Bachelor’s degree in Bioinformatics

Principles of Mathematics

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Important Advices

- If you need to contact me, use bruni@diag.uniroma1.it and ALWAYS use the subject Principles of Maths (or PM course)
- If you use a different subject your email may be mistaken for spam
- Always try to understand what we will see, do not learn by heart pretending that you understood
- The slides may be updated during the course, so be sure you are using the latest version
- To do the exams, you need to register first. Reservetion must be done during the reservation period, usually from 2 to 1 week in advance!

POM1 and POM2

- The course Principles of Mathematics is divided in 2 **modules**:

  Principles of Mathematics 1 and Principles of Mathematics 2
- POM1 is in the first semester, POM2 in the second semester
- The 2 modules will be taught by **2 different professors**, and you will take 2 different exams, each organized by one professor. I organize POM1.
- After completion of the 2 modules, the full exam of Principles of Mathematics 12 credits will be verbalized on **Infostud**
- Thus, POM1 alone will not be on Infostud: dates, reservations for the exam and grades will appear only in **Moodle**!

  (https://elearning.uniroma1.it/)

Brief outline of the course POM1

- Numbers and Functions
- Types of Functions
- Limits of a function
- Continuity of a function
- Derivatives
- Integrals

Material of the course POM1

- Books for extensive study, they contain more than the program of this course:
- ***Biocalculus: Calculus for Life Sciences****,* authors James Stewart, Troy Day – Cengage Learning 2015
- ***Calculus For Biology and Medicine****,* author Claudia Neuhauser - Pearson 2014
- Slides of the course, available from the home page of the professor (http://www.diag.uniroma1.it//~bruni/). They contain everything that is needed, if they are well studied and understood

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Real Numbers and Functions

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Sets of Numbers

If you count, the numbers you use are called counting numbers, or **natural numbers**. These numbers can be expressed using set notation

$$\{1, 2, 3, 4, . . . \}$$

If we include 0 we have the set of whole numbers

$$\{0, 1, 2, 3, 4, . . . \}$$

If we include also the opposites of the natural numbers we have the set of **integer numbers, or simply integers**

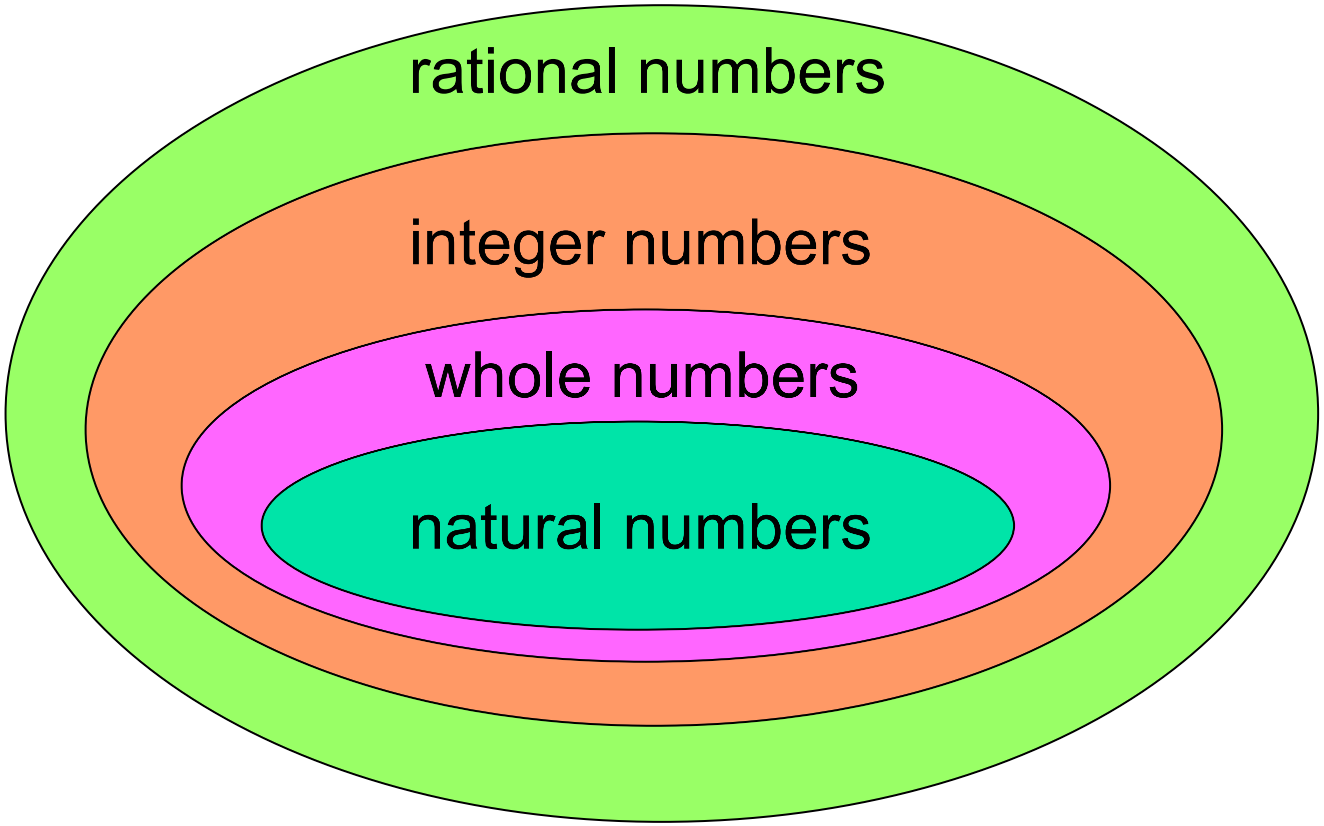
$$\{ \ldots, -3, -2, -1, 0, 1, 2, 3, . . . \}$$

Rational Numbers

If we consider a new set of the numbers obtainable as quotient of two integers (except /0), we have the set of **rational numbers**

This means to divide one integer by another or "make a fraction"

Es: ½, ¾, …



Real Numbers

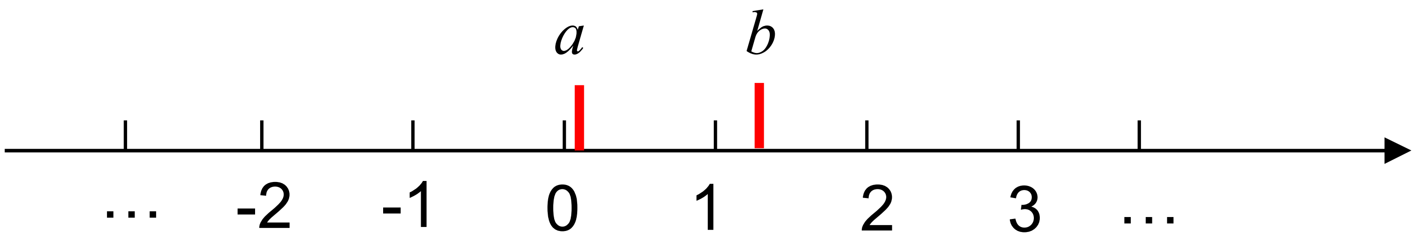
However, there are numbers that cannot be expressed as the quotient of two integers. These are called **irrational numbers**

Examples are π $\sqrt{2}$

The rational numbers together with the irrational numbers make the set of **real numbers**

The Line of Real Numbers

The line of real numbers is a graphical representation of the continuity of real numbers: every point of the line corresponds to a real number



a and b are two real numbers and, since a is on the left of b, we have $a < b$

In each interval of the real line there is an infinity of points, hence an infinity of real numbers

Sets

- Sets of real numbers are typically denoted by the capital letters **A, B, C,** etc. To describe a set **A**, we either write all the numbers in it (may be cumbersome), or we write

  - **A = {x : condition}**

- where "condition" defines which numbers are in A. We may read it as **the set of all real numbers x such that condition is verified**

- Example: the set of even numbers is {x: x divisible by 2}

Intervals

- An important type of sets are **intervals**. Given two numbers **a < b**, then
- this is an **open** interval **(a, b) = {x : a < x < b}**
  (it does not contain its extremes **a** and **b**)
- this is a **closed** interval **[a, b] = {x : a ≤ x ≤ b}**
  (it contains also its extremes)
- We can also use **half-open** intervals:
- **[a, b) = {x : a ≤ x < b}** and **(a, b] = {x : a < x ≤ b}**

