Notes on Linear Control Systems: Module XII

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Abstract—**Transient output response design in frequency domain.**

I. Transient performances

One of the most important characteristics of a control system is the transient error response. As we have seen, since the aim of a control system is to guarantee some tracking performances (steady-state errors within given tolerances, disturbance compensation and so on), the transient error response must be modified if necessary until it is satisfactory according to given criteria (rise time, settling time, maximal overshooting). If the transient error response is not satisfactory, the process $\mathbf{P}(s)$ must be modified, for example, in a feedback interconnection in such a way to achieve the desired performances. In general, it is possible to modify the transient error response by inserting in the control loop a controller $\mathbf{G}(s)$ in series interconnection with the process $\mathbf{P}(s)$. The transient perfomances of the closed-loop forced response can be evaluated, for instance, on the base of the Bode plot of the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$. The most significant and commonly used parameters on the Bode plots are the following.

Definition *1.1:* *The resonance peak M_R is the maximal value of $|\mathbf{W}(j\omega)|_{dB} - |\mathbf{W}(0)|_{dB}$ for $\omega \geq 0$. The frequency $\omega = \omega_R$ at which $|\mathbf{W}(j\omega)|_{dB} - |\mathbf{W}(0)|_{dB}$ is maximal is the resonance frequency. The cut-off frequency ω_F is the frequency at which $|\mathbf{W}(j\omega)|_{dB} - |\mathbf{W}(0)|_{dB} = -3dB$.*

Resonance peak and cut-off frequency of the closed-loop system $\mathbf{W}(s)$ can be related (through experimental relations or look-up tables) to rise time and maximal overshooting, defined in time domain for the closed-loop forced output response to step inputs. On the other hand, resonance peak and cut-off frequency of the closed-loop system can be related to the cross-over frequency and phase margin of the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$. This connection can be established on the *Nichols chart* (the study of Nichols chart goes beyond the scope of these notes and will be omitted).

We limit ourselves to study how it is possible to change the cross-over frequency and phase margin of the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ through elementary control actions $\mathbf{G}(s)$ which, through the Nichols chart, introduce corresponding variations in the resonance peak and cut-off frequency of the closed-loop system $\mathbf{W}(s)$ and, as already noticed, in rise time and maximal overshooting of the closed-loop forced output response to step inputs. The class of control actions we consider

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consists of series interconnections of three elementary control actions: *proportional*, *anticipative* or *attenuative* actions.

An *anticipative* action is modeled as

$$\mathbf{G}(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s} \qquad (1)$$

with $\tau_a > 0$ and $m_a > 1$, which corresponds (approximately) to the combination of a derivative with a proportional action. The Bode plot of the function (1) versus frequency $\omega_N := \omega \tau_a$ with parameters $\tau_a > 0$ and $m_a > 1$ is given in Figure 1. Here, ω_N is the (normalized to τ_a) frequency scale in the Bode plot of $\mathbf{G}(s)$ in (1) while ω is the frequency scale of the Bode plot of $\mathbf{P}(s)$. The parameter m_a represents one of the numbers which label the curves in Figure 1, therefore by selecting m_a we select different pairs of curves (magnitude and phase) in Figure 1. By setting $\tau_a = \frac{\omega_N^*}{\omega^*}$ we establish that the frequency ω^* on the Bode plots of $\mathbf{P}(s)$ corresponds to the frequency ω_N^* on the Bode plots of $\mathbf{G}(s)$ in Figure 1. At the same time we establish that there is a scale factor τ_a between the frequency scale of the Bode plot of $\mathbf{P}(s)$ and the frequency scale of the curves (magnitude and phase) of $\mathbf{G}(s)$ in Figure 1 with label m_a. This also means that we can obtain the Bode plot of $\mathbf{P}(s)\mathbf{G}(s)$ simply by summing the Bode plot of $\mathbf{G}(s)$ in Figure 1 with label m_a at each frequency ω_N to the Bode plot of $\mathbf{P}(s)$ at frequency $\omega = \frac{\omega_N}{\tau_a}$. For example, if the open-loop system is

$$\mathbf{P}(s) = \frac{1}{s(1+s)} \qquad (2)$$

and we select a $\mathbf{G}(s)$ with $m_a = 10$ and $\tau_a = 2$

$$\mathbf{G}(s) = \frac{1 + 2s}{1 + \frac{1}{5}s} \qquad (3)$$

the Bode plot of the series interconnection

$$\mathbf{P}(s)\mathbf{G}(s) = \frac{1}{s(1+s)}\frac{1+2s}{1+\frac{1}{5}s} \qquad (4)$$

is the sum of the magnitude and the phase curves in Figure 1 with label $m_a = 10$ at each frequency ω_N with the magnitude and the phase of (2) at each frequency $\omega = \frac{\omega_N}{2}$ (see Figure 2). In particular, with respect to the Bode plot of $\mathbf{P}(s)$, at $\omega = 1$ rad/sec we have a magnitude increment $+6dB$ (which is the value of the magnitude curve in (3) with label $m_a = 10$ at $\omega_N = 2$ rad/sec) and a phase increment $+52°$ (which is the value of the phase curve in (3) with label $m_a = 10$ at $\omega_N = 2$ rad/sec).
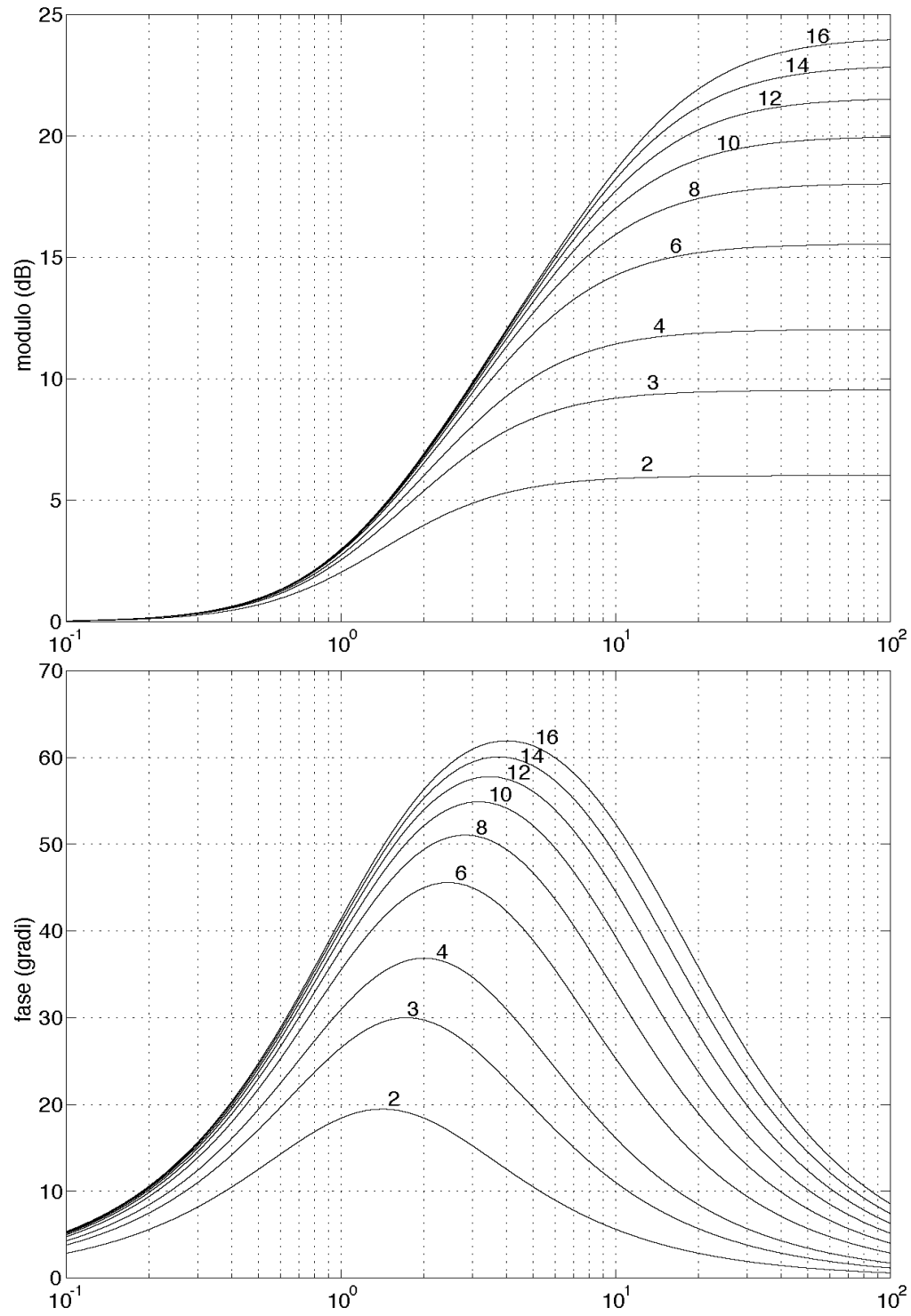
An *attenuative* action is modeled by

$$\frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s} \qquad (5)$$

with $\tau_i > 0$ and $m_i > 1$. The Bode plot of the function (5) versus frequency $\omega_N := \omega \tau_i$ with parameters $\tau_i > 0$ and $m_i > 1$

Diagrammi di Bode delle reti anticipatrici al variare di m_a. In ascissa appare la pulsazione normalizzata $\omega \tau_a$.

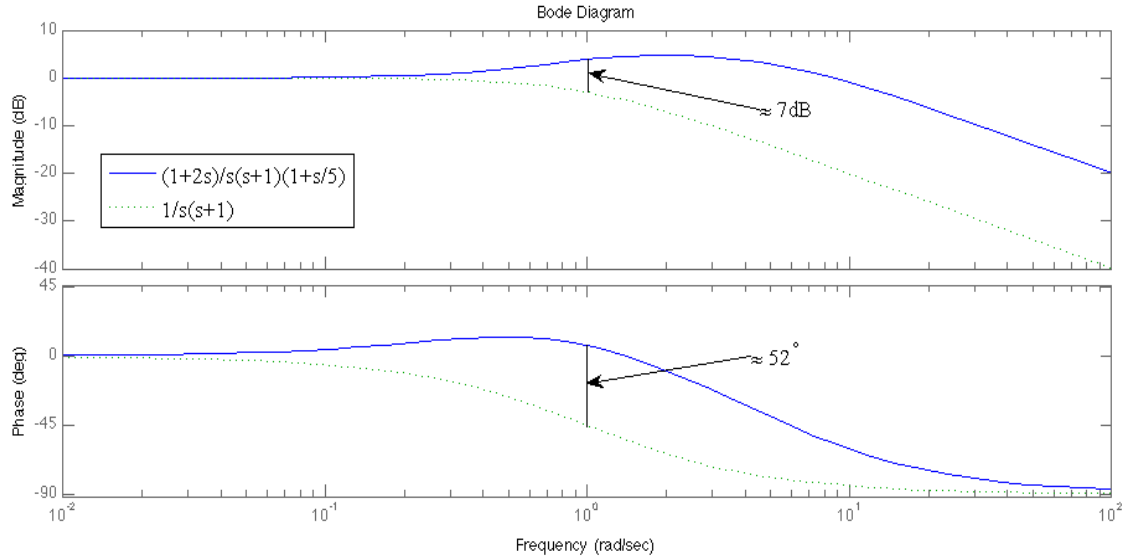Figure 1. Anticipative control actions.

Figure 2. Bode plots of $\mathbf{P}(s) = \frac{1}{s(1+s)}$ and the compensated $\mathbf{P}(s)\mathbf{G}(s) = \frac{1}{s(1+s)}\frac{1+2s}{1+\frac{1}{5}s}$.
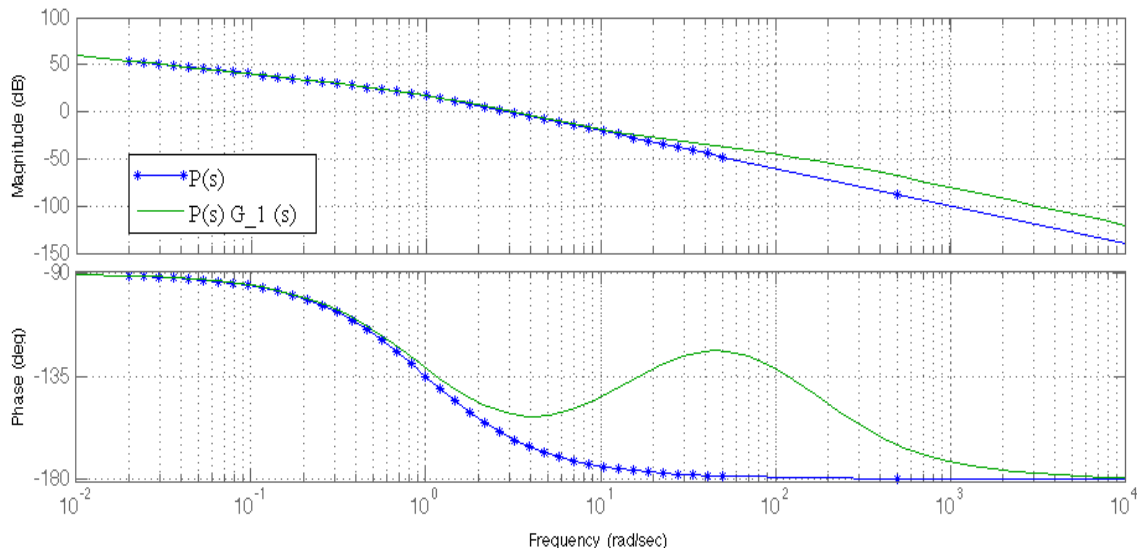

Figure 3. Bode plots of $\mathbf{P}(s) = \frac{1}{s(s+1)}$ and $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{1+\frac{0.2}{3}s}{1+\frac{0.02}{3}s}$.

can be derived from the curves of Figure 1 simply by changing the sign of the values of the magnitude and phase plotted in Figure 1. All the remarks discussed for the anticipative action can be extended to the attenuative action, by using m_i and τ_i (instead of m_a and τ_a, respectively) as parameters.

