\infty, K_B)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its right, while for $K \in (K_B, 0)$ the Nyquist plot of $K\mathbf{P}$ crosses the negative real axis leaving the point $-1+0j$ to its left. In the first situation, we see from Figure 16 that the number of counterclockwise tours of $-1+0j$ is -1 while in the second situation the number of counterclockwise tours of $-1+0j$ is 0. Since $n^{+}(\mathbf{p}) = 0$, for $K \in (-\infty, K_B) \cup (0, K_A)$ the closed-loop system is asymptotically stable, while for other vaues of $K \in (K_B, 0) \cup (-\infty, K_B)$ the closed-loop system is unstable.

Note that the closed-loop system with proportional control action $u = -Ky$ is

$$W(s) = \frac{K\mathbf{P}(s)}{1+K\mathbf{P}(s)} = \frac{K}{K+(s+1)^2(s+2)} \tag{28}$$

The poles of the closed-loop system are the roots of $\mathbf{w}(s) = K + (s+1)^2(s+2)$. Therefore, we can study how these roots

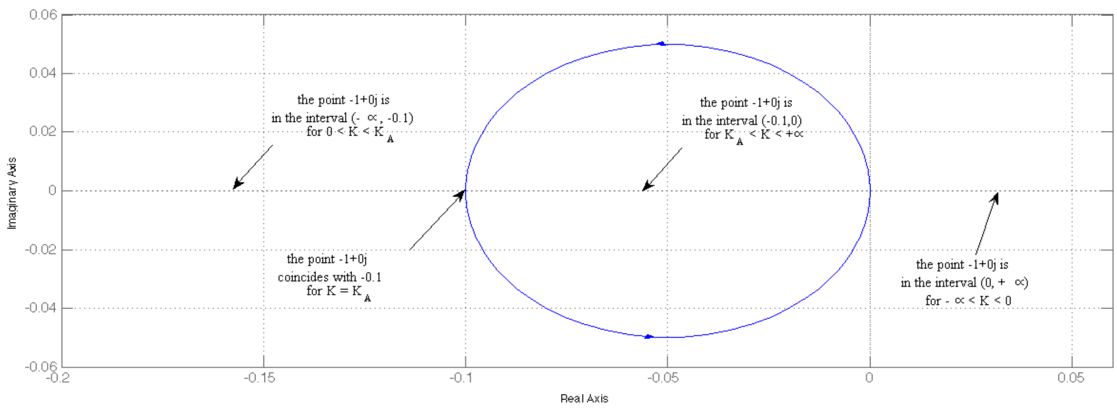


Figure 17. Nyquist plot of $\mathbf{P}(s) = K\frac{1}{s-10}$.

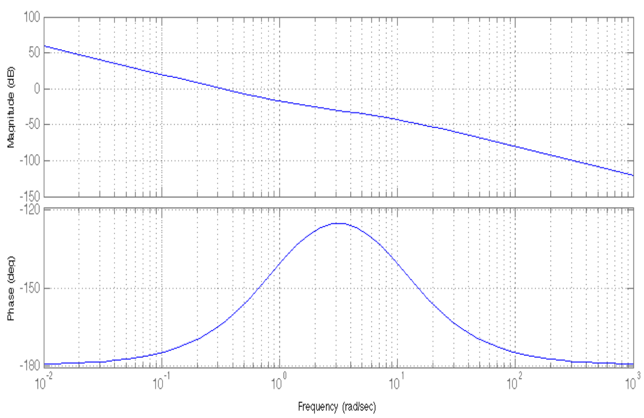


Figure 18. Bode diagrams of to $\mathbf{P}(s) = \frac{s+1}{s^2(s+10)}$.