Unbounded Intervals

- Some intervals may be **unbounded**, in other words they are sets of the form **{x : x > a}**. Here are the possible cases:
- **[a, ∞) = {x : x ≥ a}**
- **(−∞, a] = {x : x ≤ a}**
- **(a, ∞) = {x : x > a}**
- **(−∞, a) = {x : x < a}**
- Since ∞ is generally considered **not a real number**, we cannot use intervals closed over the ∞
- The real number line can be expressed as **R= (−∞, ∞)**

Functions

- Science often studies relationships between quantities (for instance, how the measure of a tree is related to its age, etc.). To describe such relationships mathematically, we use functions

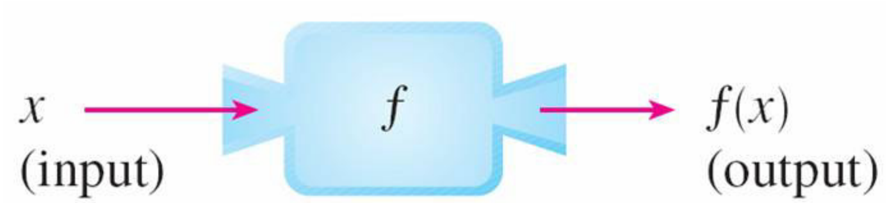
DEFINITION

- A **function** f is a rule that assigns to each element x in the set A exactly one element y in the set B
- The element y is called the **image** or **value** of x under f and is denoted by $f(x)$ (read " *f* of *x*")
- The set A is called the **domain** of f, the set B is called the **codomain** of f
- The set $f(A) = \{y : y = f(x)$ for $x \in A\}$ is called the **range** of f (all the values taken by y as x varies all over the domain)

- Example: The area a of a circle depends on the radius r of the circle. The rule that connects r and a is given by the equation $a = \pi r^2$. For each value of r there is a value of a, so a is a *function* of r

Functions

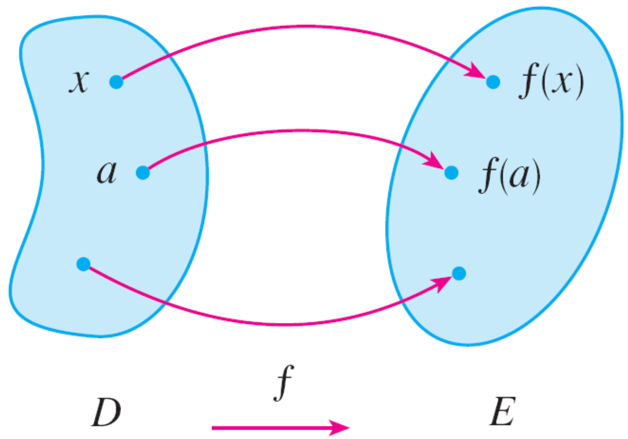
Since the concept of function is so widespread, it can be viewed in several ways. One can think of a function as a machine, and we have the machine diagram of a function



x is also known as the independent variable and $y = f(x)$ as the dependent variable.

Functions

Another way to picture a function is by an **arrow diagram**



Each arrow connects an element of the domain *D* to an element of the codomain *E*. For example, *f* (*x*) is associated with *x*, *f* (*a*) is associated with *a*, and so on. We often write something like this to describe a function

- $f : D \to E$

  $x \to f(x)$

Functions

There are several possible ways to provide a function. For example:

- **numerically** (by giving a table of values)
- **algebraically** (by giving a formula – more used)
- **visually** (by giving a graph – more used)
- **verbally** (by giving a description in words)

Function given by a table

- We define a function by providing a table with all pairs x and f(x), or at least some (for example, those that could be measured)

Example. The human population of the world P changes with time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

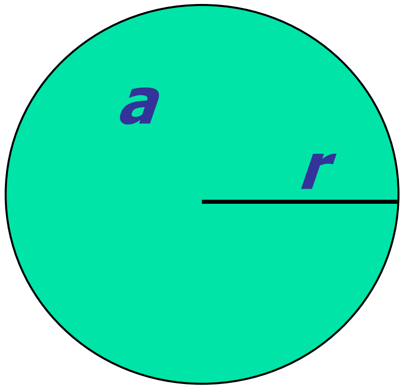
$$P(1950) \approx 2{,}560{,}000{,}000$$

Since for each value of the time t there is a corresponding value of P, and we can say that P is a function of t

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

Function given by a formula

- We define a function by providing the mathematical rule connecting each pair x and f(x)



- Example: The area a of a circle depends on the radius r of the circle. The rule that connects r and a is given by the equation $a = \pi r^2$

Graph of a function

The most common method for visualizing a function is its **graph**. If ***f*** is a function with domain ***D***, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

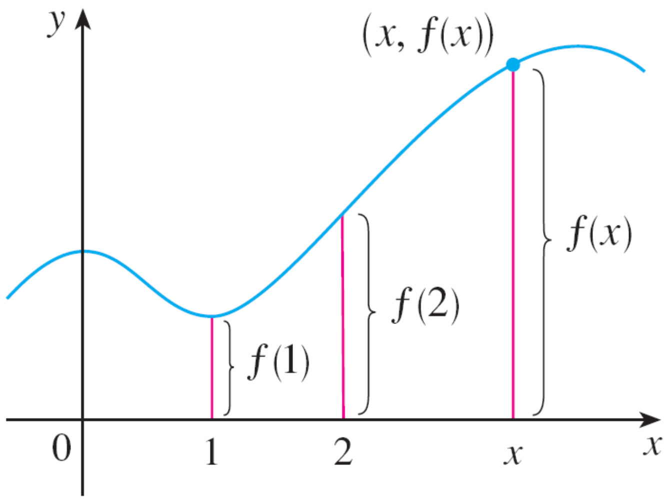
In other words, the graph of *f* consists of **all points *(x, y)* in the coordinate plane such that *y = f (x)*** and ***x*** is in the domain of ***f***

The graph of a function gives us a complete picture of the behavior of the function

Graph of a function

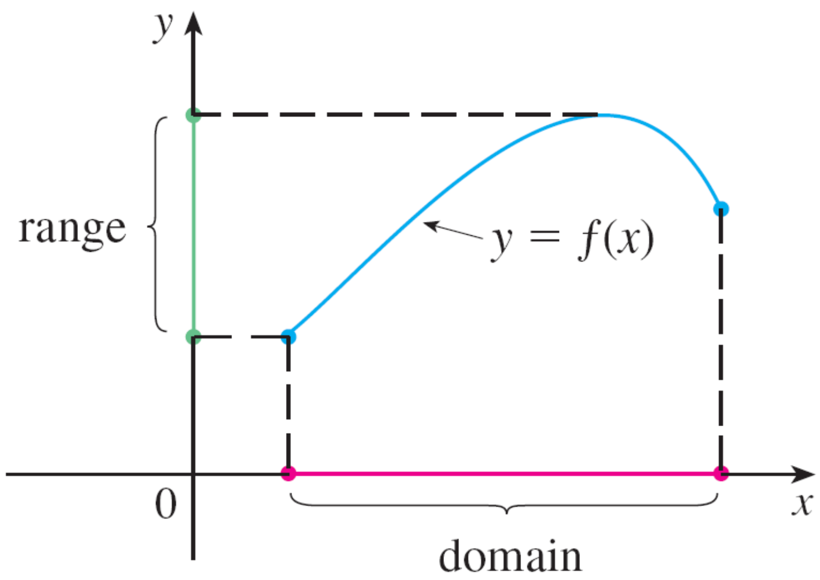
On the coordinate plan ***x y***, we consider the **line** (not necessarily straight neither necessarily continuous) composed of the points ***(x, y)*** such that each ***y = f(x)***

In other words, **for each value of *x***, we can read the **corresponding value of *f(x)*** from the graph as the height of the graph above the point ***x***



Graph of a function

The graph of *f* also allows to see the domain of *f* on the *x*-axis and its range on the *y*-axis