With proportional, anticipative and attenuative actions, we can distinguish four different basic situations in which we can change the cross-over frequency and/or the phase margin of the open-loop system:

(i) increase the phase margin while maintaining the same cross-over frequency
(ii) increase the phase margin by decreasing the cross-over frequency
(iii) increase the phase margin and the cross-over frequency
(iv) increase the phase margin and decreasing the cross-over frequency.

We will discuss each of these situations, showing how each situation corresponds to a different choice of the parameters m_a and τ_a (respectively, m_i and τ_i).
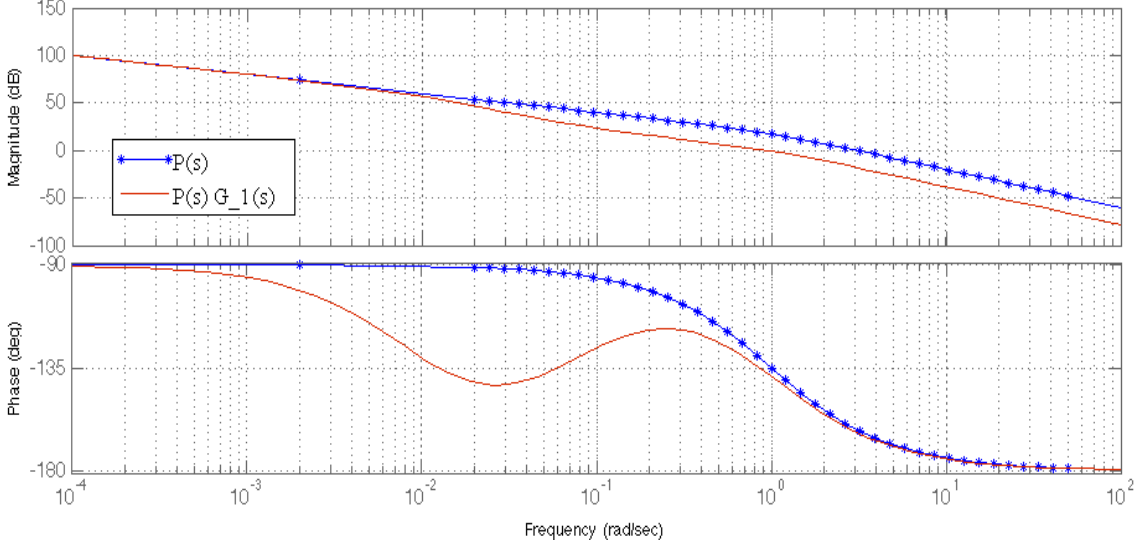
Figure 4. Bode plots of $\mathbf{P}(s) = \frac{10}{s(s+1)}$ and $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{10}{s(s+1)}\frac{1+\frac{25}{2}s}{1+100s}$.

A. Increase the phase margin while maintaining the same cross-over frequency

Let

$$\mathbf{P}(s) := \frac{10}{s(s+1)} \qquad (6)$$

be the open-loop system and assume that desired values of cross-over frequency and phase margin are $\omega_t^* \approx 3$ rad/sec and, respectively, $m_f^* \geq 28°$. Notice that (see Figure 3) the magnitude of (6) at $\omega = 3$ rad/sec is ≈ 0 dB and the phase of (6) at $\omega = 3$ rad/sec is $-162°$. Therefore, to achieve a phase margin $m_f^* \geq 28°$ we must increase the phase of (6) by at least $10°$ at $\omega = 3$ rad/sec without altering the magnitude significantly.

We increase the phase of (6) by means of an anticipative action (1) and selecting τ_a and m_a in such a way to gain the required phase increment at $\omega = 3$ rad/sec in the series interconnection of (1) and (6). Notice (see 1) that the phase of (1) for $m_a = 10$ is $\approx 10°$ at $\omega_N = 0.2$ rad/sec with a magnitude of ≈ 0dB. Therefore, select $m_a = 10$ and set $\omega_N^* = 0.2$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = \frac{0.2}{3}$. With this choice of τ_a we make correspond ω_N^* to ω_t^* and the increment of $\approx 10°$ (contributed by the anticipative action) at the frequency $\omega_t^* = 3$ rad/sec in the Bode plot of $\mathbf{P}(s)$. Define

$$\mathbf{G}_1(s) := \frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s} = \frac{1+\frac{0.2}{3}s}{1+\frac{0.02}{3}s} \qquad (7)$$

The resulting series interconnection $\mathbf{P}(s)\mathbf{G}_1(s)$ has the desired values of cross-over frequency $\omega_t^* \approx 3$ rad/sec and phase margin $m_f^* \geq 28°$ (see Figure 3).

B. Increase the phase margin by decreasing the cross-over frequency.

Let

$$\mathbf{P}(s) := \frac{10}{s(s+1)} \qquad (8)$$

be the open-loop system and assume that desired values of the phase margin is $m_f^* \geq 40°$. Notice that (see 4) the magnitude of (8) at $\omega = 1$ rad/sec is ≈ 17 dB and the phase of (8) at $\omega = 1$ rad/sec is $-135°$. Therefore, to achieve a phase margin $m_f^* \geq 45°$ we can decrease the magnitude of (8) by 17 dB at $\omega = 1$ rad/sec without decreasing the phase significantly.

We decrease the magnitude of (8) by means of an attenuative action (5) and selecting τ_i and m_i in such a way to have the required magnitude decrement at $\omega = 1$ rad/sec in the series interconnection of (5) and (8). Notice (see Figure 1) that the phase of (5) for $m_i = 8$ is $\approx -4°$ at $\omega_N = 100$ rad/sec with a magnitude of ≈ -18dB. Therefore, select $m_i = 8$ and $\omega_N^* = 100$ rad/sec and, consequently, $\tau_i = \frac{\omega_N^*}{\omega_t^*} = 100$. Define

$$\mathbf{G}_1(s) := \frac{1+\frac{\tau_i}{m_i}s}{1+\tau_i s} = \frac{1+\frac{25}{2}s}{1+100s} \qquad (9)$$

The resulting series interconnection $\mathbf{P}(s)\mathbf{G}_1(s)$ has the desired values of cross-over frequency $\omega_t^* = 1$ rad/sec and phase margin $m_f^* \geq 45°$ (see Figure 4).

C. Increase the phase margin and the cross-over frequency.

Let

$$\mathbf{P}(s) := \frac{100}{s^2(s+1)} \qquad (10)$$

be the open-loop system and assume that desired values of cross-over frequency and phase margin are $\omega_t^* = 10$ rad/sec and, respectively, $m_f^* \geq 60°$. Notice that (see Figure 5) the magnitude of (10) at $\omega = 10$ rad/sec is -20 dB and the phase of (10) at $\omega = 10$ rad/sec is $-264°$. Therefore, to achieve cross-over frequency and phase margin $\omega_t^* = 10$ rad/sec and, respectively, $m_f^* \geq 60°$ we must increase the magnitude of (10) by 20 dB at $\omega = 10$ rad/sec and increase the phase of (10) at least by $144°$ at $\omega = 10$ rad/sec.
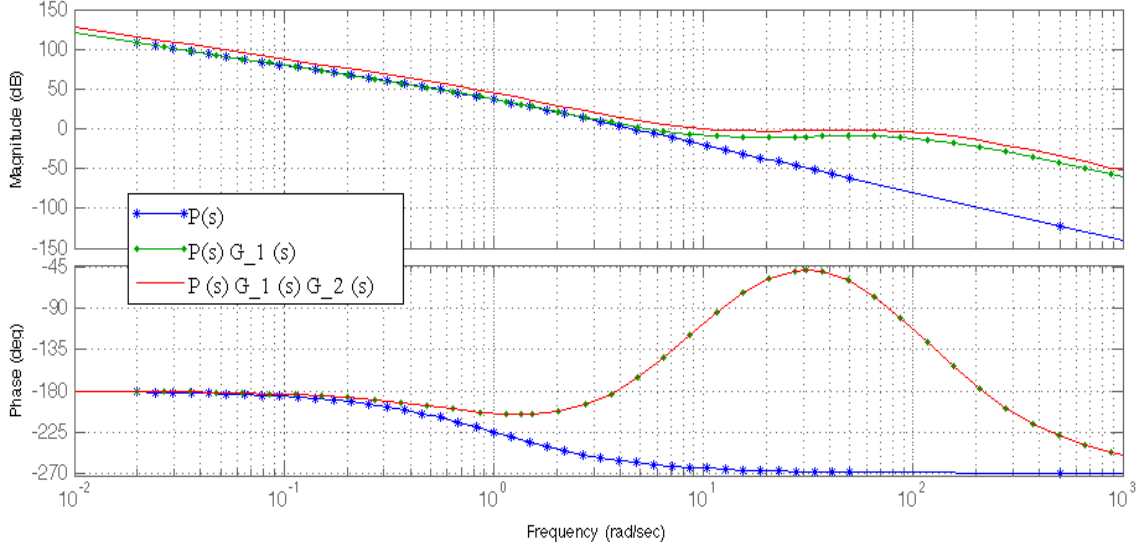
Figure 5. Bode plots of $\mathbf{P}(s) = \frac{100}{s^2(s+1)}$, $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s^2(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = 2.5119\frac{100}{s^2(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$.
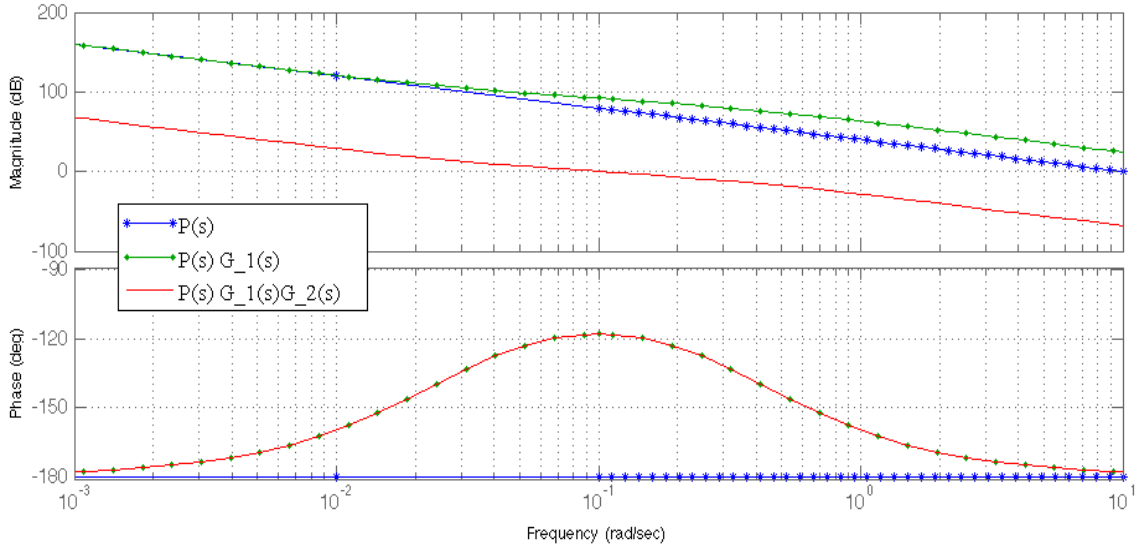

Figure 6. Bode plots of $\mathbf{P}(s) = \frac{100}{s^2}$, $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s^2}\frac{1+40s}{1+\frac{5}{2}s}$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = 2.5119e^{-5}\frac{100}{s^2}\frac{1+40s}{1+\frac{5}{2}s}$.

First, we increase the phase of (10) by means of an anticipative action (1) and selecting τ_a and m_a in such a way to have the needed phase increment at $\omega = 10$ rad/sec in the series interconnection of (1) and (10). Notice (see Figure 1) that the phase of (1) for $m_a = 10$ is $\approx 40°$ at $\omega_N = 1$ rad/sec with a magnitude increase of ≈ 3dB. Therefore, select $m_a = 10$ and set $\omega_N^* = 1$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 0.1$. Therefore, to obtain a phase increment of $\approx 160° > 144°$ we take 4 anticipative actions (1)

$$\mathbf{G}_1(s) := \left(\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}\right)^4 = \left(\frac{1+0.1s}{1+0.01s}\right)^4 \qquad (11)$$

Next, we increment the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)$ at $\omega = 10$ rad/sec by means of a gain increment. On account of the fact that the magnitude of the series interconnection of $\mathbf{P}(s)\mathbf{G}_1(s)$ is $-20+12 = -8$ dB at $\omega = 10$ rad/sec and the phase is $-264° + 160° = 104° \geq 60°$ at $\omega = 10$, define

$$\mathbf{G}_2(s) := K = 8\text{dB} = 2.5119 \qquad (12)$$

The resulting series interconnection $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ has the desired values of cross-over frequency $\omega_t^* = 10$ rad/sec and phase margin $m_f^* \geq 60°$ (see Figure 5). The resulting controller is the series interconnection $\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s)$.
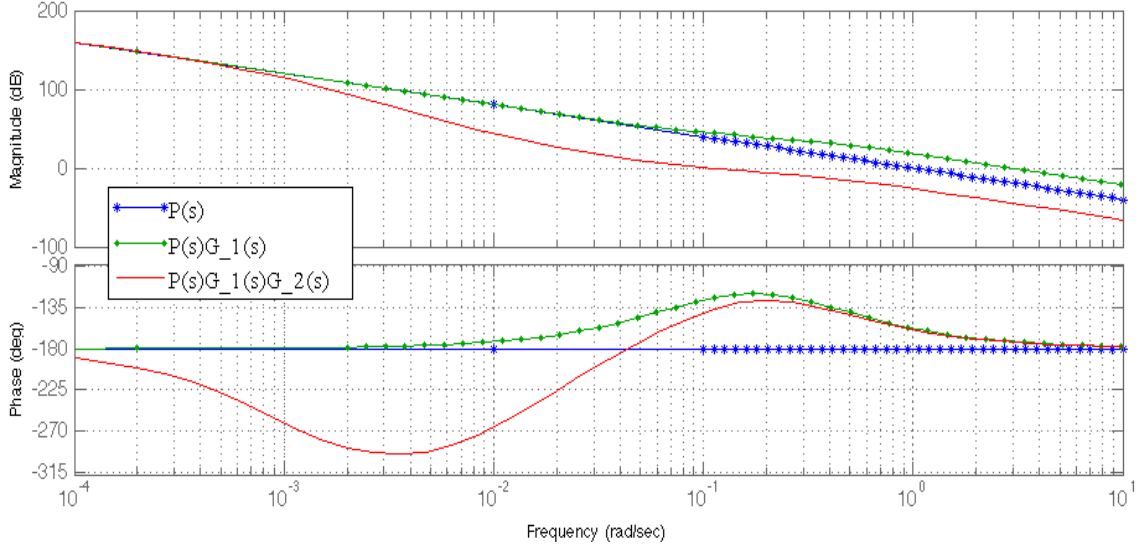
Figure 7. Bode plots of $\mathbf{P}(s) = \frac{1}{s^2}$, $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{1}{s^2}\left(\frac{1+10s}{1+\frac{10}{3}s}\right)^2$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{1}{s^2}\left(\frac{1+10s}{1+\frac{10}{3}s}\right)^2 \frac{1+\frac{500}{7}s}{1+1000s}\frac{1+\frac{250}{3}s}{1+1000s}$.

