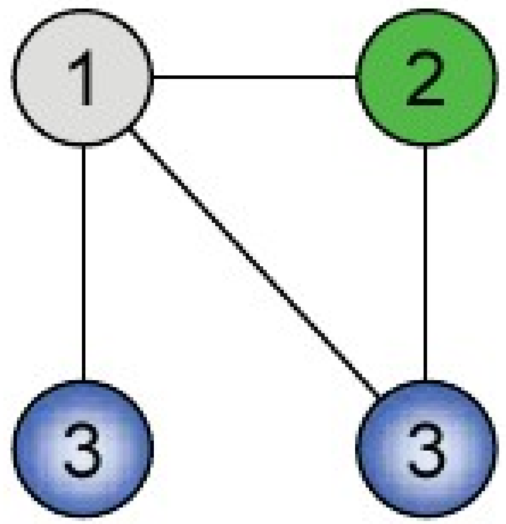
Vertex coloring

Problem

Problem 1.1 (Vertex Coloring). *Given an undirected graph $G = (V, E)$, assign a color c_u to each vertex $u \in V$ such that the following holds: $e = (v, w) \in E \Rightarrow c_v \neq c_w$.*



A non distributed solution

Assumption 1.2 (Node Identifiers). *Each node has a unique identifier, e.g., its IP address. We usually assume that each identifier consists of only* $\log n$ *bits if the system has* n *nodes.*

Algorithm 1 Greedy Sequential

1: **while** $\exists$ uncolored vertex v **do**

2: color v with the minimal color (number) that does not conflict with the already colored neighbors

3: **end while**

Performance

Theorem 1.5 (Analysis of Algorithm 1). *The algorithm is correct and terminates in n "steps". The algorithm uses $\Delta + 1$ colors.*

- Δ is the maximum degree of the graph
- The *chromatic number* Ξ(G) of G is the minimum number of colours in a proper vertex coloring of G

A useful subroutine

Procedure 2 First Free

Require: Node Coloring {e.g., node IDs as defined in Assumption 1.2}

Give v the smallest admissible color {i.e., the smallest node color not used by any neighbor}

- Caveat: no two nodes are coloured at the same time

Algorithm for synchronous case

Algorithm 3 Reduce

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Assume that initially all nodes have ID's (Assumption 1.2)
Each node v executes the following code
node v sends its ID to all neighbors
node v receives IDs of neighbors
while node v has an uncolored neighbor with higher ID do
    node v sends "undecided" to all neighbors
    node v receives new decisions from neighbors
end while
node v chooses a free color using subroutine First Free (Procedure 2)
node v informs all its neighbors about its choice
```

- Thm.: algorithm is correct and has time complexity n. It uses $\Delta + 1$ colours

Trees

Lemma 1.9. $\chi(Tree) \leq 2$

Algorithm 4 Slow Tree Coloring

Color the root 0, root sends 0 to its children
Each node v concurrently executes the following code:
if node v receives a message x (from parent) **then**
    node v chooses color $c_v = 1 - x$
    node v sends c_v to its children (all neighbors except parent)
end if

- Caveat: how do we choose a root?

Trees/cont.

- The previous algorithm also works in the asynchronous case
  - Time complexity is the tree height
  - Message complexity is n-1
- Is it possible to do better?
- log*n time is possible!