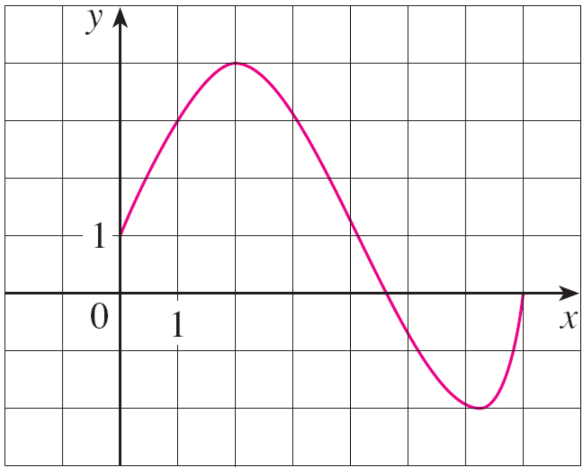


Example 1

The graph of a function *f* is shown in the figure

(a) Find the values of $f(1)$ and $f(5)$

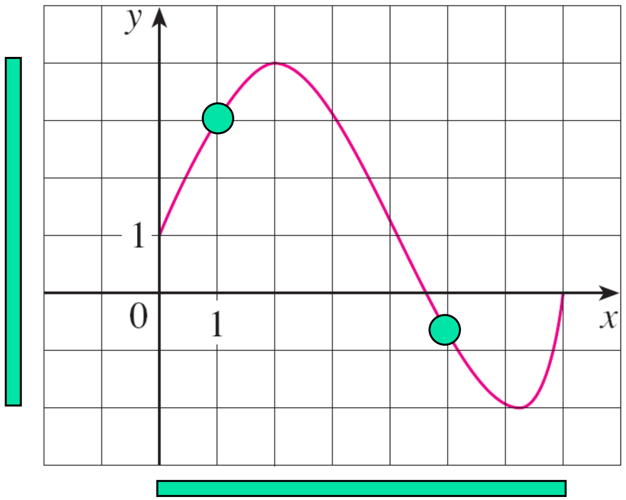
(b) What are the domain and range of *f* ?



Example 1 – *Solution*

(a) $f(1) = 3$ and $f(5) = -0.7$

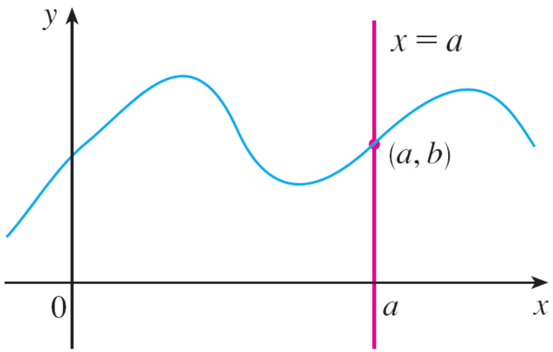
(b) the domain of f is $[0,7]$, the range of f is $[-2, 4]$

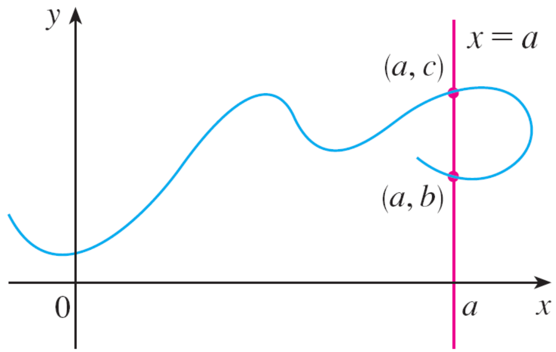


Test for the graph

The graph of a function is a curve in the *xy*-plane. But which curves in the *xy*-plane are graphs of functions? We have a test:

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.





Indeed, if each vertical line $x = a$ intersects the graph **only once**, then exactly one functional value is defined for $f(a)$

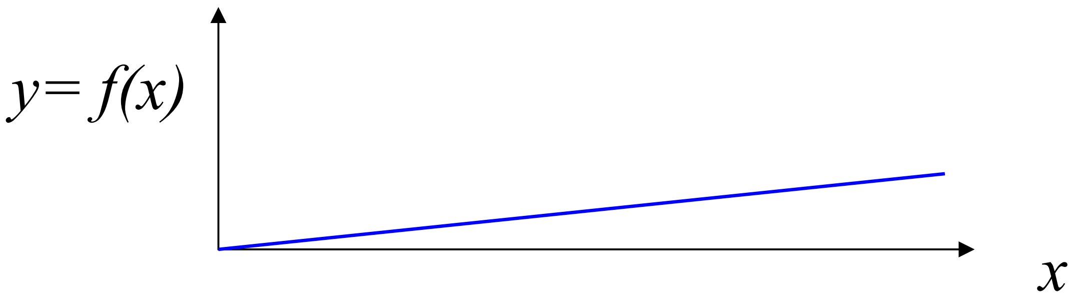
On the contrary, if a line $x = a$ intersects the curve **twice** (or more) then the curve can't represent a function because a function can't assign two different values to a

Linear Functions

In many cases, the relationship represented by a function is **linear**, in the sense that it is given by a **linear equation**, and the graph is a **straight line**

Example

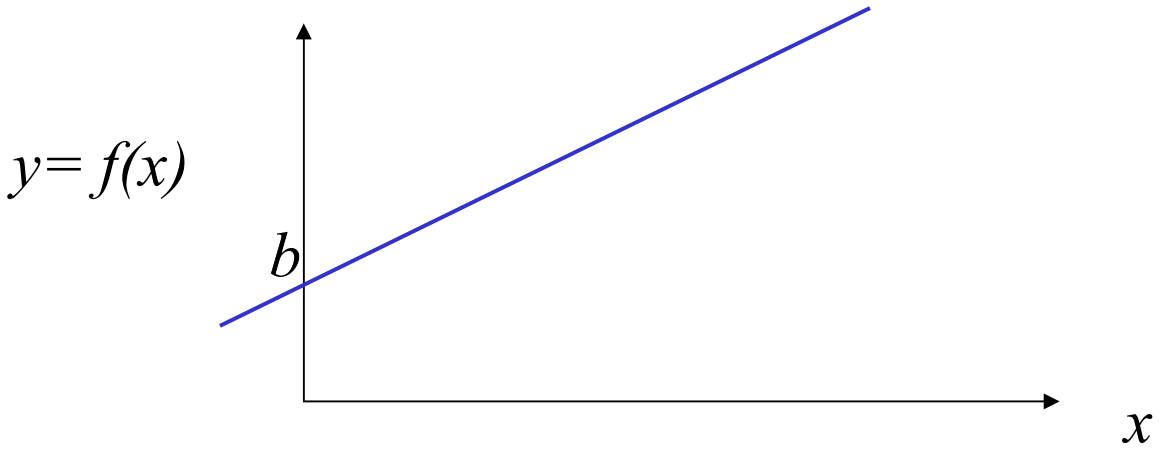
The amount of fuel y in liters consumed by a car can be seen as a linear function of the distance x in kilometers.
The function would be $y = c\,x$ with c equal for example to $1/20$
Note that, in many cases, reality can be seen as linear by allowing some approximation!



Linear Functions

In general, a linear function is $y = mx + b$

where m is the **slope** and b is the **y-intercept**, which is the point of intersection of the line with the y-axis which has coordinates $(0, b)$