D. Increase the phase margin and decreasing the cross-over frequency.

Let

$$\mathbf{P}(s) := \frac{100}{s^2} \tag{13}$$

be the open-loop system and assume that desired values of cross-over frequency and phase margin are $\omega_t^* = 0.1$ rad/sec and, respectively, $m_f^* \geq 60°$. Notice that (see Figure 6) the magnitude of (13) at $\omega = 0.1$ rad/sec is 80 dB and the phase of (13) at $\omega = 0.1$ rad/sec is $-180°$. Therefore, to achieve cross-over frequency and phase margin are $\omega_t^* = 0.1$ rad/sec and, respectively, $m_f^* \geq 60°$ we must decrease the magnitude of (13) by 80 dB at $\omega = 0.1$ rad/sec and increase the phase of (13) at least by $60°$ at $\omega = 0.1$ rad/sec.

First, we increase the phase of (13) by means of an anticipative action (1) and selecting τ_a and m_a in such a way to have the required phase increment at $\omega = 0.1$ rad/sec in the series interconnection of (1) and (13). Notice (see Figure 1) that the phase of (1) for $m_a = 16$ is $\approx 62°$ at $\omega_N = 4$ rad/sec with a magnitude increase of ≈ 12dB. Therefore, select $m_a = 16$ and set $\omega_N^* = 4$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 40$. Define

$$\mathbf{G}_1(s) := \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s} = \frac{1 + 40s}{1 + \frac{5}{2}s} \tag{14}$$

Next, the magnitude can be decreased by means of either a gain decrement or a an attenuative action (5). However, the gain decrement cannot be used if a reduction of the steady-state error response is required (see module XI).

If a gain reduction is allowed, on account of the fact that the magnitude of the series interconnection of $\mathbf{G}_1(s)$ and (13) has a magnitude of $80 + 12 = 92$ dB at $\omega = 0.1$ rad/sec we define

$$\mathbf{G}_2(s) := K = -92dB = 0.000025 \tag{15}$$

The resulting series interconnection $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ has the desired values of cross-over frequency $\omega_t^* = 0.1$ rad/sec and phase margin $m_f^* \geq 60°$ (see Figure 6).

If a gain reduction is not allowed, we will decrease the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)$ by means of an attenuative action (5). However, if so, when attenuative the magnitude by (5) we have at the same a phase attenuation. Therefore, when using the series of an anticipative action with an attenuative action, it is necessary to increase the phase a little more to compensate for the phase decrement due to the attenuative action. Let

$$\mathbf{P}(s) := \frac{1}{s^2} \tag{16}$$

be the open-loop system and assume that desired values of cross-over frequency and phase margin are $\omega_t^* = 0.1$ rad/sec and, respectively, $m_f^* \geq 30°$. Notice that (see Figure 7) the magnitude of (16) at $\omega = 0.1$ rad/sec is 40 dB and the phase of (16) at $\omega = 0.1$ rad/sec is $-180°$. Therefore, to achieve cross-over frequency and phase margin are $\omega_t^* = 0.1$ rad/sec and, respectively, $m_f^* \geq 30°$ we must decrease the magnitude of (16) by 40 dB at $\omega = 10$ rad/sec and increase the phase of (16) at least by $30°$ at $\omega = 10$ rad/sec.

First, we increase the phase of (16) by means of an anticipative action (1) and selecting ω_N and m_a in such a way to have the required phase increment at $\omega = 0.1$ rad/sec in the series interconnection of (1) and (16). Notice (see 1) that the phase of (1) for $m_a = 3$ is $\approx 26°$ at $\omega = 1$ rad/sec with a magnitude increase of ≈ 2.5dB. Therefore, select $m_a = 3$ and set $\omega_N^* = 1$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 10$. In order to obtain a phase increment of $\approx 50°$ ($20°$ more than necessary, since these extra phase will be lost when attenuative the magnitude) we take two anticipative actions (1)

$$\mathbf{G}_1(s) := \left(\frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}\right)^2 = \left(\frac{1 + 10s}{1 + \frac{10}{3}s}\right)^2 \tag{17}$$
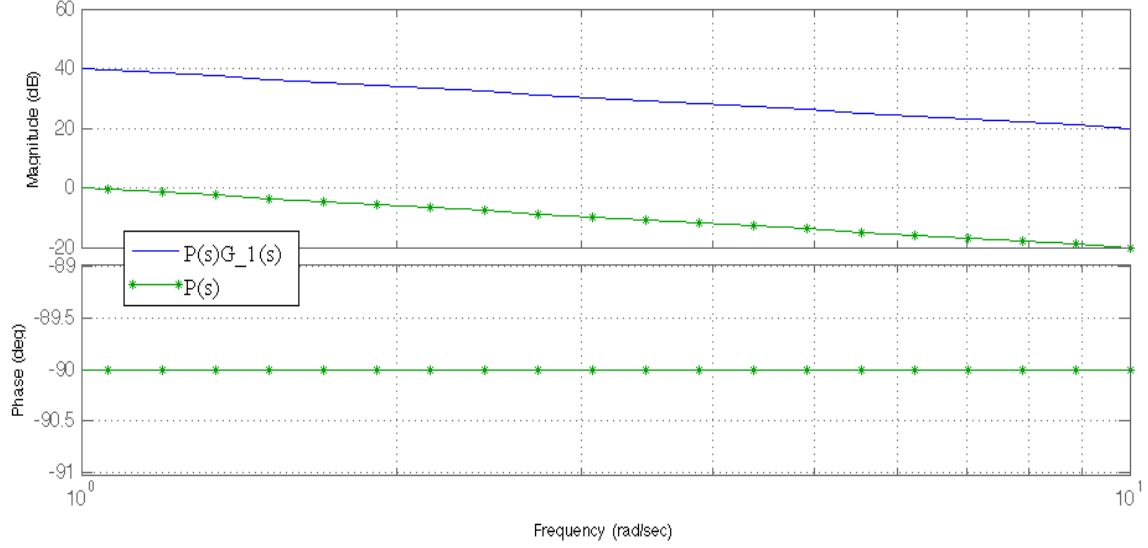
Figure 8. Bode plots of $\mathbf{P}(s) = \frac{1}{s}$ and $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s}$ (exercize 2.1).
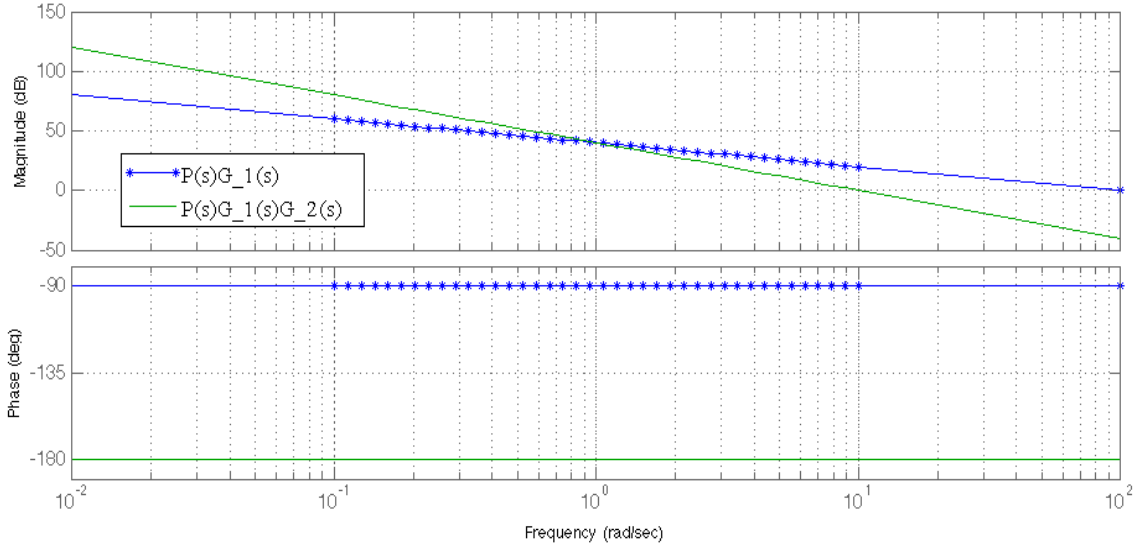

Figure 9. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s}$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{100}{s^2}$ (exercize 2.1).

Next, we decrement the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)$ at $\omega = 0.1$ rad/sec. On account of the fact that the magnitude of the series interconnection of $\mathbf{P}(s)\mathbf{G}_1(s)$ is $40 + 5 = 45$ dB at $\omega = 0.1$ rad/sec and since (see 1) the magnitude of (5) for $m_i = 14$ (resp. $m_i = 12$) at $\omega = 100$ rad/sec is ≈ -23 dB (resp. ≈ -22) dB with phase $\approx -9°$ (resp. $\approx -8°$), in order to obtain a magnitude decrement of ≈ 45 dB we take two attenuative actions (5): for the first, select $m_i' = 14$ and set $\omega_N'^* = 100$ rad/sec and, consequently, $\tau_i' = \frac{\omega_N'^*}{\omega_t^*} = 1000$, for the second select $m_i'' = 12$ and set $\omega_N''^* = 100$ rad/sec and, consequently,

$\tau_i'' = \frac{\omega_N''^*}{\omega_t^*} = 1000$. Define

$$\mathbf{G}_2(s) := \frac{1 + \frac{\tau_i'}{m_i'}s}{1 + \tau_i's}\frac{1 + \frac{\tau_i''}{m_i''}s}{1 + \tau_i''s} = \frac{1 + \frac{500}{7}s}{1 + 1000s}\frac{1 + \frac{250}{3}s}{1 + 1000s} \quad (18)$$

The ensuing phase decrement is $\approx -17°$, therefore since (17) has been designed with an extra phase increment of $20°$, the phase after the attenuation of (18) is $\geq -150°$ (which corresponds to the desired phase margin). The resulting series interconnection $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)$ has the desired values of cross-over frequency $\omega_t^* = 0.1$ rad/sec and phase margin $m_f^* \geq 30°$ (see Figure 7). The resulting controller is the series interconnection $\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s)$.
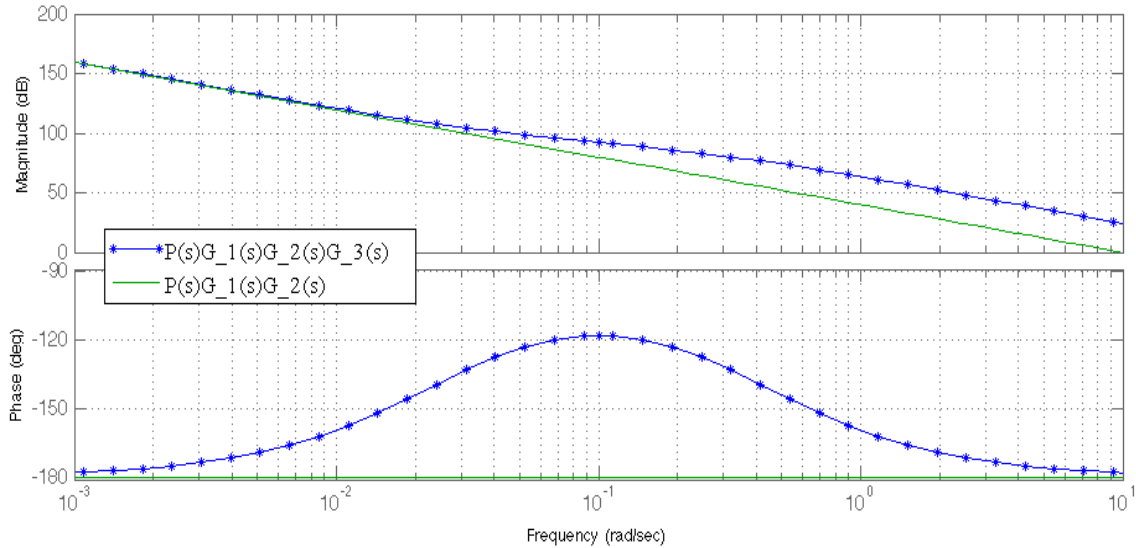
Figure 10. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{100}{s^2}$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s) = \frac{100}{s^2}\frac{1+40s}{1+\frac{5}{2}s}$ (exercize 2.1).
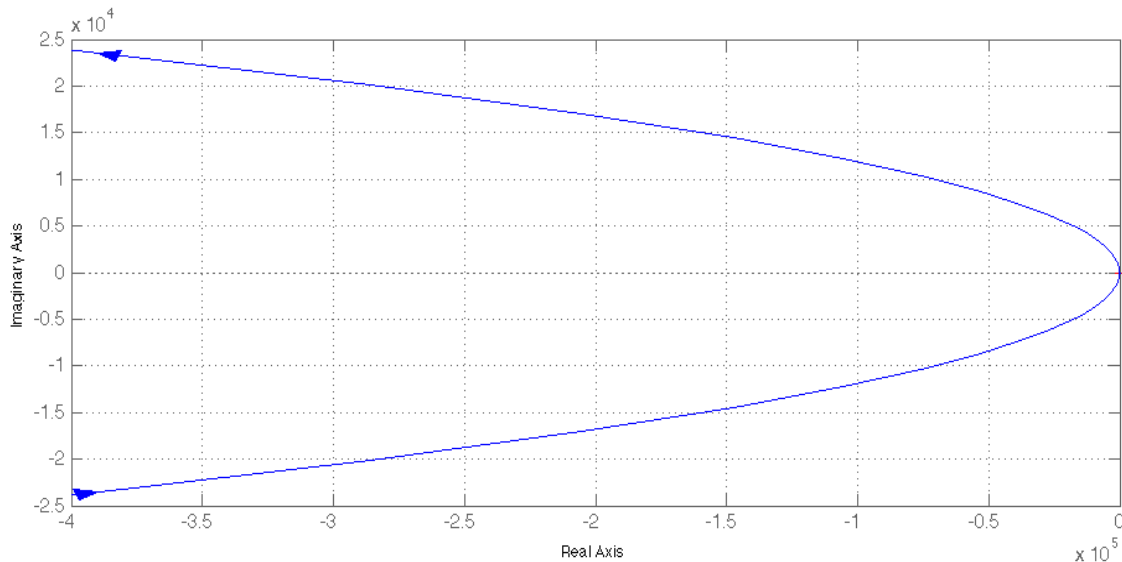

