A biological solution to a fundamental distributed computing problem

Maximal Independent set

- Given graph $G = (V, E)$

- Maximal Independent Set
  - Subset $A$ of the nodes such that:
    1) Each node is either in $A$ or is adjacent to a node in $A$
    2) If $u$ and $v$ both belong to $A \rightarrow u$ and $v$ are not adjacent

- Notation
  - $n$: upper bound on $|V|$
  - $D$: upper bound on number of active (see further) neighbours of any node (possibly $n$)
Distributed MIS

- Identical nodes
- Synchronous model
- In a round, a node can only tell whether or not it received a message
  - It cannot count the number of messages it received
Previous work

- Distributed MIS impossible using deterministic algorithms [Cohen et al. 1984]

- Polylogarithmic time probabilistic algorithms [Luby 1986, Alon et al. 1986]
  - Require knowledge of number of active neighbours (nodes who have not yet been assigned to A or V-A)
  - Require messages of size function of the number of nodes in the network
A biological perspective

- Sensory organ precursors of the fly's bristles
- SOPs form a MIS computed over a set of initially indifferentiated cells
- The MIS is computed using a biological process that roughly follows the beacon algorithm described in the next slide
Beacon algorithm

1. Algorithm: MIS \((n, D)\) at node \(u\)
2. For \(i = 0\): \(\log D\)
   3. For \(j = 0\): \(M \log n\) \(/ M\) is constant derived below
   4. * exchange 1*
   5. \(v = 0\)
   6. With probability \(\frac{1}{2^{\log D - 1}}\) broadcast \(B\) to neighbors and set \(v = 1\) // \(B\) is one bit
   7. If received message from neighbor, then \(v = 0\)
   8. * exchange 2 *
   9. If \(v = 1\) then
      10. Broadcast \(B\); join MIS; exit the algorithm
   11. Else
      12. If received message \(B\) in this exchange, then mark node \(u\) inactive; exit the algorithm
13. End
14. End
15. End

- **Active node**: a node for which a decision has not been made yet
Generic round: first possibility

- Black node joins A, yellow nodes don't
- All three nodes become inactive
Both red and green node send beacons
All three nodes remain active
Generic round: third possibility

- No broadcast
- All three nodes remain active
Properties of the algorithm

- **Lemma 1**: No two nodes in A are connected to each other
- **Lemma 2**: If node w becomes inactive and it does not belong to A, then it is adjacent to a node in A
- **Corollary 3**: If run forever, the algorithm eventually produces a MIS for G
- Proofs of lemmas 1 and 2: see previous two slides
Theorem 4: with probability at least $1 - \frac{\log D}{n^2}$, all nodes are either in A or adjacent to a node in A by the end of the algorithm.

Corollary 5: with high probability, the algorithm computes a MIS for G in $O(\log^2 n)$ rounds.
Proof of thm. 4

1) Lemmas 1 and 2 ensure that the only reason why the algorithm does not compute a MIS for G is that there are still active nodes when it terminates.

2) The proof follows from the following Lemma 6:
   with probability at least \( 1 - \frac{i}{n^2} \), there are no nodes with degree \( > \frac{D}{2^i} \) at the end of phase \( i \).
   - Phase: an iteration of lines 3 – 14 of the algorithm.
   - Degree of u in a phase: \# u's active neighbours + 1
Proof of Lemma 6/1

- By induction on \( i \)
- Trivial for \( i = 0 \)
- Assume true for \( i - 1 \) and consider node \( v \) with > \( D/2^i \) neighbours. Then:

\[
P(v \lor \text{neighbour of } v \text{ broadcasts}) \geq 1 - \frac{1}{e}
\]
Proof of Lemma 6/2

- On the other hand, as a node broadcast a message:

\[ P(\text{No collisions occur}) \geq \left( 1 - \frac{1}{D/2^i} \right)^{2D/2^i} \approx \frac{1}{e^2} \]

- As a consequence, in any round of phase i:

\[ P(\text{v removed}) \geq \left( 1 - \frac{1}{e} \right) \frac{1}{e^2} \]
Proof of Lemma 6/3

- As a consequence:

\[ P(v \text{ removed during i-th phase}) \geq 1 - \frac{1}{n^3} \]

- Proof of lemma then follows since i) at most \( n \) nodes, ii) above bound and iii) induction hypotheses
Proof of Thm. 4/cont.

- From Lemma 6, all nodes left in the algorithm at the end of phase $\log D$ have degree 1 with probability at least $1 - \frac{\log D}{n^2}$.

- They have no active neighbours and thus they will insert themselves into $A$. 