Linear Functions

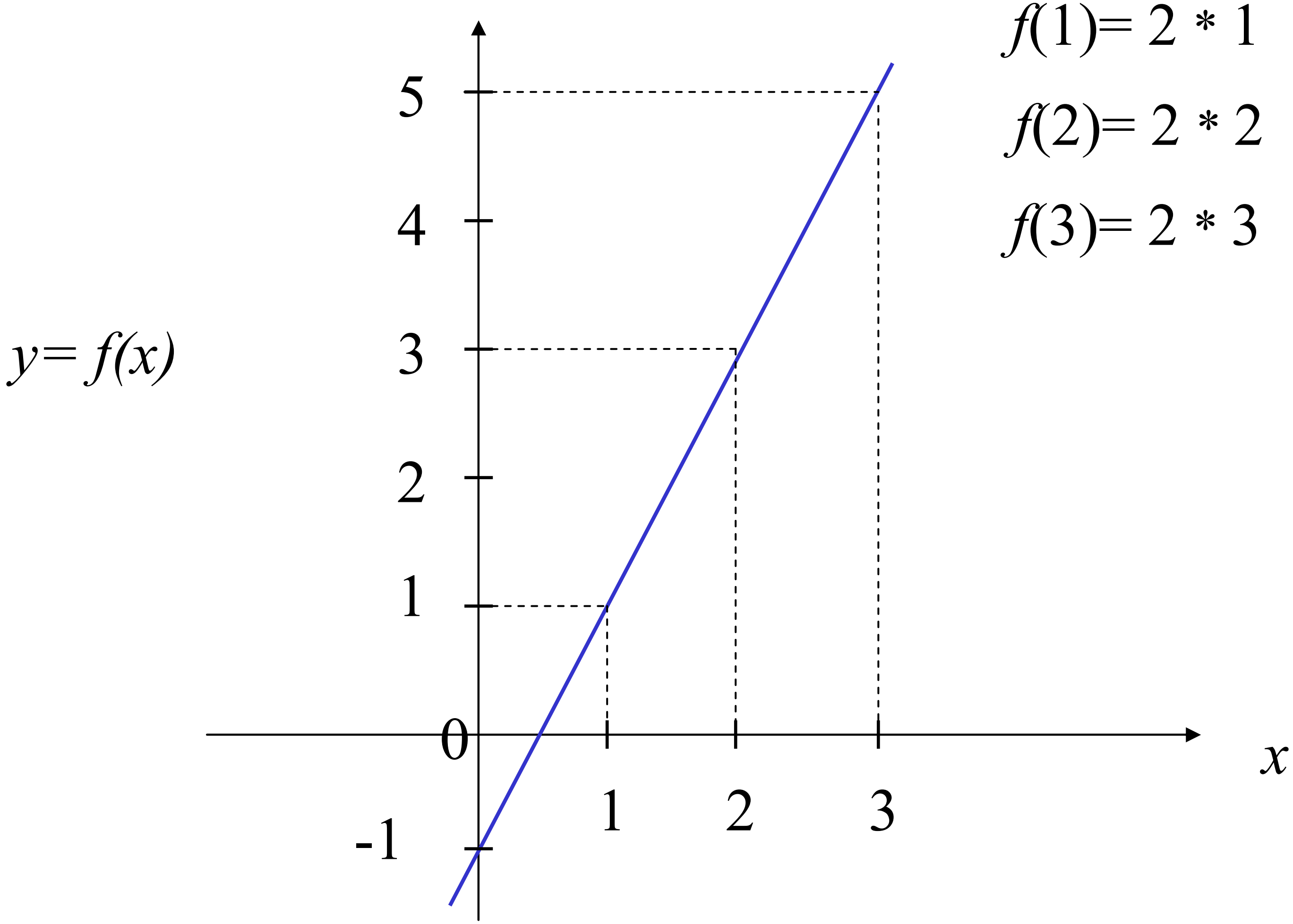
Example $y = 2x - 1$

$f(0) = 2 * 0 - 1 = -1$

$f(1) = 2 * 1 - 1 = 1$

$f(2) = 2 * 2 - 1 = 3$

$f(3) = 2 * 3 - 1 = 5$



Linear Functions

Example $y = -1/2\, x + 2$

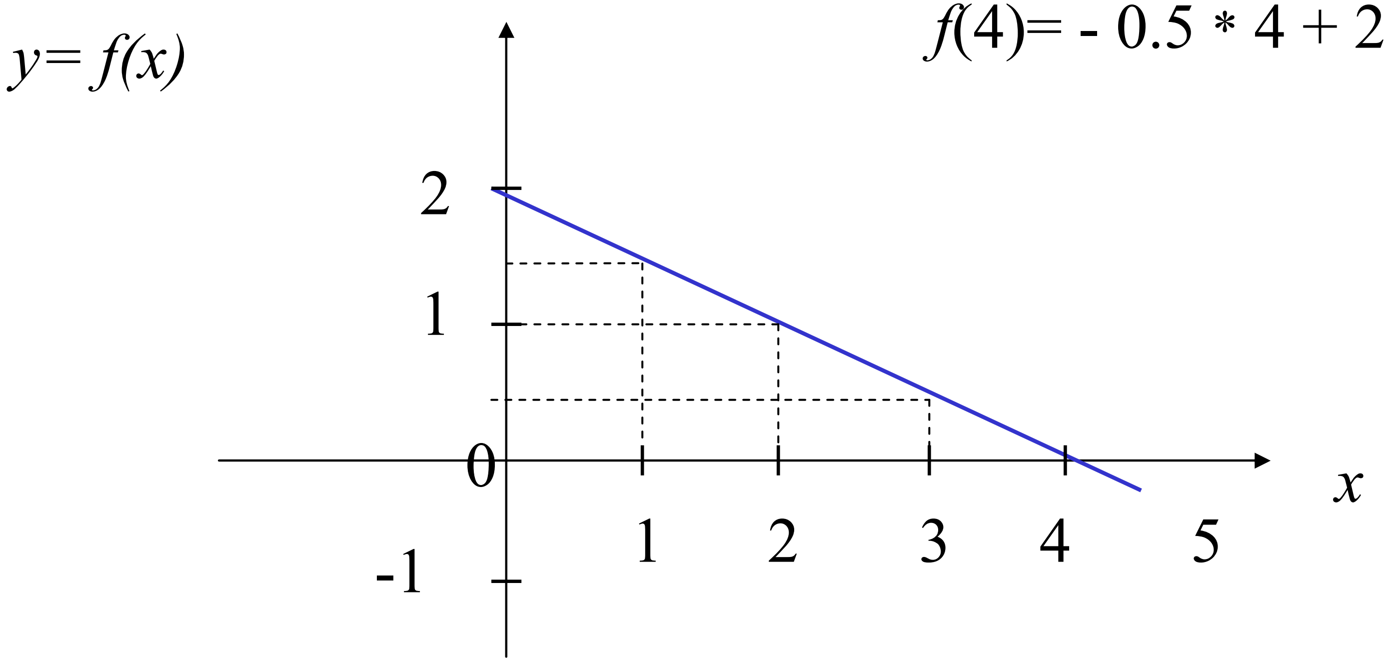
$f(0) = -0.5 * 0 + 2 = 2$

$f(1) = -0.5 * 1 + 2 = 1.5$

$f(2) = -0.5 * 2 + 2 = 1$

$f(3) = -0.5 * 3 + 2 = 0.5$

$f(4) = -0.5 * 4 + 2 = 0$



Piecewise Defined Functions

Some functions have a graph composed of **several** parts, and not of a single line or curve. These functions are called **Piecewise Functions**

Example

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph

Example

$$1 - x \quad \text{if } x \le -1$$

$$x^2 \quad \text{if } x > -1$$

Solution:

Remember that a function is a rule. For this particular function the rule is the following:

First look at the value of the input x. If it happens that $x \le -1$, then the value of $f(x)$ is $1 - x$

On the other hand, if $x > -1$, then the value of $f(x)$ is x^2

Example – *Solution*

Since $-2 \le -1$, we have $f(-2) = 1 - (-2) = 3$

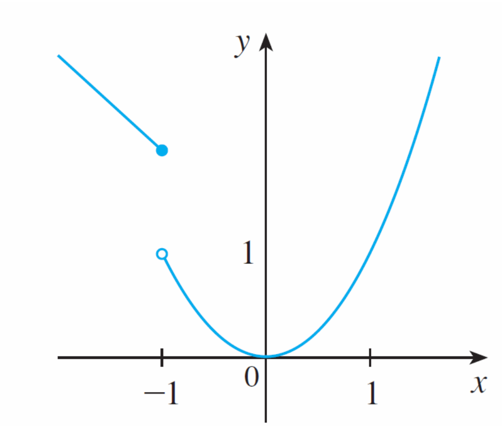
Since $-1 \le -1$, we have $f(-1) = 1 - (-1) = 2$

Since $0 > -1$, we have $f(0) = 0^2 = 0$

How do we draw the graph of f ? We observe that if $x \le -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1

Example – *Solution*

If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. Hence:



The solid dot indicates that the point (–1, 2) is included in the graph; the open dot indicates that the point (–1, 1) is excluded from the graph

Piecewise Defined Functions

Another example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \qquad \text{for every number } a$$

For example,

$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$

$|3 - \pi| = \pi - 3$

Piecewise Defined Functions

In general, for the absolute value we have

$$|a| = a \quad \text{if } a \geqslant 0$$
$$|a| = -a \quad \text{if } a < 0$$

(Remember that if *a* is negative, then –*a* is positive)