Figure 11. Nyquist plot of $\mathbf{P}(s)\mathbf{G}(s) = \frac{0.0025}{s^2}\frac{1+40s}{1+\frac{5}{2}s}$ (exercize 2.1).

II. MIXED STEADY-STATE AND TRANSIENT PERFORMANCES REQUIREMENTS

A first class of exercizes we consider is the one which combines steady-state (tracking error and disturbances compensation) with transient performance requirements (phase margin and crossover frequency requirements).

Exercize 2.1: Given

$$\mathbf{P}(s) := \frac{1}{s} \qquad (19)$$

find $\mathbf{G}(s)$ such that the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ is asymptotically stable with

(i) the absolute value of the steady-state error response to $\mathbf{v}(t) := t$ is ≤ 0.01

(ii) zero steady-state response to input disturbances $\mathbf{d}(t) := 1$

and the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ has

(iii) crossover frequency $\omega_t^* = 0.1$ rad/sec and phase margin $m_f^* \geq 60°$.

Let $e^{(ss,1)}$ denote the steady-state error response to $\mathbf{v}(t) := t$ and $y^{(ss,0)}$ the steady-state output response to an input disturbance $\mathbf{d}(t) := 1$. Since $\mathbf{P}(s)$ has a pole in $s = 0$ then the closed-loop system $\frac{\mathbf{P}(s)}{1+\mathbf{P}(s)}$ is of type 1 and $e^{(ss,1)}$ is constant

and non-zero. In particular,

$$e^{(ss,1)} = \frac{1}{(s\mathbf{P})|_{s=0}} = 1 \qquad (20)$$

and therefore $|e^{(ss,1)}| = 1 > 0.01$. We must reduce $e^{(ss,1)}$ by a proportional control action $\mathbf{G}_1(s) = K_1$ (see Figure 8):

$$e^{(ss,1)} = \frac{1}{(s\mathbf{P}\mathbf{G}_1)|_{s=0}} = \frac{1}{K_1} \qquad (21)$$

Therefore, for the closed-loop system $\frac{\mathbf{G}_1(s)\mathbf{P}(s)}{1+\mathbf{G}_1(s)\mathbf{P}(s)}$

$$|e^{(ss,1)}| \le 0.01 \Leftrightarrow |K_1| \ge 100 \qquad (22)$$

Define

$$\mathbf{G}_1(s) := K_1 = 100 \qquad (23)$$

Despite the fact that $\mathbf{P}(s)\mathbf{G}_1(s)$ has a simple pole in $s = 0$ and since there are no integral control action before the entrance point of the disturbance in the control loop , the steady-state response to input disturbances $\mathbf{d}(t) := 1$ tends to infinity as time increases. We must introduce an additional integral control action $\mathbf{G}_2(s) = \frac{1}{s}$ before the entrance point of the disturbance or equivalently the product $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ must have a pole in $s = 0$ (see Figure 9). The closed-loop system

$$\frac{\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}{1+\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}$$

has a zero steady-state response to input disturbances $\mathbf{d}(t) := 1$.

Since $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ has now two poles in $s = 0$ the closed-loop system

$$\frac{\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}{1+\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}$$

is of type 2 and $e^{(ss,1)} = 0$. Therefore, (22) is not any more necessary and the gain of $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ can be decreased by subsequent series proportional control actions.

Notice that at $\omega = 0.1$ rad/sec the magnitude of $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ is 80dB and the phase is $-180°$. Therefore, in order to have $\omega_t^* = 0.1$ rad/sec and $m_f^* \ge 60°$ it is necessary to decrease the magnitude at $\omega = 0.1$ rad/sec by 80dB and increase the phase by $60°$. First, we increase the phase (at least) by $60°$: select $m_a = 16$, $\omega_N^* = 4$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 40$. Define

$$\mathbf{G}_3(s) := \frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s} = \frac{1+40s}{1+\frac{5}{2}s} \qquad (24)$$

(see Figure 10)

Note that the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)$ at $\omega = 0.1$ rad/sec is $80 + 12 = 92$dB and the phase is $-180° + 62° = -118°$ (the phase margin is $62° \ge 60°$). Finally, we decrease the magnitude by 92dB: define

$$\mathbf{G}_4(s) := -92dB = 0.000025. \qquad (25)$$

The final formula for the controller is

$$\mathbf{G}(s) = \mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s) = \frac{0.0025}{s^2}\frac{1+40s}{1+\frac{5}{2}s}.$$

The stability of the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ can be ascertained by means of the Nyquist criterion (Figure 11): indeed, the number of poles of $\mathbf{G}(s)\mathbf{P}(s)$ in $\mathbb{C}^+$ is zero and zero is the number of counterclockwise encirclements of $-1 + 0j$ on behalf of $\mathbf{GP}(j\omega)$.

Exercize 2.2: Given

$$\mathbf{P}(s) := \frac{1}{s(s+1)} \qquad (26)$$

find $\mathbf{G}(s)$ such that for the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ is asymptotically stable with
(i) the absolute value of the steady-state error response to $\mathbf{v}(t) := t$ is ≤ 0.01
(ii) the absolute value of the steady-state response to input disturbances $\mathbf{d}(t) := t$ is ≤ 0.01
 and the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ has
(iii) crossover frequency $\omega_t^* = 10$ rad/sec and phase margin $m_f^* \ge 60°$.

Let $e^{(ss,1)}$ denote the steady-state error response to $\mathbf{v}(t) := t$ and $y^{(ss,1)}$ the steady-state output response to an input disturbance $\mathbf{d}(t) := t$. Since $\mathbf{P}(s)$ has a pole in $s = 0$ then the closed-loop system $\frac{\mathbf{P}(s)}{1+\mathbf{P}(s)}$ is of type 1 and $e^{(ss,1)}$ is constant and non-zero. In particular,

$$e^{(ss,1)} = \frac{1}{(s\mathbf{P})|_{s=0}} = 1 \qquad (27)$$

and therefore $|e^{(ss,1)}| = 1 > 0.01$. We must reduce $|e^{(ss,1)}|$ by a proportional control action $\mathbf{G}_1(s) = K_1$ (see Figure 12):

$$e^{(ss,1)} = \frac{1}{(s\mathbf{P}\mathbf{G}_1)|_{s=0}} = \frac{1}{K_1} \qquad (28)$$

Therefore, for the closed-loop system $\frac{\mathbf{G}_1(s)\mathbf{P}(s)}{1+\mathbf{G}_1(s)\mathbf{P}(s)}$

$$|e^{(ss,1)}| \le 0.01 \Leftrightarrow |K_1| \ge 100 \qquad (29)$$

Define

$$\mathbf{G}_1(s) := K_1 = 100 \qquad (30)$$

Despite the fact that $\mathbf{P}(s)\mathbf{G}_1(s)$ has a simple pole in $s = 0$ and since there are no integral control action before the entrance point of the disturbance in the control loop, the steady-state response to input disturbances $\mathbf{d}(t) := t$ tends to infinity as time increases. We must introduce an additional integral control action $\mathbf{G}_2(s) = \frac{1}{s}$ before the entrance point of the disturbance or equivalently the product $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ must have a pole in $s = 0$ (see Figure 13).

The closed-loop system

$$\frac{\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}{1+\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}$$

has a constant and non-zero steady-state response to input disturbances $\mathbf{d}(t) := t$. Since

$$y^{(ss,1)} = \frac{1}{s}\frac{\mathbf{P}(s)}{1+\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)}\Big|_{s=0} = \frac{1}{K_1} \qquad (31)$$

for the closed-loop system $\frac{\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s)}{1+\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{P}(s)}$

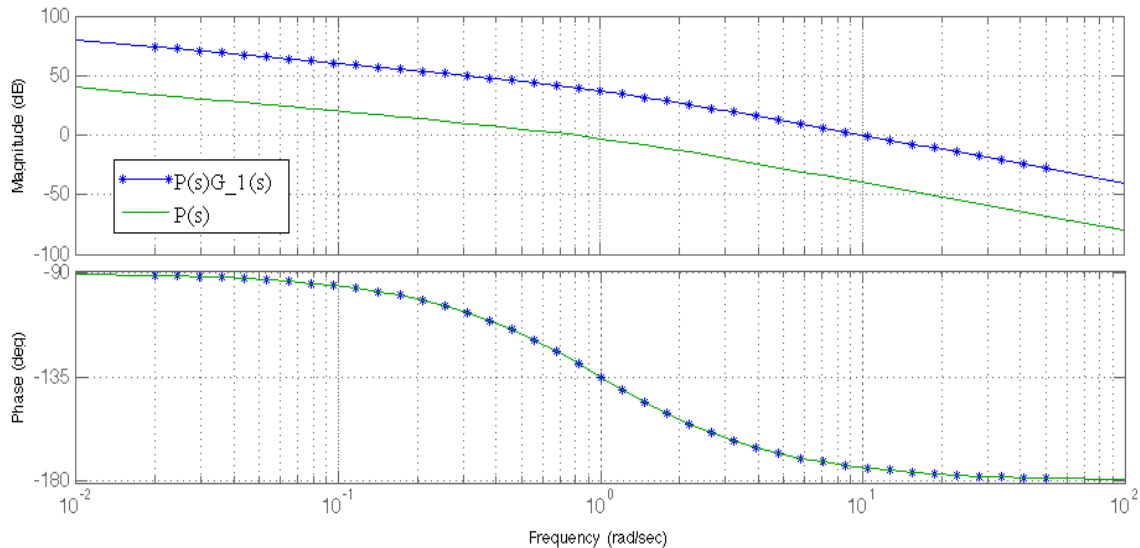$$|y^{(ss,1)}| \le 0.01 \Leftrightarrow |K_1| \ge 100 \qquad (32)$$

Figure 12. Bode plots of $\mathbf{P}(s) = \frac{1}{s(s+1)}$ and $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s(s+1)}$ (exercize 2.2).
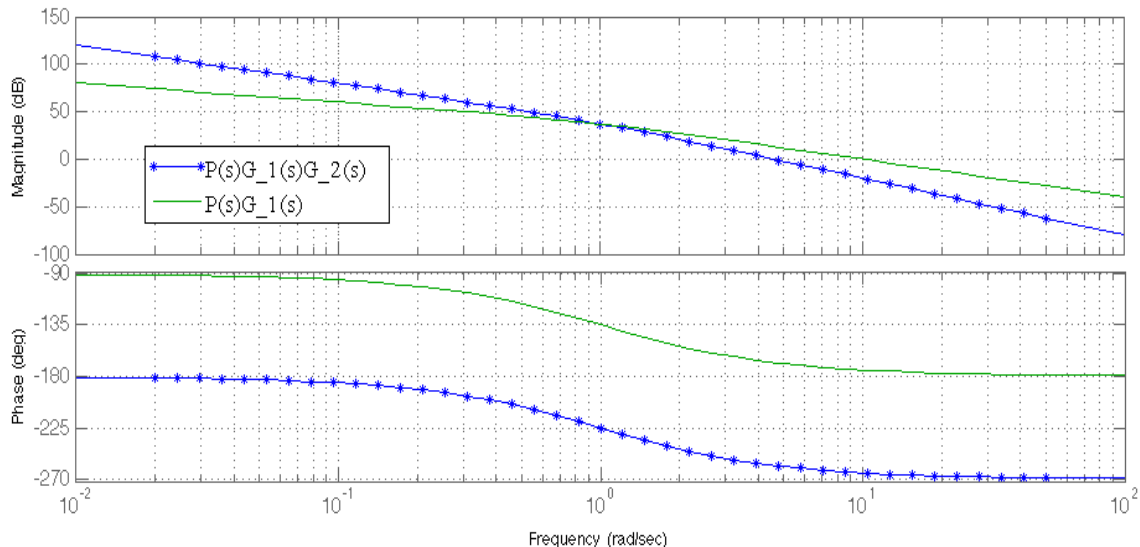

Figure 13. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{100}{s(s+1)}$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{100}{s^2(s+1)}$ (exercize 2.2).

Therefore, we do not need to reduce $|y^{(ss,1)}|$.