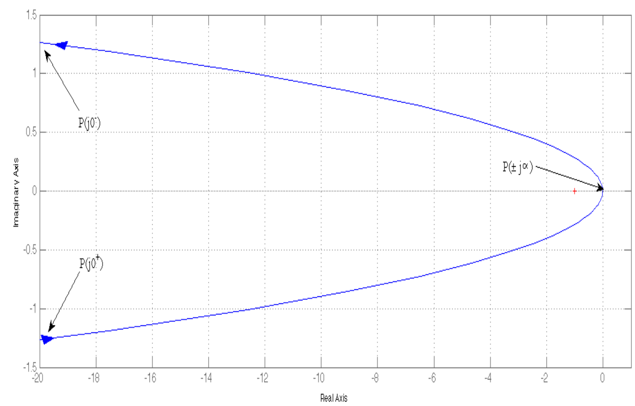


Figure 19. Polar plot of $\mathbf{P}(s) = \frac{s+1}{s^2(s+10)}$.

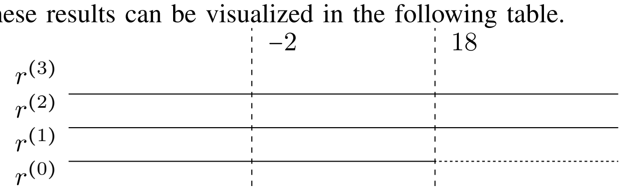
vary with K by calling upon the Routh criterion. The Routh table generated by $\mathbf{w}(s)$ is

$$\begin{array}{c|cc} r^{(3)} & 1 & 5 \\ r^{(2)} & 4 & 2+K \\ r^{(1)} & -K+18 & \\ r^{(0)} & K+2 & \end{array} \quad (29)$$

Note that the values of K for which the elements of the first column are zero determine the values of K_A and K_B. We can discuss the number of variations and permanencies in the first column of the Routh table as follows. First, we discuss the sign of each $r_1^{(j)}$, $j = 1, \dots, 3$:

- $r_0^{(0)} = 0$ for $K = -2$, $r_1^{(1)} = 0$ for $K = 18$
- $r_1^{(3)}$ and $r_1^{(2)}$ are positive for all k
- $r_1^{(1)} > 0$ for $K < 18$
- $r_0^{(1)} > 0$ for $K > -2$

These results can be visualized in the following table.



Therefore, we have

- for $K = -2$ and $K = 18$ the table is not regular
- for $K < -2$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 2$
- for $K \in (-2, 18)$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 3$
- for $K > 18$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 1$

We conclude

- for $K < -2$ the table is regular and $N_V(\mathbf{p}) = 1$ and $N_P(\mathbf{p}) = 2 \Rightarrow p(\lambda)$ is not Hurwitz
- for $K \in (-2, 18)$ the table is regular and $N_V(\mathbf{p}) = 0$ and $N_P(\mathbf{p}) = 3 \Rightarrow p(\lambda)$ is Hurwitz
- for $k > 18$ the table is regular and $N_V(\mathbf{p}) = 2$ and $N_P(\mathbf{p}) = 1 \Rightarrow p(\lambda)$ is not Hurwitz

It is clear that $K_A = 18$ and $K_B = -2$, which are the critical values of K for which te closed-loop system changes its stability behaviour.

Exercize *2.3:* (*Conditionally low/high gain feedback control*). Consider the system