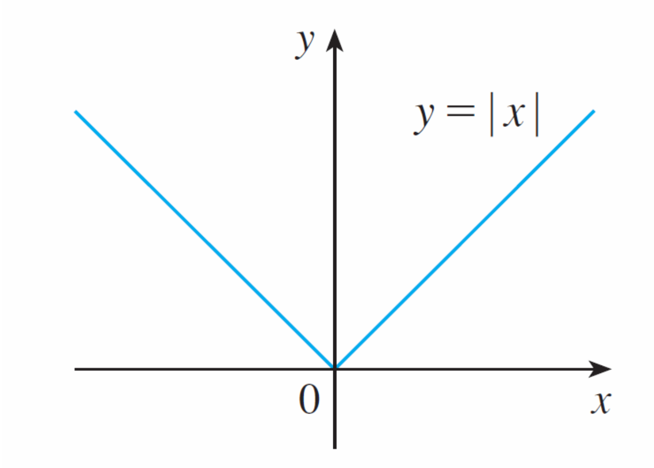
Example

Sketch the graph of the absolute value function $f(x) = |x|$

Solution: From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis



Example

consider the cost $C(w)$ of mailing a large envelope with weight w

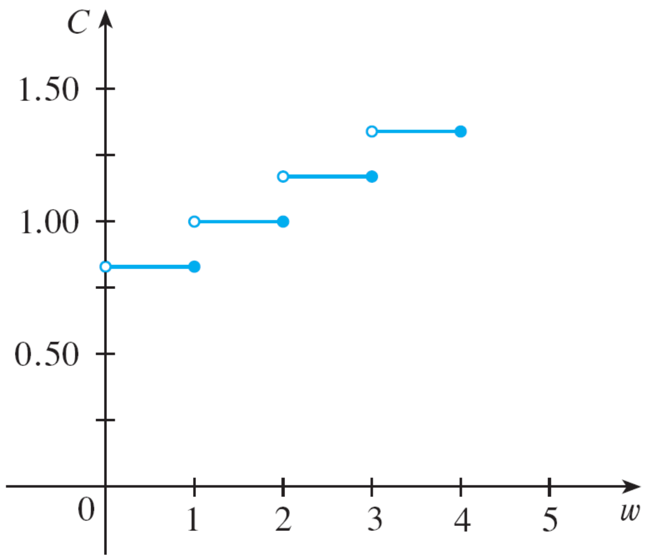
In effect, this is a piecewise defined function because, for each interval of weight there is a different cost, as follows (note that the numbers are just examples)

$$C(w) = \begin{cases} 0.88 \text{ Euro if } 0 < w \leq 1 \\ 1.05 \text{ Euro if } 1 < w \leq 2 \\ 1.22 \text{ Euro if } 2 < w \leq 3 \\ 1.39 \text{ Euro if } 3 < w \leq 4 \\ \vdots \end{cases}$$

Example

cont'd

The graph is the following



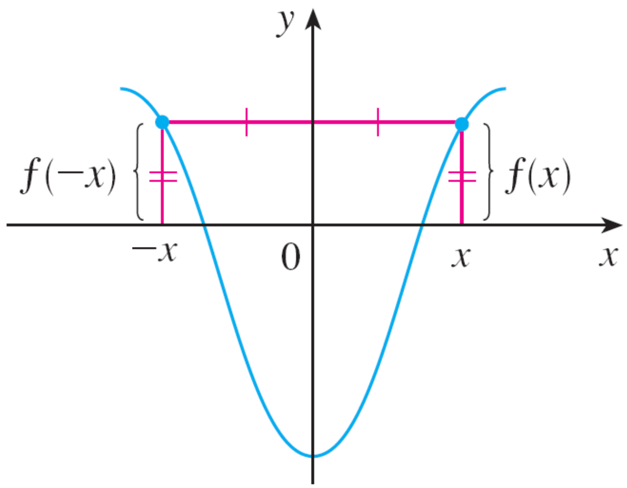
You can see why functions similar to this one are called **step functions**—they jump from one value to the next

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even

Geometrically, this means its graph is symmetric with respect to the y-axis

So, if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis



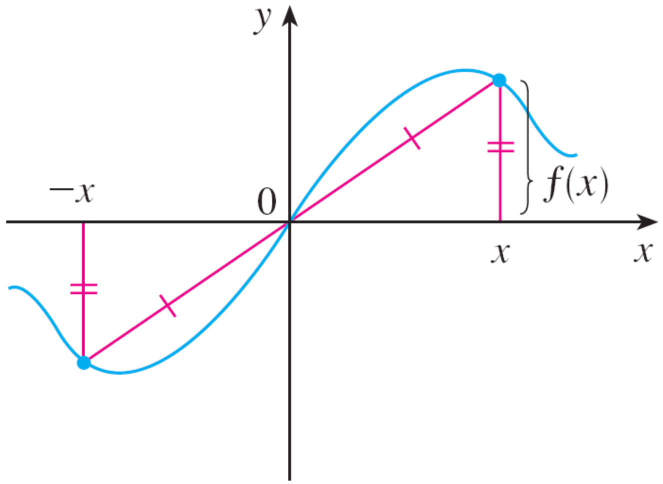
An even function

Symmetry

On the contrary, if f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd

The graph of an odd function is symmetric about the origin

If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin



An odd function

Example

Determine whether each of the following functions is even, odd, or neither even nor odd

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Solution:

(a) $f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$

$= -x^5 - x = -(x^5 + x)$

$= -f(x)$

Therefore f is an odd function

Example – *Solution*

cont'd

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

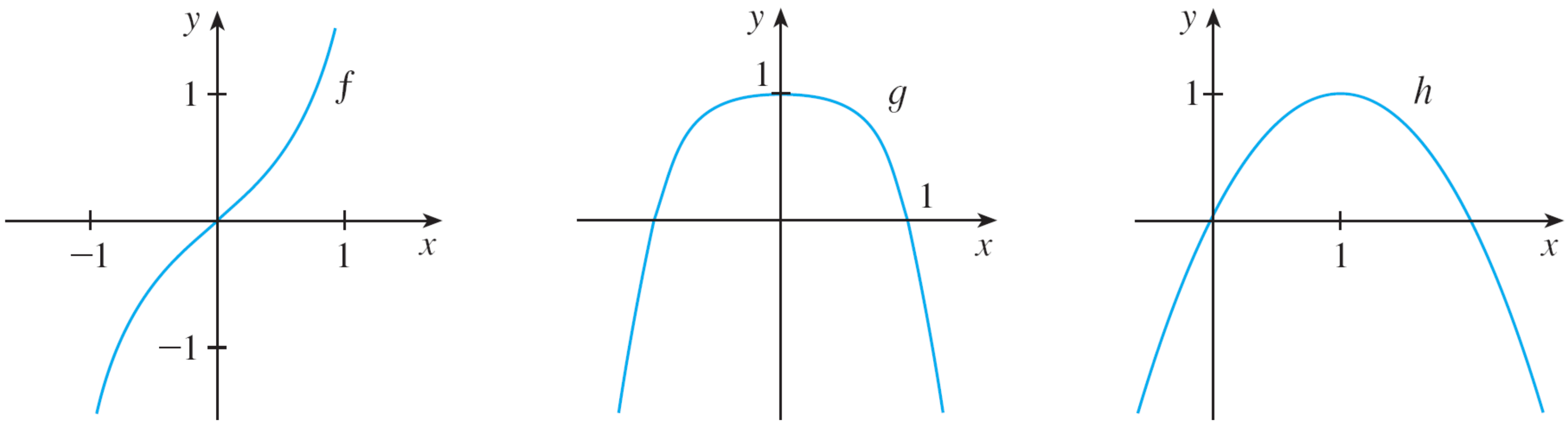
So g is even

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd

Example – *Solution*

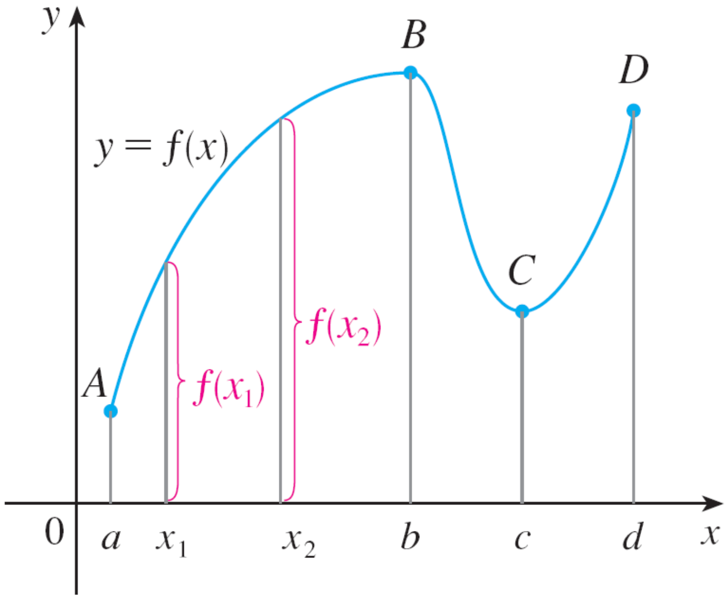
These are the graphs of the functions examined before. Notice that the graph of *h* is symmetric neither about the *y*-axis nor about the origin



(a) (b) (c)

Increasing & decreasing functions

The graph shown here rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be increasing on the interval [*a*, *b*], decreasing on [*b*, *c*], and increasing again on [*c*, *d*].