Since $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ has now two poles in $s = 0$ the closed-loop system

$$\frac{\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}{1 + \mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)}$$

is of type 2 and $e^{(ss,1)} = 0$. Therefore, (29) is not any more necessary. However, the gain of $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ cannot be decreased by subsequent series proportional control actions for the active constraint (32).

Notice that at $\omega = 10$ rad/sec the magnitude of $\mathbf{G}_2(s)\mathbf{G}_1(s)\mathbf{P}(s)$ is -20dB and the phase is $-264°$. Therefore, in order to have $\omega_t^* = 10$ rad/sec and $m_f^* \geq 60°$ it is

necessary to increase the magnitude at $\omega = 10$ rad/sec by 20dB and increase the phase by $144°$.

First, we increase the phase (at least) by $144°$: select $m_a = 10$, $\omega_N^* = 1$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 0.1$. Define

$$\mathbf{G}_3(s) := \left(\frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a}s}\right)^4 = \left(\frac{1 + 0.1s}{1 + 0.01s}\right)^4 \quad (33)$$

(see Figure 14)

Note that the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)$ at $\omega = 10$ rad/sec is $-20+12 = -8$dB and the phase is $-264°+160° = -104°$ (the phase margin is $76° \geq 60°$). Finally, we increase
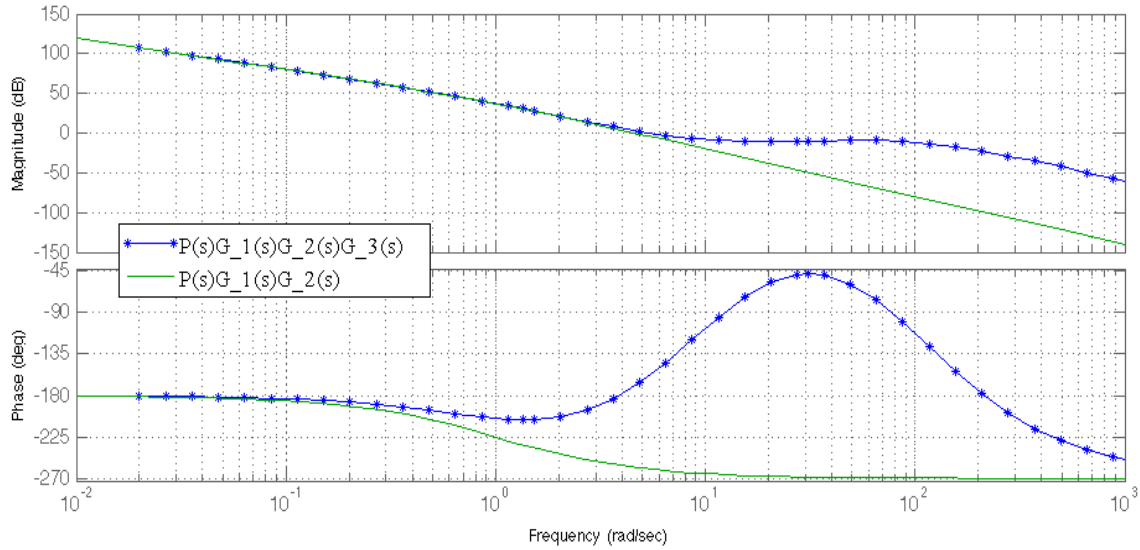
Figure 14. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = \frac{100}{s(s+1)}$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s) = \frac{100}{s(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$ (exercize 2.2).
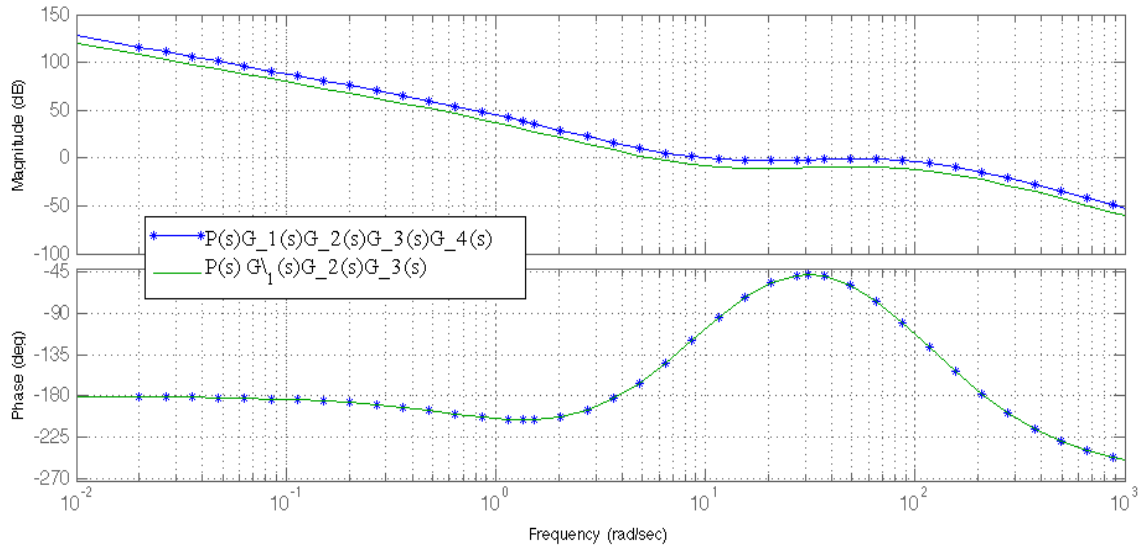

Figure 15. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s) = \frac{100}{s(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s) = 2.5119\frac{100}{s(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$ (exercize 2.2).

the magnitude by 8dB: define

$$\mathbf{G}_4(s) := 8dB = 2.5119 \qquad (34)$$

(see Figure 15). The final controller $\mathbf{G}(s)$ is $\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s)$.

The stability of the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ can be ascertained by means of the Nyquist criterion (Figure 16): indeed, the number of poles of $\mathbf{G}(s)\mathbf{P}(s)$ in $\mathbb{C}^+$ is zero and zero is the number of counterclockwise encirclements of $-1 + 0j$ on behalf of $\mathbf{GP}(j\omega)$.

Exercize 2.3: Consider the feedback loop of Figure 17 with

$$\mathbf{P}_1(s) := \frac{1}{s(s+2)}, \quad \mathbf{P}_1(s) := \frac{s-2}{(s+1)^2} \qquad (35)$$

Design the controller $\mathbf{G}(s)$ such that such that the closed-loop system is asymptotically stable with

(i) the absolute value of the steady-state error response to $\mathbf{v}(t) := t$ is ≤ 0.1,

(ii) the steady-state output response to constant disturbances $\mathbf{d}(t)$ is zero

and the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ has

(iii) crossover frequency $\omega_t^* \ge 0.5$ rad/sec and phase margin $m_f^* \ge 45°$.

First of all,it is convenient to compute the transfer function $\mathbf{P}(s)$ from u to y (setting $d = 0$). We obtain (notice, before $\mathbf{P}_2(s)$, the parallel interconnection of the proportional system
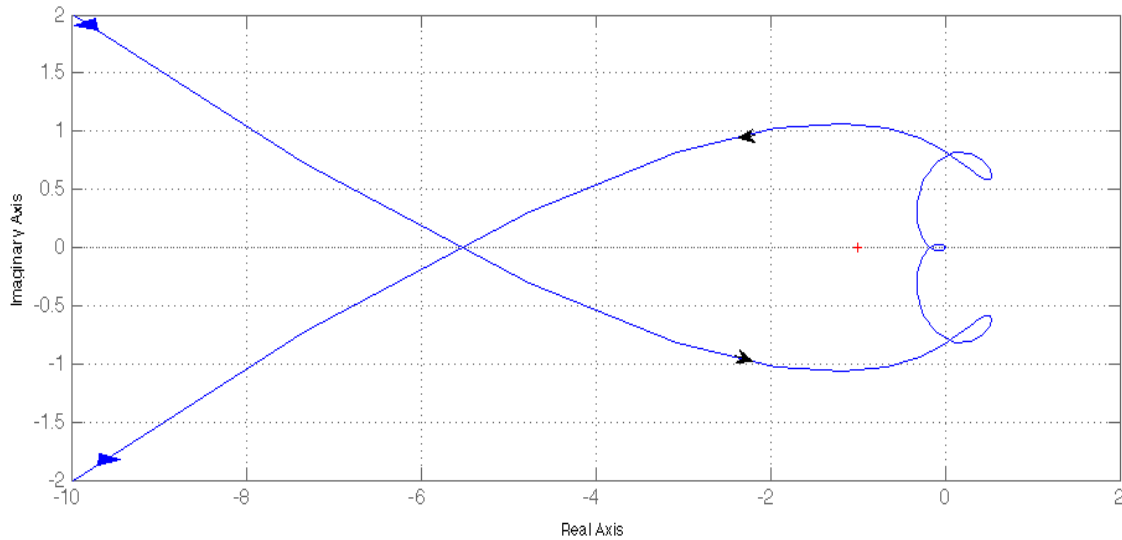
Figure 16. Nyquist plot of $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s)\mathbf{G}_3(s)\mathbf{G}_4(s) = 2.5119\frac{100}{s(s+1)}\left(\frac{1+0.1s}{1+0.01s}\right)^4$ (exercize 2.2).
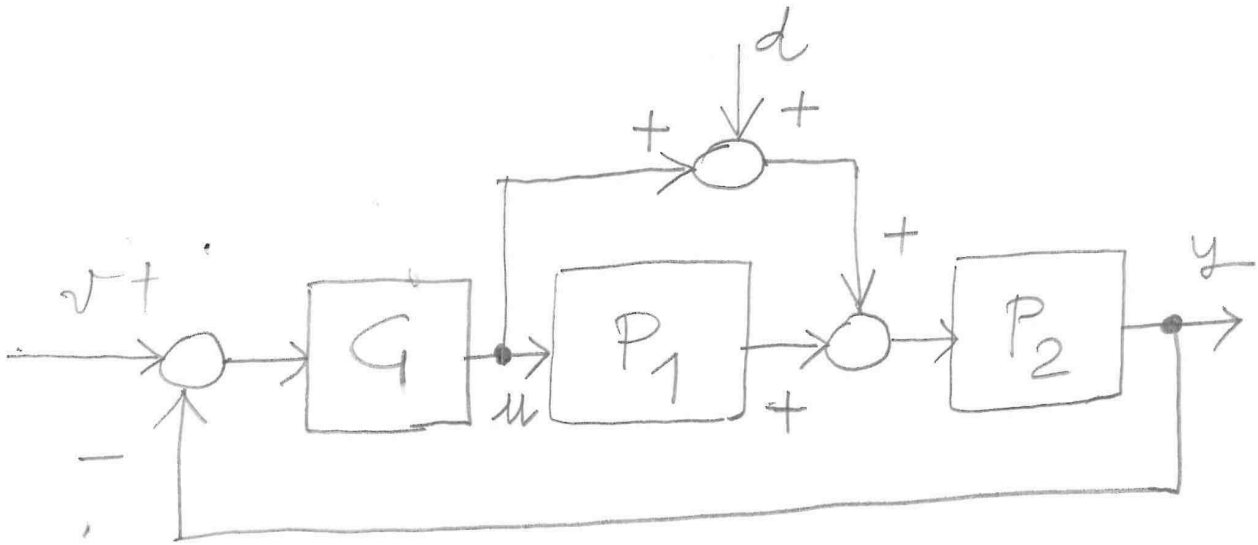
Figure 17. Feedback loop of exercize 2.3.

1 with $\mathbf{P}_1(s)$)

$$
\begin{aligned}
\mathbf{P}(s) &= (1 + \mathbf{P}_1(s))\mathbf{P}_2(s) \\
&= \frac{(s+1)^2(s-2)}{s(s+2)(s+1)^2} = \frac{s-2}{s(s+2)}
\end{aligned}
\tag{36}
$$

Notice that the zero-pole cancelation of $(s+1)^2$ is not preventing the closed-loop system from being asymptotically stable. Therefore, the I/O transfer function $\mathbf{W}(s)$ (from v to y) is

$$
\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1 + \mathbf{G}(s)\mathbf{P}(s)}
\tag{37}
$$

Also, in view of the requirement (ii) and since the way the disturbance affects the feedback loop is not a canonical one (i.e. additive in either the output y or the input u), it is convenient to compute the closed-loop transfer function $\mathbf{W}_d(s)$ from d to y (setting $v = 0$). We obtain

$$
\mathbf{W}_d(s) = \frac{\mathbf{P}_2(s)}{1 + \mathbf{G}(s)\mathbf{P}_1(s)\mathbf{P}_2(s)}
\tag{38}
$$

Since $\mathbf{P}_1(s)$ has a pole at $s = 0$, then $\mathbf{W}_d(s)$ has a zero at $s = 0$ which is the condition for which (ii) is satisfied.