$$\mathbf{P}(s) = \frac{(s+5)^2}{s(s+1)(s+0.1)(s+75)^2} \quad (30)$$

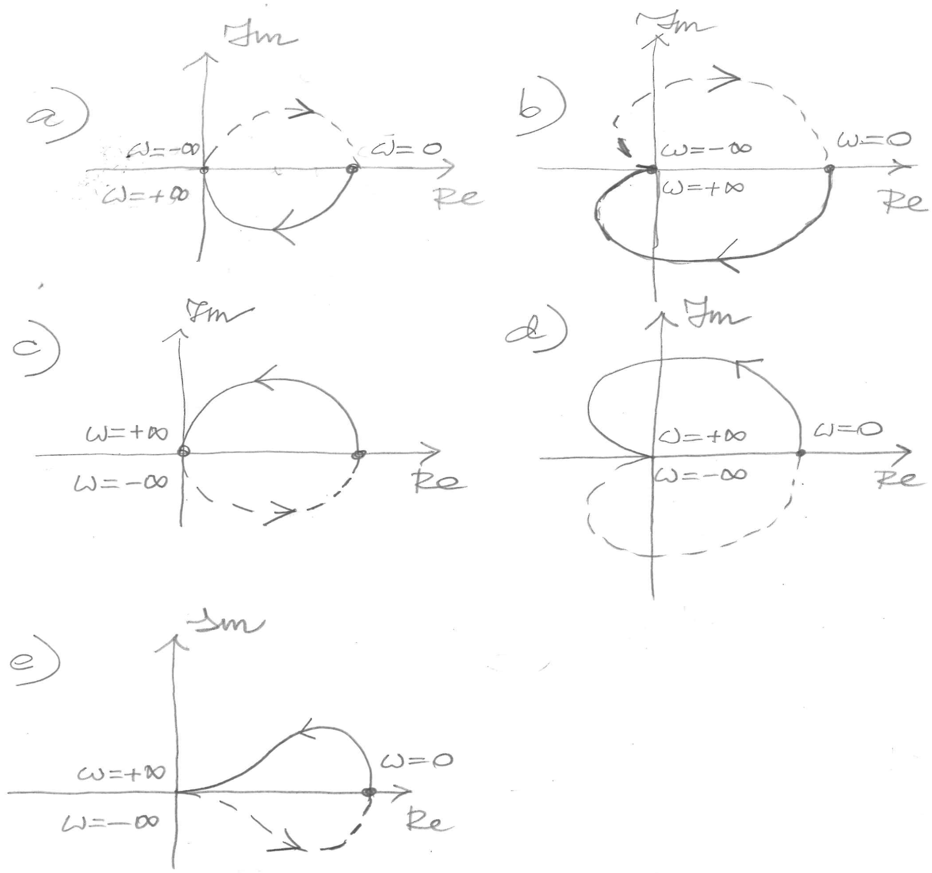


Figure 20. Nyquist plots of a), b), c), d) and e) of exercize 2.5 (for $K = 1$).

and discuss the stability of the closed-loop system under the action of a proportional control action $u = -Ky$, with $K \in \mathbb{R}$.

From the Nyquist plot we have for some $K_1, K_2 K_3 > 0$:

- if $0 < K < K_1$ the number of counterclockwise tours on behalf of $\mathbf{P}(j\omega)$ around $-1 + 0j$ is 0
- if $K_1 < K < K_2$ the number of counterclockwise tours on behalf of $\mathbf{P}(j\omega)$ around $-1 + 0j$ is -2
- if $K_2 < K < K_3$ the number of counterclockwise tours on behalf of $\mathbf{P}(j\omega)$ around $-1 + 0j$ is 0
- if $K_3 < K < +\infty$ the number of counterclockwise tours on behalf of $\mathbf{P}(j\omega)$ around $-1 + 0j$ is -2
- if $K < 0$ the number of counterclockwise tours on behalf of $\mathbf{P}(j\omega)$ around $-1 + 0j$ is -1

We conclude that since $n^+(\mathbf{p}) = 0$

- if $0 < K < K_1$ or $K_2 < K < K_3$ the closed-loop system is asymptotically stable
- for any other value of K the closed-loop system is not asymptotically stable.