Increasing & decreasing functions

Notice that if x_1 and x_2 are any two numbers between *a* and *b* with $x_1 < x_2$, then $f(x_1) < f(x_2)$

We use this as the defining property of an increasing function

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

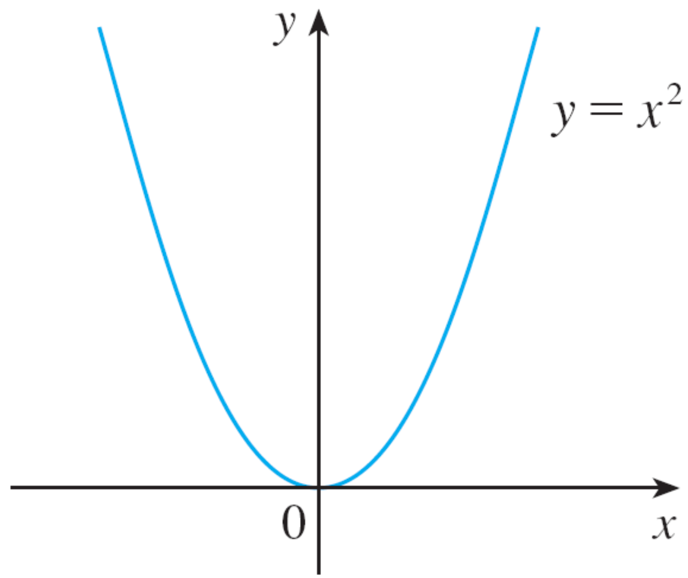
It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Increasing & decreasing functions

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$

For example the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$

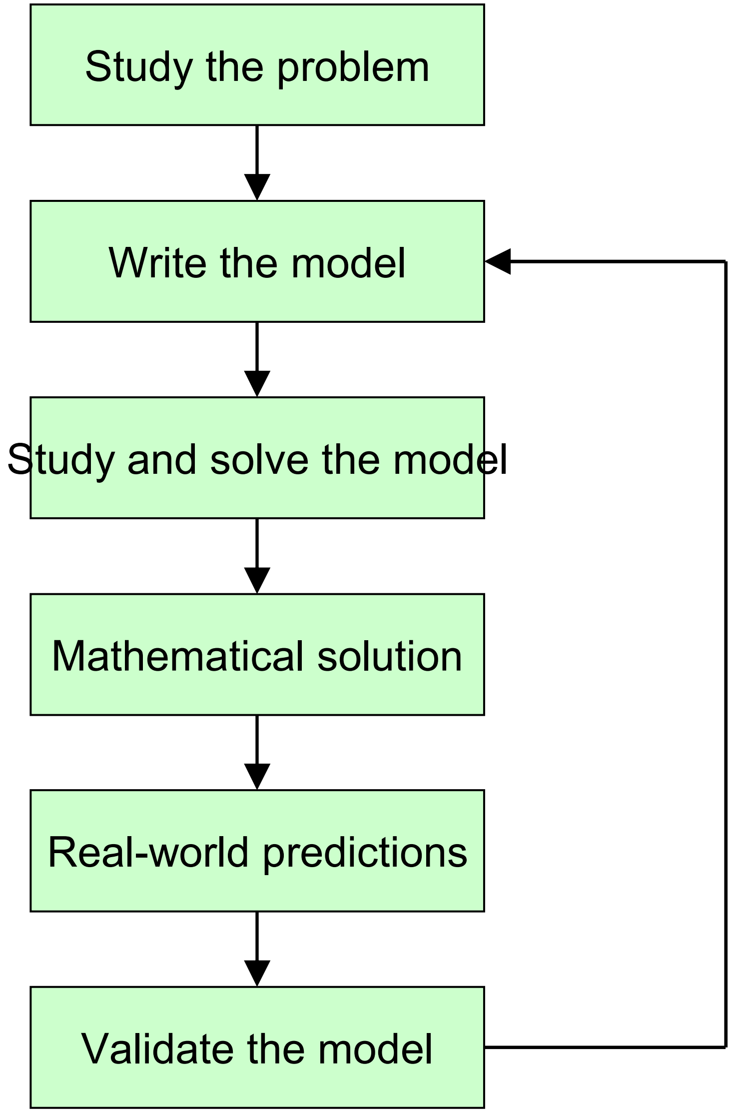


Functions and models

- There exist many types of functions, we will review some of the most common
- In many cases, functions are created to build mathematical models of something we want to study
- A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, etc.
- The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior

How to make a model ?

- Real world relationships among quantities are represented by **mathematical relationships (=functions)**
- A model must contain **all** and **only** the essential aspects of the phenomenon
- When we compute a matematical solution, we **evaluate if it is reasonable**. If it is not, we probably forgot some essential aspect of the problem in the definition of the model. We need to **go back to model definition** and solve again
- Example: we obtain a negative value for something that must be >=0? We forgot to specify non-negativity in the model



Advantages of the model

- We use the **power of mathematics** to find a solution
- We may mathematically **discover important properties** of the practical problem (for example, we discover that a quantity a is always double than b, and this was previously unknown)
- We may use mathematical **simulations** (for example, we do need to build a bridge and see whether it falls down or not, we simulate its behavior)

- **Criticisms** to the use of mathematical models
- the quality of the answer depends on the **quality of the data** (garbage in, garbage out) but this is inevitable
- Not everything can be **quantified** (for example, subjective evaluations). However, we can do our best…

Accuracy of models

A mathematical model is often a not completely accurate representation of a physical reality, especially if the reality is complex — it is an *idealization.* A good model **simplifies reality enough to permit mathematical calculations** but is **accurate enough to provide valuable conclusions**

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data

We seek a curve that “fits” the data in the sense that it captures the basic trend of the data points

Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Example 1 (a) – *Solution*

Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \bullet 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \bullet 1 + 20$$

The slope of the line is therefore $m = 10 – 20 = –10$ and the required linear function is

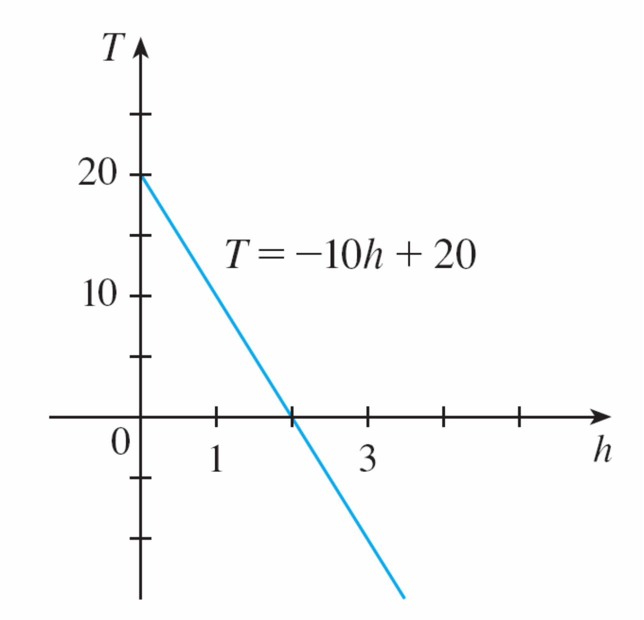
$$T = –10h + 20$$

Example 1(b) – *Solution*

cont'd

The graph is sketched in Figure

The slope is $m = -10°\text{C/km}$, and this represents the rate of change of temperature with respect to height.