Also, since $\mathbf{P}(s)$ has a pole at $s = 0$, the closed-loop system is of type 1 and

$$
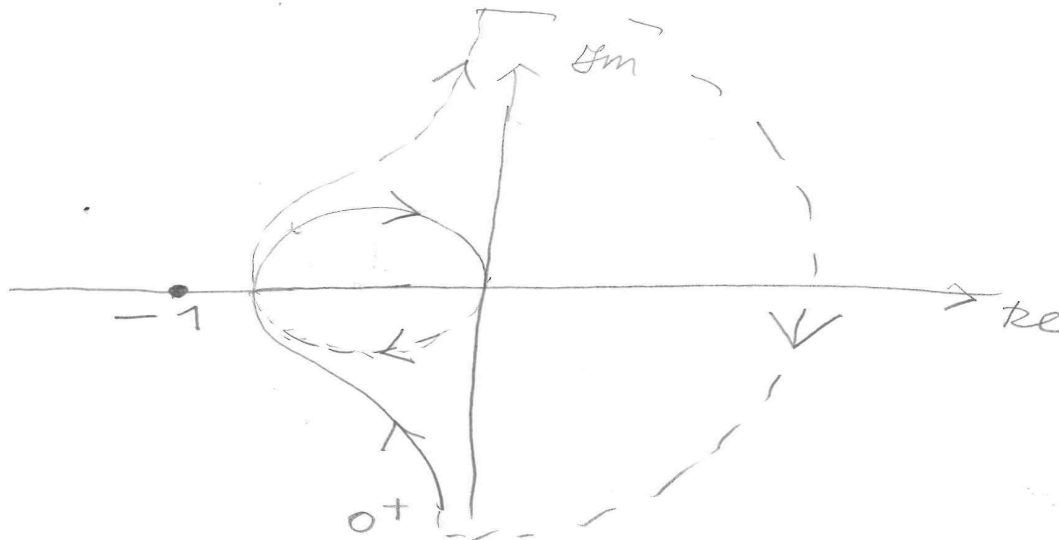e^{(ss,1)} = \left.\frac{\mathbf{W}_e(s)}{s}\right|_{s=0}
\tag{39}
$$

Figure 18. Nyquist plot of $\mathbf{P}(s)\mathbf{G}(s) = -10\frac{s-2}{s(s+2)}\frac{1+\frac{200}{16}s}{1+200s}$ (exercize 2.3).
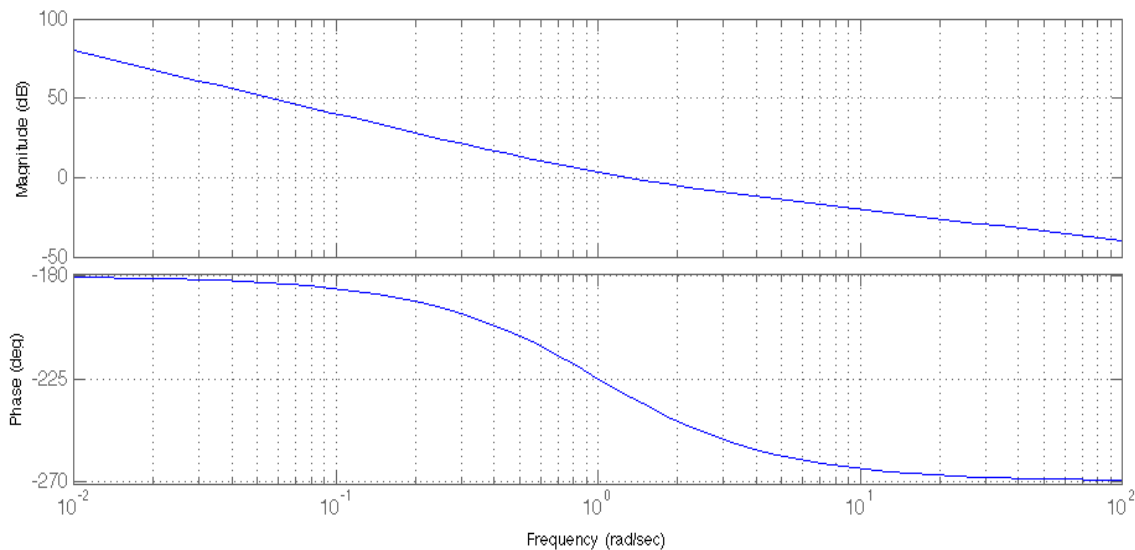

Figure 19. Bode plots of $\mathbf{P}(s) = \frac{1-s}{s^2}$ (exercize 3.1).

where

$$\mathbf{W}_e(s) = 1 - \mathbf{W}(s) = \frac{1}{1+\mathbf{G}(s)\mathbf{P}(s)} \qquad (40)$$

Therefore,

$$e^{(ss,1)} = \left.\frac{1}{s(1+\mathbf{G}(s)\mathbf{P}(s))}\right|_{s=0} = \frac{1}{\mathbf{G}(0)(s\mathbf{P})|_{s=0}} = -\frac{1}{\mathbf{G}(0)}$$

The requirement (i) gives the condition

$$\left|e^{(ss,1)}\right| \le 0.1 \Leftrightarrow |\mathbf{G}(0)| \ge 10. \qquad (41)$$

We give $\mathbf{G}(s)$ the structure

$$\mathbf{G}(s) = K\mathbf{G}_1(s)$$

where $\mathbf{G}_1(s)$ will be designed to satisfy (iii). Notice that at this point, in view of the constraint (41), both the minimal choices $K = 10$ or $K = -10$ are potentially correct. However, by inspection of the root locus of $\mathbf{P}(s)$ it is possible to see that only the negative locus is stabilizable (with a proportional action). Therefore, this leads us to the mandatory choice $K = -10$. We stress the fact that the correct sign of the proportional action to be introduced, before the introduction of anticipative/attenuative actions, can be in general ascertained from inspection of the root locus, in particular by discovering which locus is stabilizable. The sign of the stabilizable locus points out the correct sign of the proportional action to be introduced.
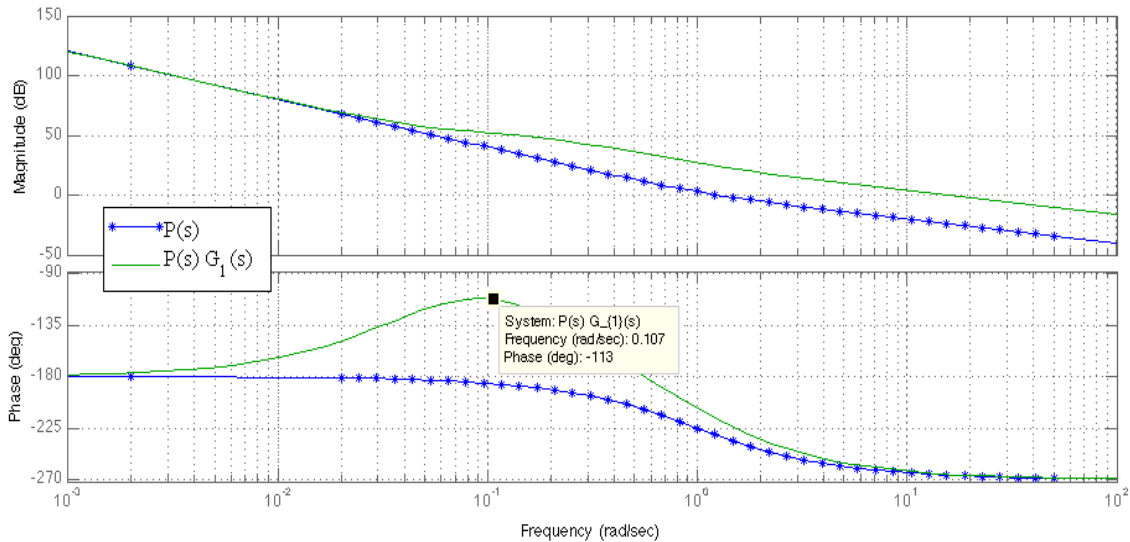
Figure 20. Bode plots of $\mathbf{P}(s) = \frac{1-s}{s^2}$ and $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{1-s}{s^2}\left(\frac{1+20s}{1+5s}\right)^2$ (exercize 3.1).
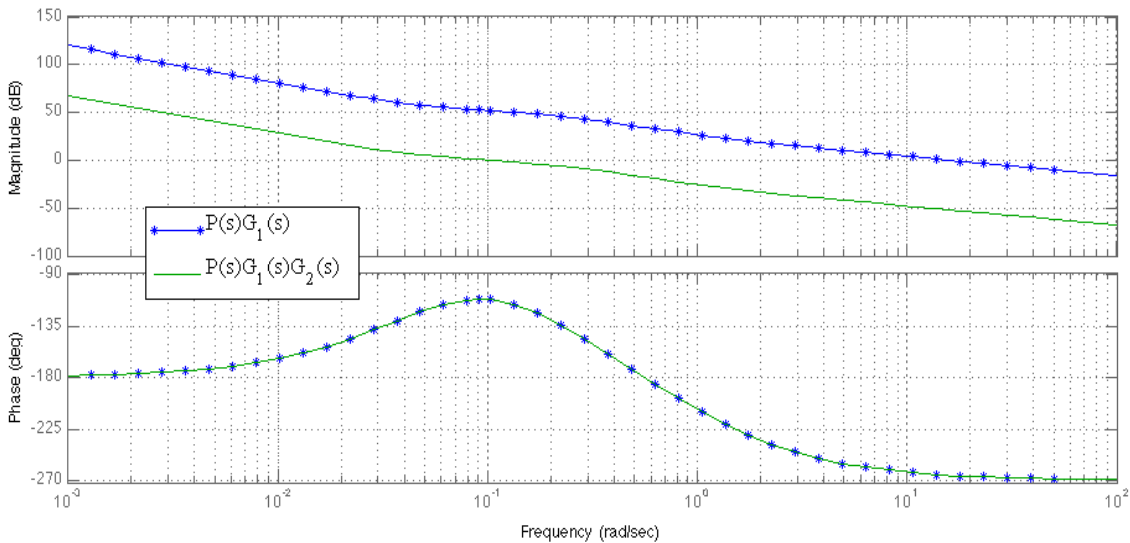

Figure 21. Bode plots of $\mathbf{P}(s)\mathbf{G}_1(s) = \frac{1-s}{s^2}\left(\frac{1+20s}{1+5s}\right)^2$ and $\mathbf{P}(s)\mathbf{G}_1(s)\mathbf{G}_2(s) = 0.0025\frac{1-s}{s^2}\left(\frac{1+20s}{1+5s}\right)^2$ (exercize 3.1).

We proceed by designing $\mathbf{G}_1(s)$. It is easy to see from the Bode plots of $K\mathbf{P}(s)$ that $\omega(t) = 10$ rad/sec and $m_f \approx -67°$. At the least admissible value of the new crossover frequency, i.e. $\omega_t^* = 0.5$ rad/sec, we have

$$|K\mathbf{P}(j\omega_t^*)| \approx 26\text{dB}, \ \text{Arg}\{K\mathbf{P}(j\omega_t^*)\} = -118°.$$

Since the value of the phase at $\omega_t^* = 0.5$ rad/sec largely guarantees a phase margin $m_f^* \geq 45°$, we can reduce the magnitude at $\omega = 0.5$ rad/sec in so that $\omega_t^* = 0.5$ rad/sec becomes the new crossover frequency. It is not possible doing this using a proportional attenuative action in view of the constraint (41). For this reason we use an attenuative action

$$\mathbf{G}_1(s) = \frac{1 + \frac{\tau_i}{m_i}s}{1 + \tau_i s}.$$

Inspection of the plot of Figure 1 reveals that for a $m_i = 16$ we have a maximum attenuation of the 24 dB at the normalized frequency $\omega_N = 100$ rad/sec. In order to let this attenuation correspond to the frequency $\omega_t^* = 0.5$ rad/sec in the Bode plots of $K\mathbf{P}(s)$, we set

$$\omega_t^*\tau_i = \omega_N^* = 100 \Rightarrow \tau_i = 200.$$

Notice that since. $|K\mathbf{P}(j0.5)| \approx 26\text{dB}$ the new crossover frequency will be actually slightly larger than 0.5. The phase
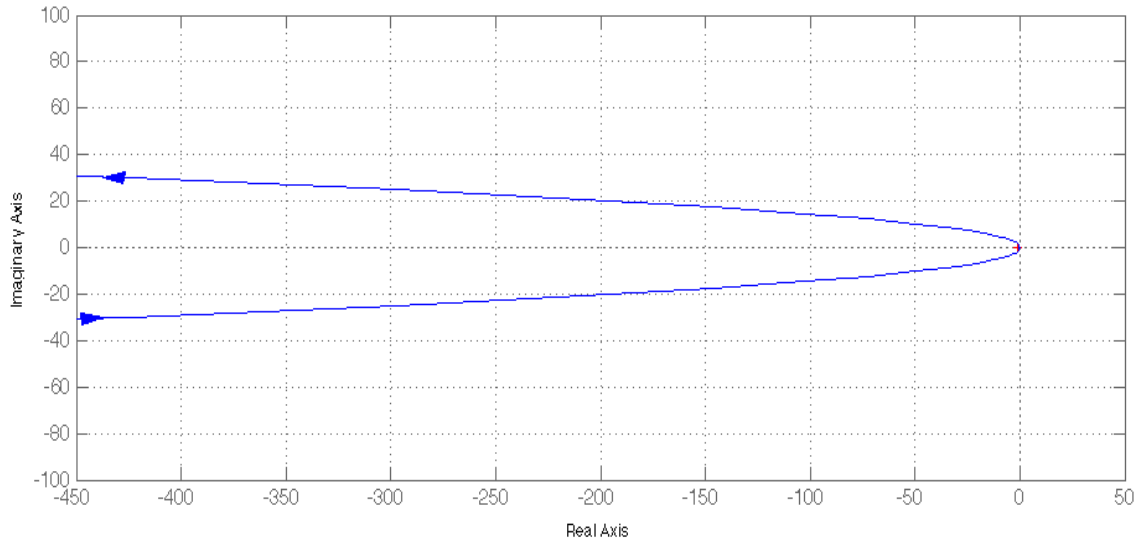
Figure 22. Nyquist plot of $\mathbf{P}(s)\mathbf{G}(s) = 0.0025\frac{1-s}{s^2}\left(\frac{1+20s}{1+5s}\right)^2$: large scale (exercize 3.1).
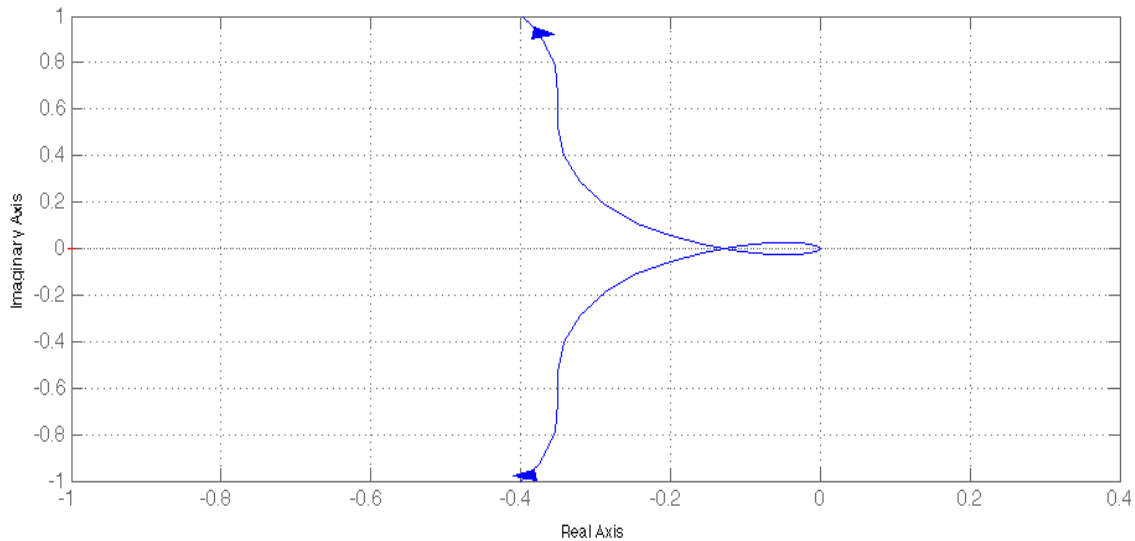

Figure 23. Nyquist plot of $\mathbf{P}(s)\mathbf{G}(s) = 0.0025\frac{1-s}{s^2}\left(\frac{1+20s}{1+5s}\right)^2$: small scale (exercize 3.1).

at the crossover frequency $\omega_t^* = 0.5$ becomes

$$\mathrm{Arg}\{K\mathbf{P}(j\omega_t^*)\} \approx -118° - 8° = -126° \qquad (42)$$

which still guarantees the requirement $m_f^* \geq 45°$.