The values of K_1, K_2, K_3 can be obtained from the denominator of the closed-loop system $\mathbf{W}(s) = \frac{K\mathbf{P}(s)}{1+K\mathbf{P}(s)}$ which is

$$\begin{aligned} &\mathrm{DEN}(\mathbf{W}(s)) = \mathrm{NUM}(1 + K\mathbf{P}(s)) \\ &= s^5 + 151.1s^4 + 5790.1s^3 \\ &+(6202.5 + K)s^2 + (562.5 + 10K)s + 25K \end{aligned} \quad (31)$$

by applying the Routh criterion. The values of K_1, K_2, K_3 are the values of K for which we have a change in the number of sign variations in the first column of the Routh table:

$$K_1 \approx 42.37,\ K_2 \approx 11063,\ K_3 \approx 644973. \quad (32)$$

Exercize *2.4:* (*Derivative plus proportional feedback control*). Consider the double integrator

$$\mathbf{P}(s) = \frac{1}{s^2} \quad (33)$$

The Nyquist plot crosses the point $-1 + 0j$ and, therefore, the closed-loop system with unitary feedback is not asymptotically

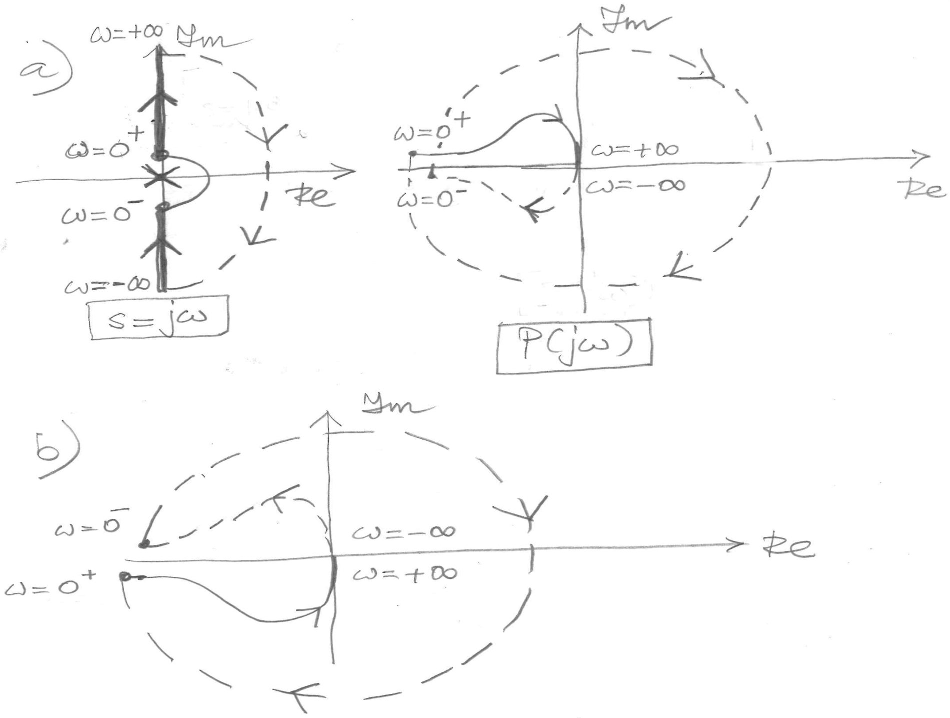


Figure 21. Nyquist plots of a), b) of exercize 2.6 (for $K = 1$). For ex. a) it is also shown the path for $s = j\omega$.

stable. Even the closed-loop system with proportional control action $u = -Ky$ is not asymptotically, since The Nyquist plot crosses the point $-1 + 0j$. Therefore, the double integrator is not stabilizable with proportional feedback action. Consider a more general feedback control action

