Computational Game Theory

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4 Nash's Theorem

Recall that a finite *normal form game* Γ is given by

- a finite set $N = \{1, \ldots, n\}$ (the set of *players*);
- for each player $i = 1, \ldots, n$, a finite set S_i (the *strategy sets*);
- for each player $i = 1, \ldots, n$, a function $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ (the *utility functions*).

A mixed strategy p_i for player i is a probability distribution on S_i. A mixed state p is a collection of mixed strategies $p = (p_1, \ldots, p_n)$, one for each player. The (expected) payoff of player i in mixed state p is

$$u_i(p) = \sum_{x \in S_1 \times \ldots \times S_n} p_1(x_1) \ldots p_n(x_n) \cdot u_i(x_i, x_{-i}).$$

Player i's mixed strategy p_i is a best response in mixed state p if

$$u_i(p_i, p_{-i}) \geq u_i(x_i, p_{-i}) \qquad \forall x_i \in S_i.$$

For a given mixed state p and $i = 1, \ldots, n$, the set of pure best responses of i is

$$B_i(p_{-i}) := \{x_i \in S : u_i(x_i, p_{-i}) \geq u_i(x_i', p_{-i}) \, \forall x_i' \in S_i\}$$

Theorem 4.1 (Nash 1950)**.** *Any finite normal form game has a mixed Nash equilibrium.*

To prove the theorem of Nash, we use a powerful mathematical result known as Brouwer's Fixed Point Theorem.

Theorem 4.2 (Brouwer 1909)**.** *Let X be a closed, bounded and convex subset of $\mathbb{R}^d$ and let $f : X \to X$ be a continuous function. Then there exists a point $x \in X$ such that $f(x) = x$.*

Notice that all conditions on X are necessary for a fixed point to exist:

- if X is not closed, take $X = (0, 1)$ and $f(x) = x/2$;
- if X is not bounded, take $X = \mathbb{R}$ and $f(x) = x + 1$;
- if X is not convex, take $X = \{(\cos\alpha, \sin\alpha) : \alpha \in [0, 2\pi)\}$ and $f((\cos\alpha, \sin\alpha)) = (\cos(\alpha + \epsilon), \sin(\alpha + \epsilon))$ for some $\epsilon \in (0, 2\pi)$.

4.1 Proof of Nash's Theorem

Consider a finite normal-form game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ where $N = \{1, \dots, n\}$.

The *mixed extension* of Γ is the game $\Delta(\Gamma) = (N, (\Delta(S_i))_{i\in N}, (u_i)_{i\in N})$ where

$$\Delta(S_i) = \{p_i \in [0,1]^{S_i} \; : \; \sum_{a\in S_i} p_i(a) = 1\}.$$

In words, the pure strategies of the mixed extension are exactly the mixed strategies of the initial game Γ. Let $\mathcal{P} = \Delta(S_1) \times \ldots \times \Delta(S_n)$ be the set of all mixed states. $\mathcal{P}$ is closed, bounded and convex (why?). The idea of the proof is to apply Brouwer's theorem by defining a continuous function $f : \mathcal{P} \to \mathcal{P}$ with the property that $f(p) = p$ if and only if p is a mixed Nash equilibrium. Then, the existence of a fixed point will imply the existence of a MNE.

For $p = (p_1, \dots, p_n) \in \mathcal{P}$, where $p_i \in \Delta(S_i)$, define $f(p) = (f_1(p), \dots, f_n(p))$, where $f_i(p) \in \Delta(S_i)$ will now be appropriately defined. Since f is a function from mixed states to mixed states, f_i will be a function from mixed states to mixed strategies of player i. We will construct f_i so that if p is a Nash equilibrium, then $f_i(p) = p_i$; otherwise, $f_i(p) = q_i \neq p_i$ where q_i is an "improvement" of p_i.

Recall that $B_i(p_{-i})$ is the set of pure best responses of player i to p_{-i}. Define

$$\alpha = \sum_{a\in B_i(p_{-i})} [u_i(a, p_{-i}) - u_i(p_i, p_{-i})].$$

Notice that $\alpha \geq 0$ (why?) and moreover $\alpha = 0$ if and only if p_i is a best response to p_{-i}. We can now define $f_i(p) = q_i$ where

$$q_i(a) = \begin{cases} \frac{p_i(a)+[u_i(a,p_{-i})-u_i(p_i,p_{-i})]}{1+\alpha} & \text{if } a \in B_i(p_{-i}), \\ \frac{p_i(a)}{1+\alpha} & \text{if } a \notin B_i(p_{-i}). \end{cases}$$

Here, $p_i(a)$ and $q_i(a)$ are the probabilities that player i chooses pure strategy a, in mixed strategy p_i and q_i, respectively. Now it follows that:

(i) $f_i(p_i, p_{-i}) = p_i$ if and only if $\alpha = 0$, if and only if p_i is a best response to p_{-i};

(ii) $f_i(p)$ is continuous because $u_i(p)$ is continuous and α, as a function of p, is continuous.