The asymptotic stability of the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{P}(s)\mathbf{G}(s)}{1+\mathbf{P}(s)\mathbf{G}(s)}$ is verified by the Nyquist criterion on $\mathbf{P}(s)\mathbf{G}(s)$: the number of counterclockwise tours around $-1+0j$ on behalf of $\mathbf{P}(j\omega)\mathbf{G}(j\omega)$ is 0 and the number of poles in $\mathbb{C}^+$ of $\mathbf{P}(s)\mathbf{G}(s)$ is also 0 (see Nyquist plot in Figure 18).

III. MIXED STEADY-STATE AND TRANSIENT PERFORMANCES REQUIREMENTS WITH CONTROLLER CONSTRAINTS

A second class of exercizes we are going to illustrate is the one which combines steady-state and transient performance requirements, on one hand, and controller constraints, on the other. The latter requirements are usually formulated on the controller $\mathbf{G}(j\omega)$ in terms of dimension (complexity) or magnitude upper bound (control energy).

Exercize *3.1: Given*
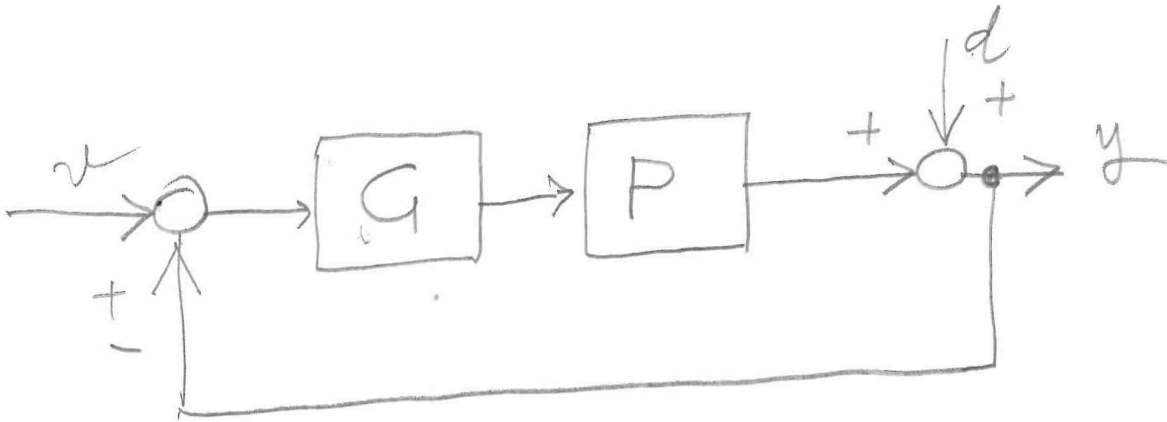
$$\mathbf{P}(s) := \frac{1-s}{s^2} \qquad (43)$$

Figure 24. Feedback loop of exercize 3.2.

find $\mathbf{G}(s)$ such that
(i) $20log_{10}|\mathbf{G}(j\omega)| \le -30$ dB for all $\omega > 0$

and for the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ is asymptotically stable with

(ii) zero steady-state response to output disturbances $\mathbf{d}(t) := t$

and the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ is such that

(iii) $m_f^* \ge 60°$.

Since $\mathbf{P}(s)$ has two poles in $s = 0$ then the closed-loop system has a zero steady-state response to output disturbances $\mathbf{d}(t) := t$.

There is no requirement on ω_t^*, which means that we can change the crossover frequency as we want. On the other hand, since it is required that $20log_{10}|\mathbf{G}(j\omega)| \le -30$ dB, we must have a magnitude attenuation of $|\mathbf{P}(j\omega)|$ of at least 30dB. Noticing that for all $\omega \le 0.3$ rad/sec the magnitude $|\mathbf{P}(j\omega)|$ is ≥ 30dB and for all $\omega \ge 0.3$ rad/sec the magnitude $|\mathbf{P}(j\omega)|$ is ≤ 30dB, we conclude that a crossover frequency ω_t° of at most 0.3 rad/sec is admissible.

Notice that at $\omega = 0.1$ rad/sec the magnitude of $\mathbf{P}(j\omega)$ is 40dB and the phase is $-186°$. Therefore, in order to have $\omega_t^* = 0.1$ rad/sec and $m_f^* \ge 60°$ it is necessary to decrease the magnitude at $\omega = 0.1$ rad/sec by 40dB and increase the phase by at least $66°$.

First, we increase the phase (at least) by $66°$: select $m_a = 4$, $\omega_N^* = 2$ rad/sec and, consequently, $\tau_a = \frac{\omega_N^*}{\omega_t^*} = 20$. Define

$$\mathbf{G}_1(s) := \left(\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}\right)^2 = \left(\frac{1+20s}{1+5s}\right)^2 \qquad (44)$$

(see Figure 14)

Note that the magnitude of $\mathbf{P}(s)\mathbf{G}_1(s)$ at $\omega = 0.1$ rad/sec is $\approx 40 + 12 = 52$dB and the phase is $\approx -186° + 64° = -112°$ (the phase margin is $\approx 68° \ge 60°$). Finally, we decrease the magnitude by 52dB: define

$$\mathbf{G}_2(s) := -52dB = 0.0025 \qquad (45)$$

(see Figure 21). The final controller $\mathbf{G}(s)$ is $\mathbf{G}_1(s)\mathbf{G}_2(s)$.

Let $\mathbf{G}(s) := \mathbf{G}_1(s)\mathbf{G}_2(s)$. The stability of the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ can be ascertianed by means of the Nyquist criterion (Figure 16): indeed, the number of poles of $\mathbf{G}(s)\mathbf{P}(s)$ in $\mathbb{C}^+$ is zero and zero is the number of counterclockwise encirclements of $-1 + 0j$ on behalf of $\mathbf{GP}(j\omega)$.

Exercize 3.2: Consider the feedback loop of Figure 24 with

$$\mathbf{P}(s) := \frac{1}{s^3} \qquad (46)$$

with $\mathbf{G}(s)$ having the following structure

$$\mathbf{G}(s) := K\left(\frac{1+\tau_1 s}{1+\tau_2 s}\right)^2 \qquad (47)$$

Determine $K, \tau_1, \tau_2 \in \mathbb{R}$ such that such that the closed-loop system $\mathbf{W}(s) = \frac{\mathbf{G}(s)\mathbf{P}(s)}{1+\mathbf{G}(s)\mathbf{P}(s)}$ is asymptotically stable with
(i) the absolute value of the steady-state output response to $\mathbf{d}(t) := \sin(\omega t)$ is ≤ 0.11 for all values of $\omega \in [0, 0.1)$ rad/sec,
(ii) $|\mathbf{G}(j\omega)|_{dB} \le 0$,

and the open-loop system $\mathbf{G}(s)\mathbf{P}(s)$ has
(iii) phase margin as large as possible.

The structure of $\mathbf{G}(s)$ consists of a proportional action and two (equal) anticipative control actions. The phase of $\mathbf{P}(j\omega)$ is equal to $-270°$ for all ω. Therefore, in order to obtain an asymptotically stable closed-loop system it will be necessary an intensive anticipative control action so that to obtain a positive phase margin. We can rewrite $\mathbf{G}(s)$ as a double anticipative plus proportional action

$$\mathbf{G}(s) := K\left(\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}\right)^2 \qquad (48)$$

with $\tau_a > 0$ and $m_a > 1$. We will choose $\tau_a > 0$ and $m_a > 1$ and L on the base of simple considerations, with the aim of obtaining the largest phase margin as possible.

First, we consider the requirement (i). Since the steady-state output response to $\mathbf{d}(t) := \sin(\omega t)$ is

$$|\mathbf{W}_d(j\omega)|\sin(t + \text{Arg}\{\mathbf{W}_d(j\omega)\})$$

where $\mathbf{W}_d(s)$ is the closed-loop transfer function from d to y, the requirement (i) can be formulated as

$$|\mathbf{W}_d(j\omega)| = \left|\frac{1}{1+\mathbf{G}(j\omega)\mathbf{P}(j\omega)}\right| \le 0.11 \qquad (49)$$

for all values of $\omega \in [0, 0.1)$ rad/sec. Equivalently,

$$|1 + \mathbf{G}(j\omega)\mathbf{P}(j\omega)| \geq \frac{100}{11} \tag{50}$$

for all values of $\omega \in [0, 0.1)$ rad/sec. Since for all ω

$$|1 + \mathbf{G}(j\omega)\mathbf{P}(j\omega)| \geq |\mathbf{G}(j\omega)\mathbf{P}(j\omega)| - 1 \tag{51}$$

for (50) it is sufficient that

$$|\mathbf{G}(j\omega)\mathbf{P}(j\omega)| \geq \frac{100}{11} + 1 \approx 10 = 20dB \tag{52}$$

for all values of $\omega \in [0, 0.1)$ rad/sec. This condition can be directly carried over to the Bode diagram by saying that the magnitude plot of $\mathbf{G}(j\omega)\mathbf{P}(j\omega)$ must not go beyond the value $20dB$ for all values of $\omega \in [0, 0.1)$ rad/sec.

The condition on $|\mathbf{G}(j\omega)\mathbf{P}(j\omega)|$ implies an upper bound on the maximal phase increment we can obtain with an anticipative action. Moreover, an anticipative action introduces an increase of the magnitude which is large to the same extent of the phase increase. In order to satisfy (iii) the value of K must be sufficiently small for compensating the large maginitude increase. However, the lower bound (52) on $|\mathbf{G}(j\omega)\mathbf{P}(j\omega)|$ implies that K cannot be arbitrarily small. Inspection of the Bode plot of $\mathbf{P}(s)$ leads to the conclusion that $K \geq -40$ dB is the admissible range of choices for the gain K. Indeed, since $|\mathbf{P}(j0.1)| = 60$ dB and the anticipative structure of (48), for $K \geq -40$ dB we have $|\mathbf{P}(j\omega)\mathbf{G}(j\omega)| \geq 60$ dB for all $\omega \in [0, 0.1)$ rad/sec.

Choose $K = -40$ dB $= 0.01$. As to the choice of τ_a, m_a, on account of requirement (ii), the choice of K implies that each anticipative action

$$\frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}$$

in $\mathbf{G}(s)$ should introduce a magnitude increase of at most 20 dB. From the plots of Figure 1 we see that this correspond to the choice of $m_a = 10$. In order to have the largest phase margin as possible, we choose the normalized frequency $\omega_N^* = 3.2$ rad/sec which corresponds to a phase increase of $\approx 54°$ on the anticipative function corresponding to $m_a = 10$. Since the corresponding magnitude increase for each anticipative action is ≈ 10 dB it is sufficient to let the phase increase amount correspond to the frequency ω_t^* such that $|K\mathbf{P}(j\omega_t^*)| = -20$ dB. By doing this ω_t^* is the new crossover frequency and we obtain a phase margin of

$$\approx -90° + 54° \times 2 = 18°.$$

From inspection of the magnitude Bode plot of $K\mathbf{P}(j\omega(t))$ we obtain $\omega_t^* = 0.5$ rad/sec. Therefore,

$$\tau_a = \frac{\omega_N^*}{\omega_t^*} = \frac{3.2}{0.5}$$

The resulting controller $\mathbf{G}(s)$ is

$$\mathbf{G}(s) := 0.01\left(\frac{1 + \frac{3.2}{0.5}s}{1 + \frac{3.2}{5}s}\right)^2 \tag{53}$$

The above procedure does not give the largest phase margin as possible but it is conceived in such a way to give results close to the optimal one, which is well enough for our purposes. For instance, it would be possible to choose

$$\mathbf{G}(s) := 0.0059\left(\frac{1 + \frac{4.2}{0.5}s}{1 + \frac{4.2}{6.5}s}\right)^2 \tag{54}$$

obtaining a crossover frequency of approx 0.42rad/sec and a phase margin of approx $28°$. This is achieved by choosing $m_a = 13$ and a normalized frequency $\omega_N^* = 4.2$ rad/sec, letting the phase increase amount correspond to the same frequency $\omega_t^* = 0.5$ rad/sec. The value of K is computed by satisfying the requirements (52) and (ii).