$$u = \frac{s+1}{s+10} y \tag{34}$$

This control action consists into the sum of two terms $\frac{s}{s+10} y$ and $\frac{1}{s+10} y$, which approximately correspond to a derivative (i.e. $\dot{y}$) and, respectively, a proportional action (i.e. y): for reasonable low frequencies the Bode diagram of $\frac{s}{s+10}$ and $\frac{1}{s+10}$ is approximately that of s and 1 (this approximation is even better if the time constant of the pole is decreased). The Bode plot of $\mathbf{P}(j\omega)$ is drawn in Figure 18 and the Nyquist plot in Figure 19. The number of counterclockwise tours around the point $-1+j0$ on behalf of the Nyquist plot of $\overrightarrow{\mathbf{P}(j\omega)}$ is 0. Since $n^{+}(\mathbf{p}) = 0$, by the Extended Nyquist criterion the closed-loop system is asymptotically stable, therefore, P can be stabilized with a derivative plus proportional feedback control action. The stability behaviour does not change if we increase the proportional control action K: this can be seen as in the above examples.

Exercize *2.5:* Draw the Nyquist plots of the following $\mathbf{P}(s)$ (see Figure 20) and discuss the stability of the closed-loop system for $K \in \mathbb{R}$:

$$\begin{aligned}
a)\mathbf{P}(s) &= \frac{K}{1+\tau s}, \ \tau > 0,\\
b)\mathbf{P}(s) &= \frac{K}{(1+\tau_1 s)(1+\tau_2 s)}, \ \tau_1, \tau_2 > 0,\\
c)\mathbf{P}(s) &= \frac{K}{1-\tau s}, \ \tau > 0,\\
d)\mathbf{P}(s) &= \frac{K}{(1-\tau_1 s)(1-\tau_2 s)}, \ \tau_1, \tau_2 > 0,\\
e)\mathbf{P}(s) &= \frac{K}{(1-\tau_1 s)(1+\tau_2 s)}, \ \tau_1 > \tau_2 > 0,
\end{aligned} \tag{35}$$

Exercize *2.6:* Draw the Nyquist plots of the following $\mathbf{P}(s)$ (see Figure 21 for a), b), Figure 22 for c), d) and Figure 23 for e), f)) and discuss the stability of the closed-loop system

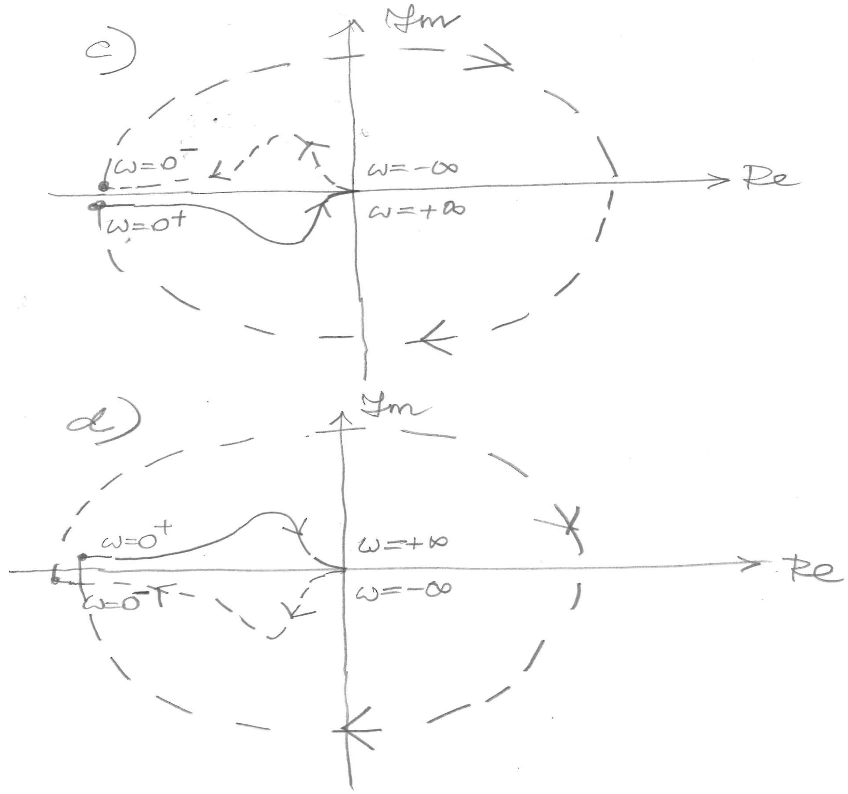


Figure 22. Nyquist plots of c), d) of exercize 2.6 (for $K = 1$).