Then from (i), $f(p) = p$ if and only if p is a MNE. From (ii) and Brouwer's Fixed Point Theorem, there exists $p \in \mathcal{P}$ such that $f(p) = p$, meaning that p is a Nash equilibrium.

4.2 Proof of Brouwer's Fixed Point Theorem for a Triangle

We will not prove Brouwer's theorem in general. However, we give a proof sketch for the case that the set X is two-dimensional, and in particular a triangle. The general case follows a similar argument.

A *triangulation* $\mathcal{T}$ of a triangle T is a set of triangles that together cover T and that mutually intersect only along their edges; see Figure 1 for an illustration. We denote by $V(\mathcal{T})$ the union of the vertices of the triangles covering T.

Lemma 4.3 (Sperner 1928)**.** *Let T be a triangle with vertices v_0, v_1, v_2, with its triangulation $\mathcal{T}$ having set of vertices $V(\mathcal{T})$. A proper coloring of $\mathcal{T}$ is a function $c : V(\mathcal{T}) \to \{0, 1, 2\}$ such that:*

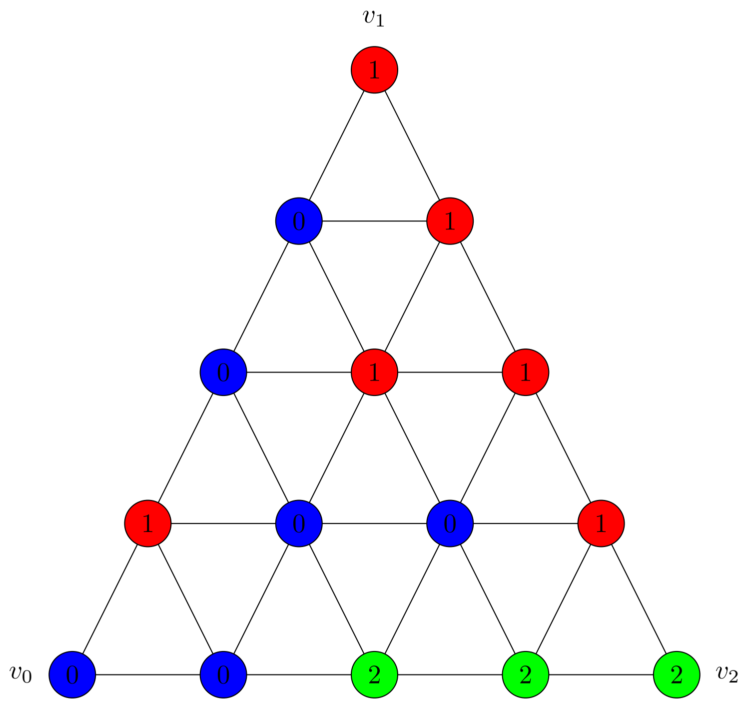

Figure 1: A properly colored, triangulated triangle

1. *$c(v_i) = i$;*
2. *if $v \in V(\mathcal{T})$ lies on the line joining v_i and v_j for $i, j \in \{0, 1, 2\}$, then $c(v)$ is either i or j;*

Then for any proper coloring of $\mathcal{T}$, there is a triangle in $\mathcal{T}$ whose vertices have all three colors – a panchromatic *triangle.*

Proof. Construct a new graph G having as set of vertices the set of triangles of $\mathcal{T}$, plus a vertex for the outer face outside $\mathcal{T}$. Put an edge between two vertices of G whenever the corresponding triangles share a 0–1 line, that is, a line that is colored 0 on one endpoint and 1 on the other. Then every vertex corresponding to a triangle that is not panchromatic has even degree in G: either 0 or 2. On the other hand, the vertex corresponding to the outer face has odd degree (why?). But the sum of the degrees of the vertices of a graph is an even number; so, there must exist another vertex with degree 1, since no vertex can have degree 3. Such a vertex corresponds to a panchromatic triangle. □