Exercize *3.3: Given the feedback system in Figure 25 with*

$$\mathbf{P}_1(s) = \frac{2.1s + 0.1}{s - 1}, \ \mathbf{P}_2(s) = \frac{1}{2.1s + 0.1}$$

design a one-dimensional controller $\mathbf{G}_1(s)$ and two-dimensional controller $\mathbf{G}_2(s)$ such that the closed-loop system (from $\mathbf{d}_1, \mathbf{d}_2, \mathbf{v}$ to $\mathbf{y}$) is asymptotically stable with

(ii) steady-state error response to constant inputs $\mathbf{v}(t)$ equal to zero,

(iii) steady-state output response to constant disturbances $\mathbf{d}_1(t)$ and $\mathbf{d}_2(t)$ and steady-state output response to ramp disturbances $\mathbf{d}_1(t) = t$ all equal to zero

and the open-loop system (from $\mathbf{e}$ to $\mathbf{y}$) has the largest as possible phase margin. We determine the disturbance-to-output, input-to-output and input-to-error transfer functions. To this aim, first the (open-loop) output response is given in Laplace domain by

$$\mathbf{y}(s) = \mathbf{W}_{e,y}(s)\mathbf{e}(s) + \mathbf{W}_{d_2,y}(s)\mathbf{d}_2(s) + \mathbf{W}_{d_1,y}(s)\mathbf{d}_1(s)$$

with (open loop) transfer function (from $\mathbf{e}$ to $\mathbf{y}$)

$$\mathbf{W}_{e,y}(s) = \mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s),$$

(open loop) transfer function (from $\mathbf{m}$ to $\mathbf{y}$)

$$\mathbf{W}_{m,y}(s) = \frac{\mathbf{G}_1(s)\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)},$$

and (open loop) transfer function (from $\mathbf{d_1}$ and, respectively, $\mathbf{d}_2$ to $\mathbf{y}$)

$$\mathbf{W}_{d_1,y}(s) = \frac{\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)}, \ \mathbf{W}_{d_2,y}(s) = 1.$$

Therefore, the closed-loop input-to-output and input-to-error transfer functions are

$$\begin{aligned}
\mathbf{W}(s) &= \frac{\mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s)}{1 + \mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s)} \\
&= \frac{\mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{G}_1(s)\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)(1 + \mathbf{G}_2(s)\mathbf{P}_2(s))}, \\
\mathbf{W}_e(s) &= 1 - \mathbf{W}(s) \\
&= \frac{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)(1 + \mathbf{G}_2(s)\mathbf{P}_2(s))},
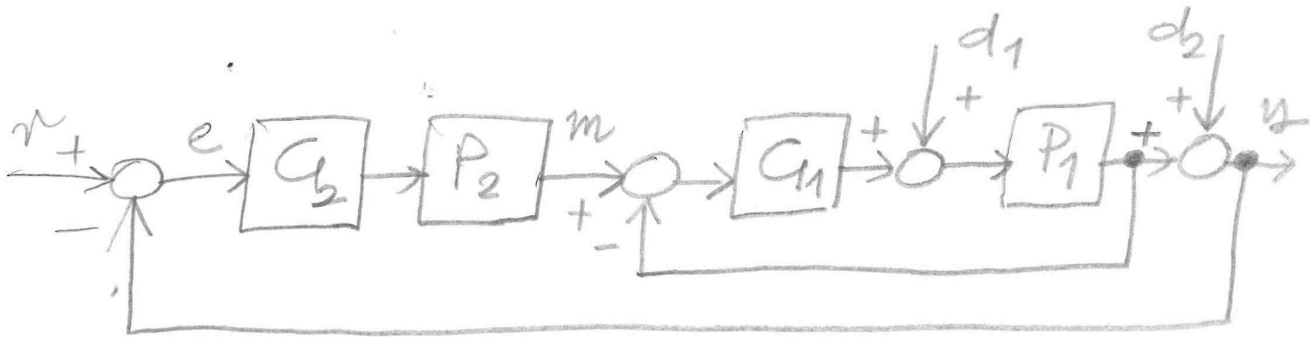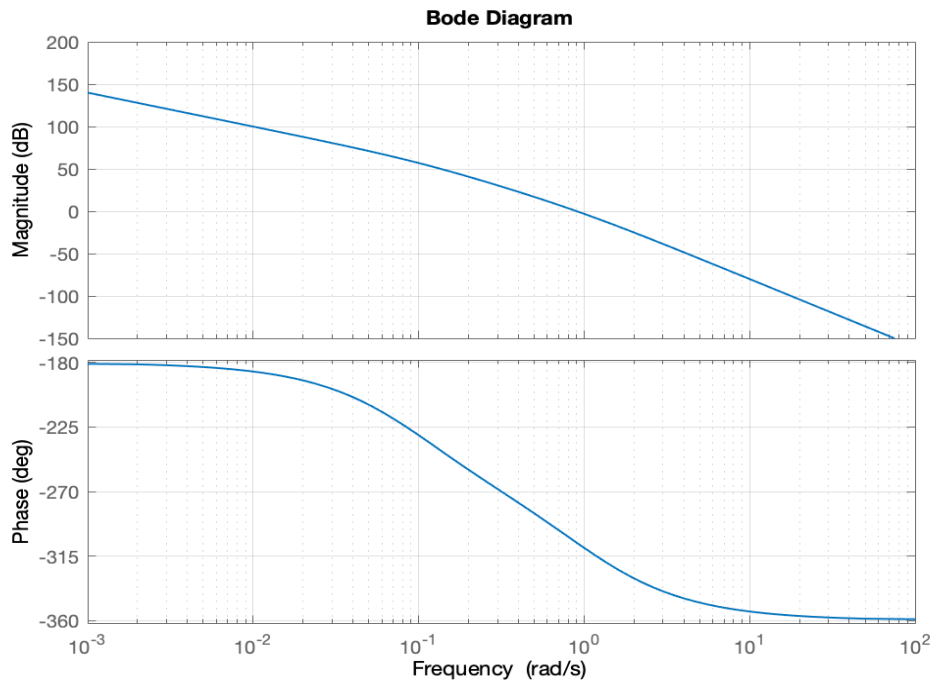\end{aligned} \tag{55}$$

Figure 25. Feedback system of exercize 3.3).

Figure 26. Bode plots of $\mathbf{L}(s)$

and the disturbance-to-output transfer functions are

$$
\begin{aligned}
\mathbf{W}_{d_1}(s) &= \frac{\mathbf{W}_{d_1,y}(s)}{1 + \mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s)} \\
&= \frac{\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)(1 + \mathbf{G}_2(s)\mathbf{P}_2(s))} \\
\mathbf{W}_{d_2}(s) &= \frac{\mathbf{W}_{d_2,y}(s)}{1 + \mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s)} \\
&= \frac{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)}{1 + \mathbf{G}_1(s)\mathbf{P}_1(s)(1 + \mathbf{G}_2(s)\mathbf{P}_2(s))}.
\end{aligned}
\tag{56}
$$

From inspection of the above transfer functions, in order to meet requirements (ii) and (iii) we assume for $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ the following structure

$$
\mathbf{G}_1(s) = \frac{1}{s}, \quad \mathbf{G}_2(s) = \frac{1}{s}\mathbf{G}_2'(s)
\tag{57}
$$

with one dimensional $\mathbf{G}_2'(s)$ (recall that $\mathbf{G}_1(s)$ is required to be one dimensional and $\mathbf{G}_2(s)$ two-dimensional). The open-loop system (from $\mathbf{e}$ to $\mathbf{y}$) is represented by

$$
\begin{aligned}
&\mathbf{G}_2(s)\mathbf{P}_2(s)\mathbf{W}_{m,y}(s) \\
&= \frac{\mathbf{G}_2'(s)}{s}\frac{1}{2.1s + 0.1}\frac{2.1s + 0.1}{s^2 + 1.1s0.1} \\
&= \frac{\mathbf{G}_2'(s)}{s}\frac{10}{s(s+1)(1+10s)} = \mathbf{G}_2'(s)\mathbf{L}(s).
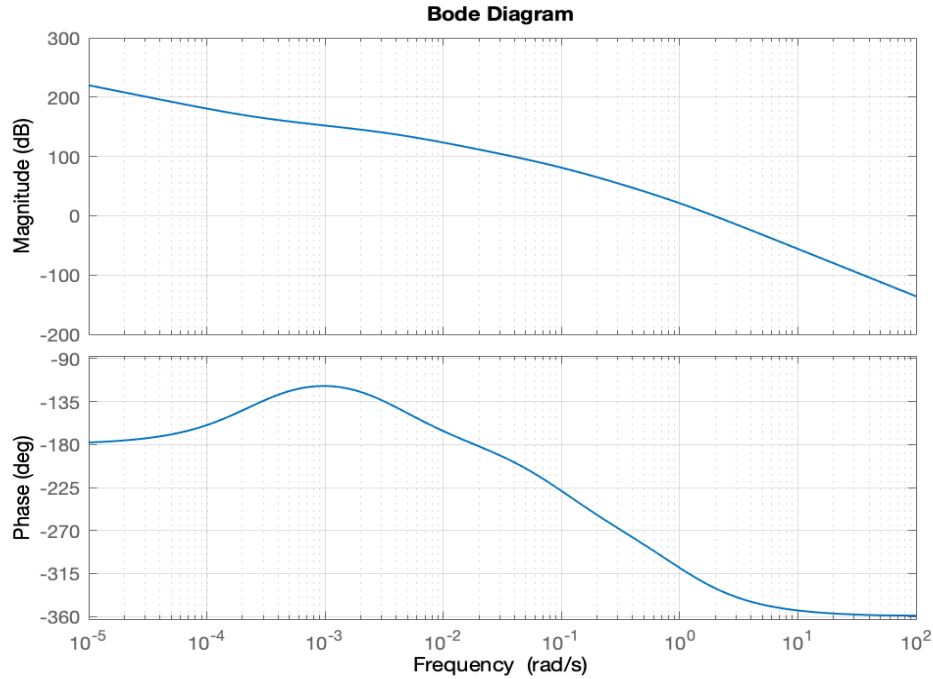\end{aligned}
\tag{58}
$$

Figure 27. Bode plots of $\mathbf{L}(s)\frac{1+4000s}{1+\frac{4000}{16}s}$

From the Bode plot of $\mathbf{L}(s)$ (Figure 26) we see that we have to increase the phase (to maximize the phase margin) using an anticipative+proportional action

$$\mathbf{G}_2'(s) = K\frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a}s}.$$

In order to maximize the phase margin, we choose $m_a = 16$ with $\omega_N = 4$ rad/sec (maximum phase value) at $\omega_t^* = 0.0001$ rad/sec (where the Bode plot of the phase of $\mathbf{L}(s)$ is higher: actually, any $\omega_t^* \leq 0.0001$ is good as well). We obtain $\tau_a = 4/0.0001 = 4000$. Therefore, the anticipative action is

$$\frac{1 + 4000s}{1 + \frac{4000}{16}s}.$$

For obtaining ω_t^* at 0.0001 rad/sec we see from the Bode plots of $\mathbf{L}(s)\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$ (Figure 27) that we need a proportional attenuation $K \approx -152dB = 2.511e^{-8}$.

The controller $\mathbf{G}_2(s)$ is given finally by

$$\mathbf{G}_2(s) = \frac{2.511e^{-8}}{s}\frac{1 + 4000s}{1 + \frac{4000}{16}s}.$$

The Bode plot of $\mathbf{G}_2(s)L(s)$ shows that a crossover frequency $\omega_t^* = 10^{-3}$ rad/sec with a phase margin $m_\phi^* \approx 150°$, while the Nyquist plot (Figure 28) shows that the closed-loop system is asymptotically stable (we have 0 counterclockwise tours around the point $-1 + 0j$).
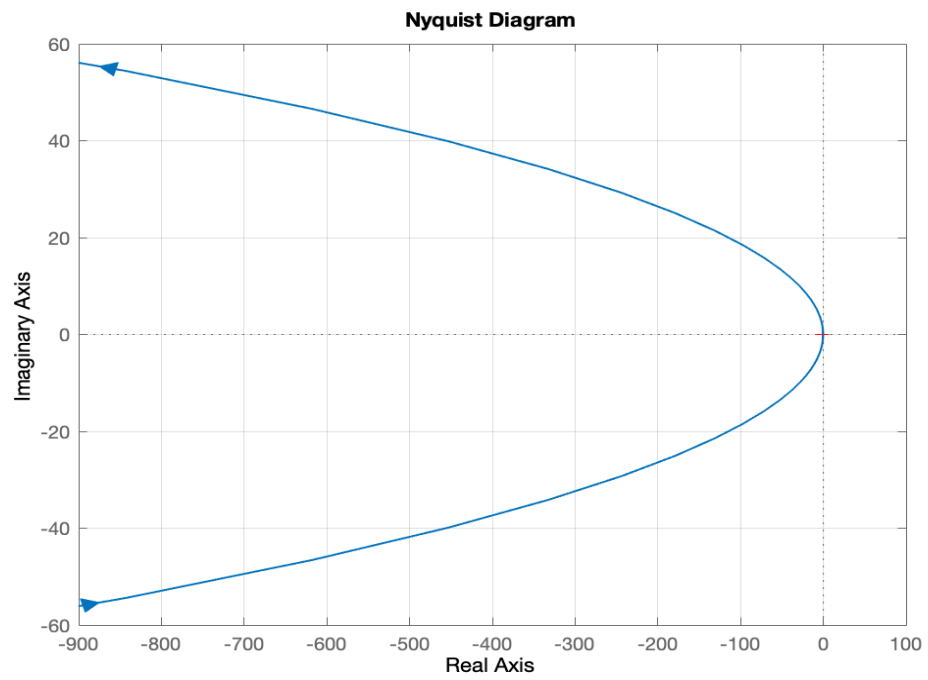
Figure 28. Nyquist plot of $\mathbf{L}(s)\mathbf{G}_2(s)$