for $K \in \mathbb{R}$:

$$\begin{aligned} a)\mathbf{P}(s) &= \frac{K(1-\tau s)}{s^2},\ \tau > 0, \\ b)\mathbf{P}(s) &= \frac{K}{s^2(1-\tau s)},\ \tau > 0, \\ c)\mathbf{P}(s) &= \frac{K(1+\tau_1 s)}{s^2(1+\tau_2 s)},\ \tau_1 > \tau_2 > 0, \\ d)\mathbf{P}(s) &= \frac{K(1+\tau_1 s)}{s^2(1+\tau_2 s)},\ \tau_2 > \tau_1 > 0, \\ e)\mathbf{P}(s) &= \frac{K(1+\tau_1 s)}{s^2(1+\tau_2 s)(1+\tau_3 s)(1+\tau_4 s)}, \\ & \tau_1 > \tau_2 > \tau_3 > \tau_4 > 0, \\ f)\mathbf{P}(s) &= \frac{K(1+\tau_1 s)}{s(\tau_2 s - 1)},\ \tau_1 > \tau_2 > 0, \end{aligned} \tag{36}$$

III. Stability margins: phase and gain margins

When the Nyquist plot has a regular behaviour in the sense that the curve of $|\mathbf{P}(j\omega)|$ has a monotonically decreasing behaviour as a function of ω and $\mathbf{P}(s)$ has no poles in $\mathbb{C}^+$ (i.e. $n^+(\mathbf{p}) = 0$), from the Nyquist plot of $\mathbf{P}$ it is possible to extract information not only on the stability of the closed-loop system $\mathbf{P}$ but also on its sensitivity to move towards instability with respect to variations of the proportional control action K. Clearly, more the Nyquist plot of $\mathbf{P}$ is far from encircling the point $-1 + 0j$ more the closed-loop system is far from instability, more the Nyquist plot of $\mathbf{P}$ is close to encircling the point $-1 + 0j$ more the closed-loop system is close to instability. By increasing (resp. decreasing) the proportional control action K the Nyquist plot of $K\mathbf{P}$ is subject to a radial expansion (resp. contraction) and for some value of K it crosses the point $-1+0j$, which corresponds to some poles of the closed-loop system on the imaginary axis. By restricting ourselves to the cases in which the Nyquist plot of $\mathbf{P}$ has a regular behaviour, it is useful to characterize to what extent the closed-loop system remains asymptotically stable under variations of the gain K through some *stability margins.*

Definition *3.1: The gain margin m_g is the ratio $\dfrac{1}{|\mathbf{P}(j\omega^\circ)|}$ where ω° : Arg$\{\mathbf{P}(j\omega^\circ)\} = -\pi$.*

The gain margin is given in dB: $m_g(dB) = -|\mathbf{P}(j\omega^\circ)|_{dB}$. Moreover, the gain margin $m_g(dB)$ is positive when the