Theorem 4.4. *Let $T \subseteq \mathbb{R}^2$ be a (closed) triangle and let $f : T \to T$ be a continuous function. There exists a point $x^* \in T$ such that $f(x^*) = x^*$.*

Proof. Let v_0,v_1,v_2 be the vertices of T. Each $x \in T$ can be uniquely represented as

$$x = x_0 v_0 + x_1 v_1 + x_2 v_2$$

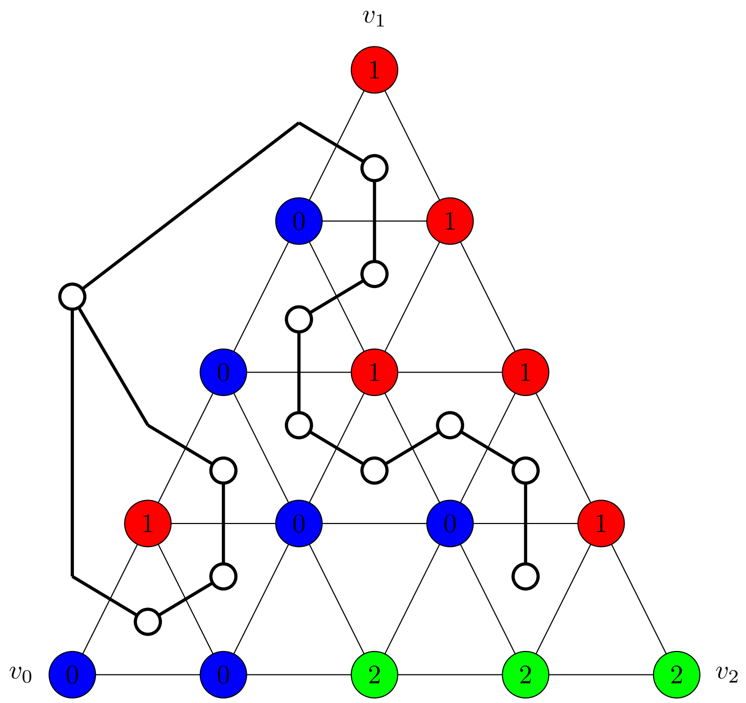

Figure 2: Proving Sperner's Lemma

where $0 \leq x_0, x_1, x_2 \leq 1$ and $x_0 + x_1 + x_2 = 1$. Then we write that x can be represented as (x_0, x_1, x_2). Observe that if x lies on the line from v_i to v_j, then $x_i + x_j = 1$ and $x_k = 0$ for $k \notin \{i, j\}$. We also denote by $x[i]$ the x_i value corresponding to point x.

We now define a coloring $g : T \to \{0, 1, 2\}$. To define $g(x)$, look at $y = f(x)$ and at the representation of y, (y_0, y_1, y_2). Define $g(x) = i$ where i is the smallest index in $\{0, 1, 2\}$ such that $y_i < x_i$. This is well-defined since we assume (by contradiction) that f has no fixed point and thus $y \neq x$. Moreover, notice that any point on the line from v_i to v_j gets colored either i or j.

Consider a sequence of triangulations $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots)$ such that $\mathcal{T}_0 = T$ and the size of the triangulating triangles decreases at each step. The coloring g restricted to $\mathcal{T}_i$ satisfies the condition of Sperner's lemma, so there is a panchromatic triangle $t_i \in \mathcal{T}_i$ for each i. Let its vertices be (v_0^i, v_1^i, v_2^i) where v_j^i is the vertex with color j.

Consider the sequence of points $(v_0^0, v_0^1, v_0^2, \ldots)$. Since T is closed and bounded (compact), there is a convergent subsequence $(v_0^{i_1}, v_0^{i_2}, \ldots)$ that converges to some point $x^* \in T$. Since the size of the triangles decreases, this also implies that the sequences $(v_1^{i_1}, v_1^{i_2}, \ldots)$ and $(v_2^{i_1}, v_2^{i_2}, \ldots)$ both converge to x^*.

By definition of the coloring, $f(v_0^{i_k})[0] < v_0^{i_k}[0]$ for each k. By continuity of f we can take limits and conclude $f(x^*)[0] \leq x^*[0]$. In the same way, $f(x^*)[i] \leq x^*[i]$ for each $i \in \{0, 1, 2\}$. But $f(x^*)[0] + f(x^*)[1] + f(x^*)[2] = 1$ and $x^*[0] + x^*[1] + x^*[2] = 1$, so the inequalities must all be equalities and we get $f(x^*) = x^*$, a contradiction. Thus f must have a fixed point. $\square$

References

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