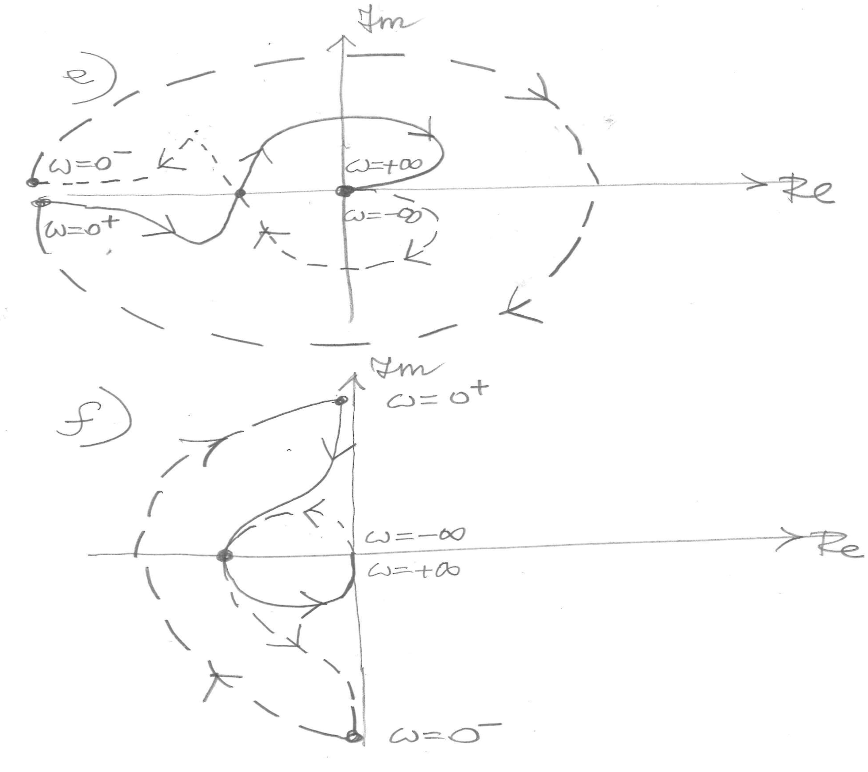


Figure 23. Nyquist plots of e), f) of exercize 2.6 (for $K = 1$).

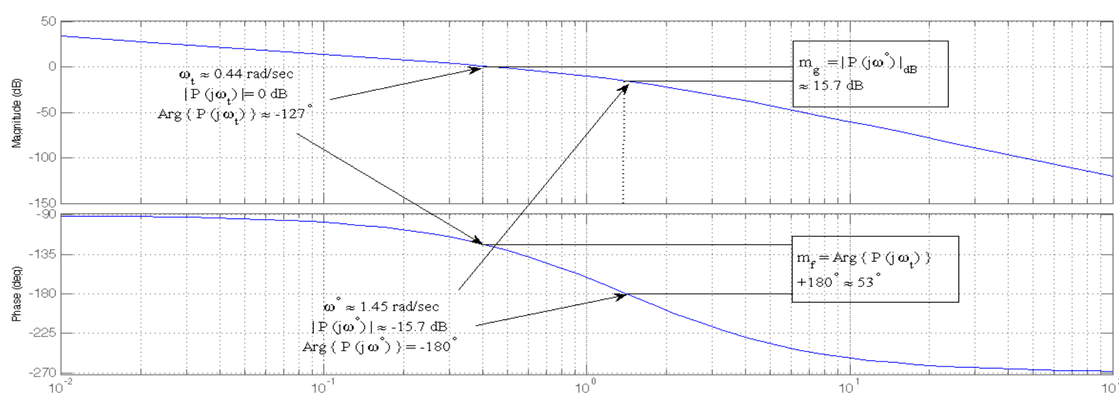


Figure 24. Phase and gain margins on the Bode diagrams of to $\mathbf{P}(s) = \frac{1}{s(s+1)(s+2)}$.

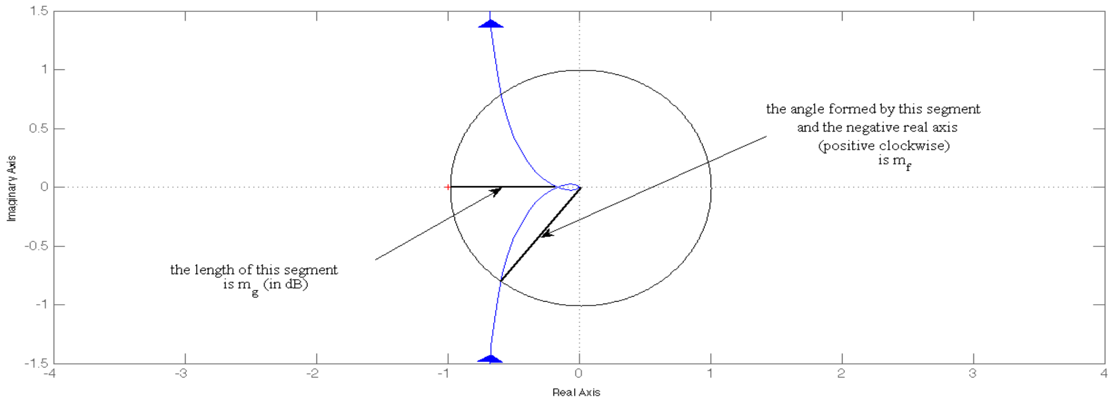


Figure 25. Phase and gain margins on polar plot of $\mathbf{P}(s) = \frac{1}{s(s+1)(s+2)}$.

Nyquist plot of $\mathbf{P}(j\omega)$ crosses the negative real axis leaving the point $-1+0j$ to its left ($|\mathbf{P}(j\omega^\circ)| < 1$) and negative when the Nyquist plot of $\mathbf{P}(j\omega)$ crosses the negative real axis leaving the point $-1+0j$ to its right ($|\mathbf{P}(j\omega^\circ)| > 1$). The gain margin gives a quantitative evaluation of how close, on the negative real axis, are the point $-1+0j$ and the point at which $\mathbf{P}(j\omega)$ crosses the negative real axis itself.

Definition *3.2: The cross-over frequency ω_t is the frequency $\omega_t : |\mathbf{P}(j\omega_t)| = 1$.*

Definition *3.3: The phase margin m_f (or m_φ) is defined as $Arg\{\mathbf{P}(j\omega_t)\} - (-\pi) = Arg\{\mathbf{P}(j\omega_t)\} + \pi$.*

The phase margin is given in degrees or radiant. Moreover, the phase margin is negative when the Nyquist plot of $\mathbf{P}(j\omega)$ crosses inward the circumference with radius 1 and center at 0 leaving the point $-1+0j$ to its right ($\text{Arg}\{\mathbf{P}(j\omega_t)\} < -180°$) and positive when the Nyquist plot of $\mathbf{P}(j\omega)$ crosses inward the circumference with radius 1 and center at 0 leaving the point $-1+0j$ to its left ($\text{Arg}\{\mathbf{P}(j\omega_t)\} > -180°$). The phase margin gives a quantitative evaluation of how close are, on the circumference with radius 1 and center at 0, the point $-1+0j$ and the point at which $\mathbf{P}(j\omega)$ crosses the circumference itself.

Gain and phase margins are easily calculated from the Bode and the Nyquist plot.

Exercize *3.1:* Consider the system

$$\mathbf{P}(s) = \frac{1}{s(s+1)(s+2)} \tag{37}$$

The phase margin is $\text{Arg}\{\mathbf{P}(j\omega^\circ)\} - (-\pi)$ for $\omega_t : |\mathbf{P}(j\omega_t)| = 1$, i.e. $\omega_t \approx 0.44$ rad/sec and $\text{Arg}\{\mathbf{P}(j\omega^\circ)\} - (-\pi) = 53°$ (see Figure 24). The gain margin m_g is $-|\mathbf{P}(j\omega^\circ)|_{dB}$ for ω° : $\text{Arg}\{\mathbf{P}(j\omega^\circ)\} = -\pi$, i.e. $\omega^\circ \approx 1.45$ rad/sec and $-|\mathbf{P}(j\omega^\circ)|_{dB} = 15.7dB$ (see Figure 24). In Figure 25 gain and phase margins are graphically pointed out.