Example 1(c) – *Solution*

cont'd

At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°\text{C}$$

Polynomial functions

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. The degree of the polynomial is given by the leading coefficient $a_n \neq 0$. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6

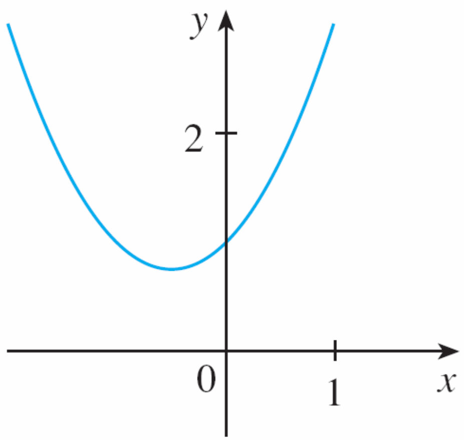
Polynomial functions

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function

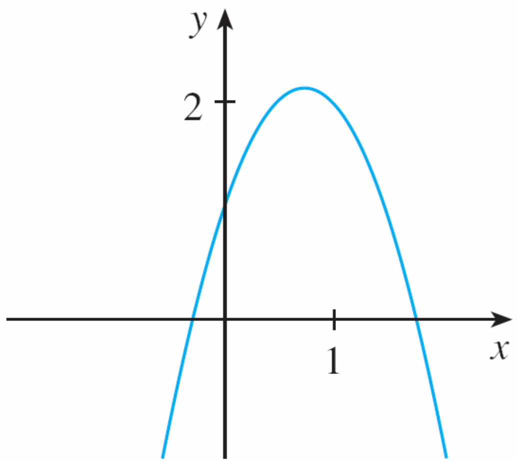
A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**

Quadratic functions

The graph of a quadratic function is always a parabola, obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$



(a) $y = x^2 + x + 1$



(b) $y = -2x^2 + 3x + 1$

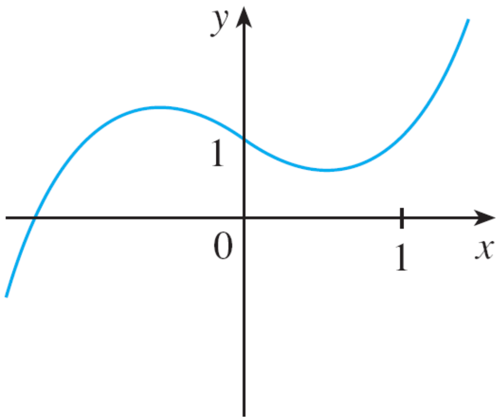
The graphs of quadratic functions are parabolas

Cubic functions

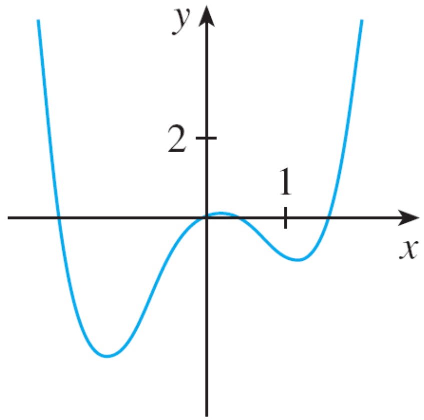
A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

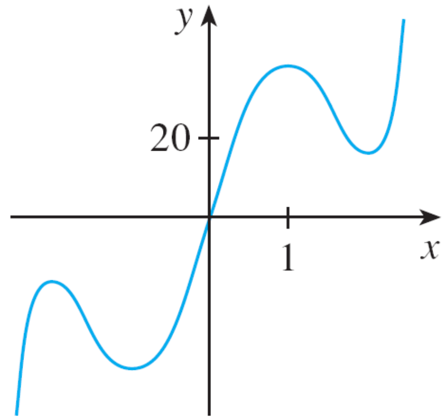
and is called a **cubic function**. We have the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c)



(a) $y = x^3 - x + 1$



(b) $y = x^4 - 3x^2 + x$



(c) $y = 3x^5 - 25x^3 + 60x$

Example

A ball is dropped from the top of a skyscraper, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2

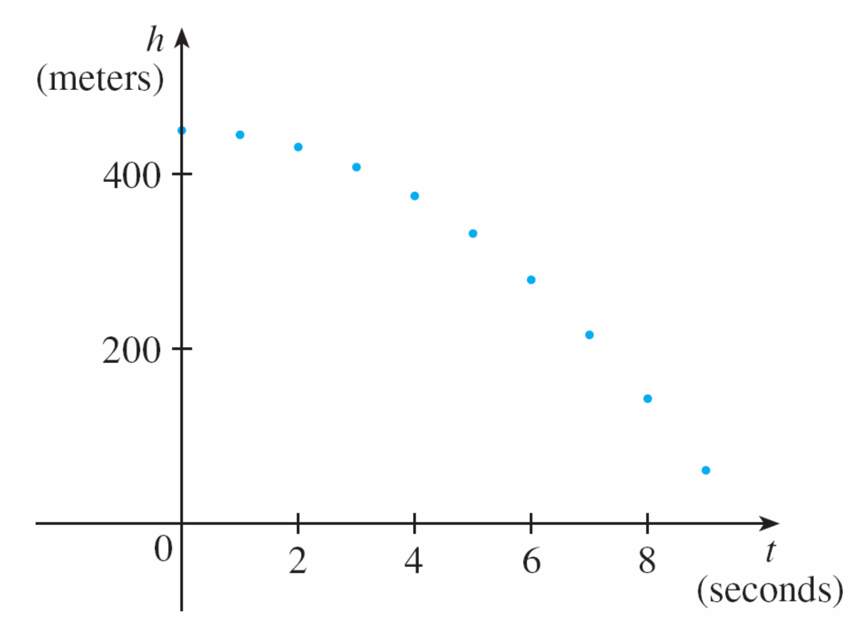
Find a model to fit the data and use the model to predict the time at which the ball hits the ground

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Example – *Solution*

We draw a scatter plot of the data and observe that a linear model is inappropriate



Scatter plot for a falling ball

Example – *Solution*

cont'd

But it looks as if the data points might lie on a parabola, so we try a quadratic model instead

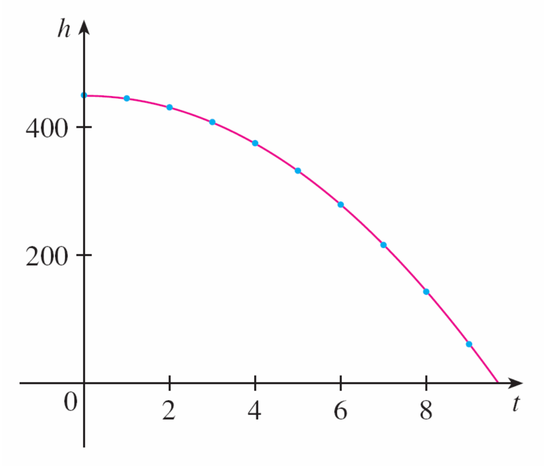
Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$h = 449.36 + 0.96t - 4.90t^2$$

Example – Solution

cont'd

We plot the graph of the quadratic function together with the data points and see that the quadratic model gives a very good fit:



Quadratic model for a falling ball

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

Example – *Solution*

cont'd

The formula for second degree equations gives

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

There are 2 solutions, but only one is positive, so only one is acceptable. That is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds

Power functions

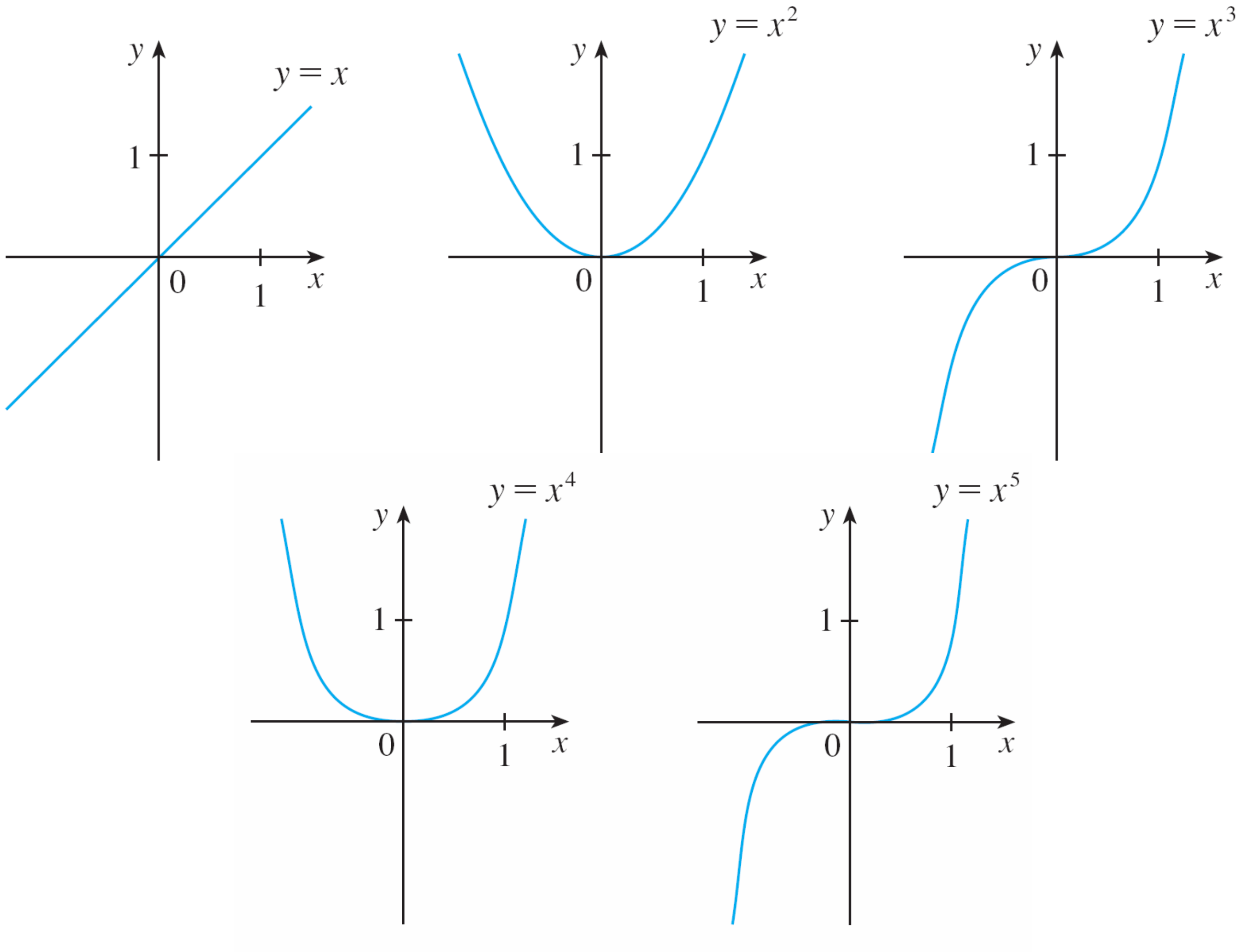
A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases

(i) ***a* = *n*, where *n* is a positive integer**

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in the next slide. (These are polynomials with only one term)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola)

Power functions



Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Power functions

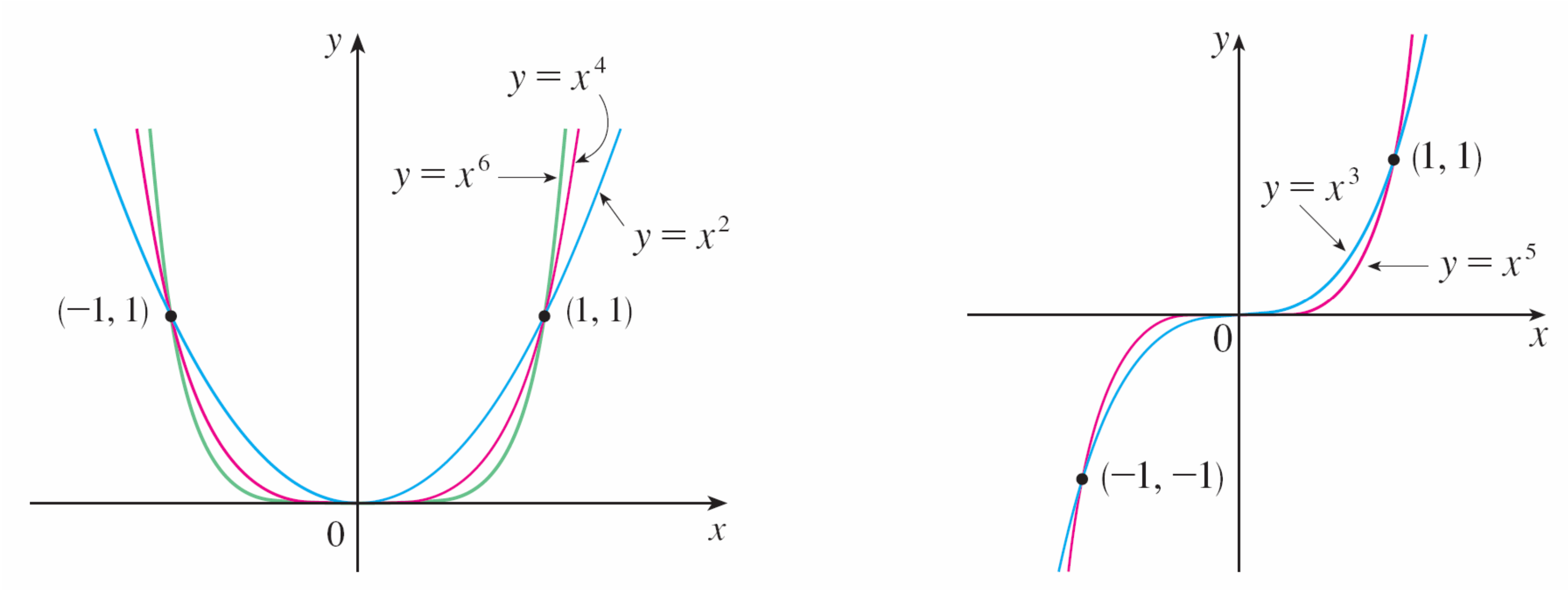
The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd

If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$

If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$

Power functions

Notice, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$.
(If x is smaller than 1, then x^2 is smaller, x^3 is even smaller, and so on. If x is larger than 1, then x^2 is larger, etc.)

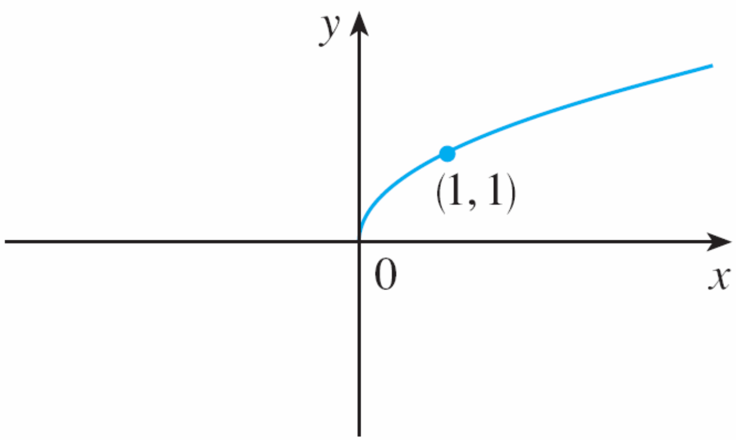


Families of power functions

Power functions

(ii) *a* = 1/*n*, where *n* is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For *n* = 2 it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$



$f(x) = \sqrt{x}$

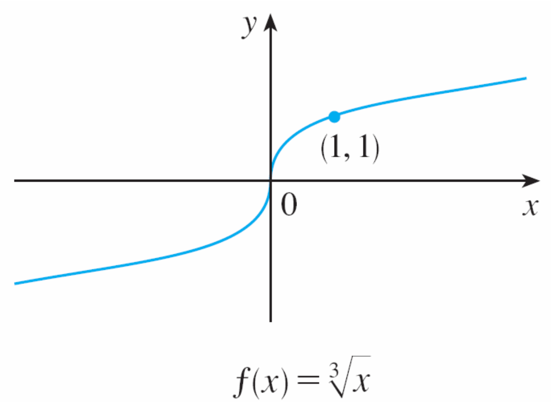
Graph of root function

Power functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) The graph of

$y = \sqrt[n]{x}$ for *n* odd (*n* > 3) is similar to that of $y = \sqrt[3]{x}$.

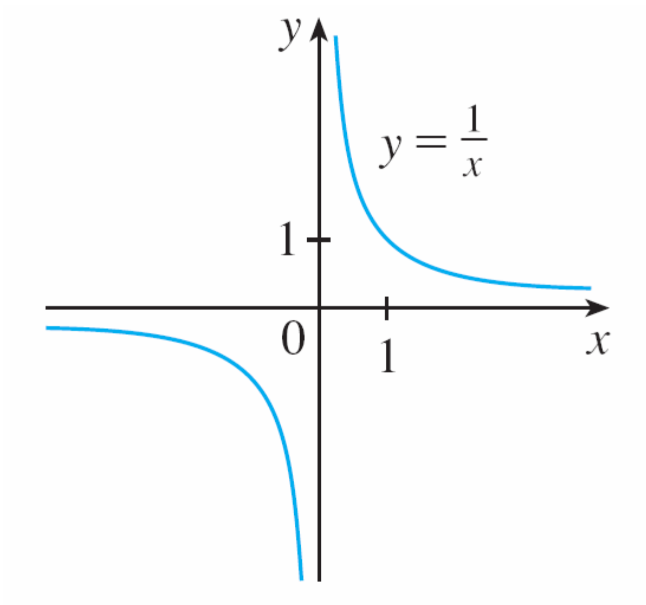


Graph of root function

Power functions

(iii) *a* = –1

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown here. It has equation $y = 1/x$, or $xy = 1$, and it is a hyperbola with the coordinate axes as its asymptotes



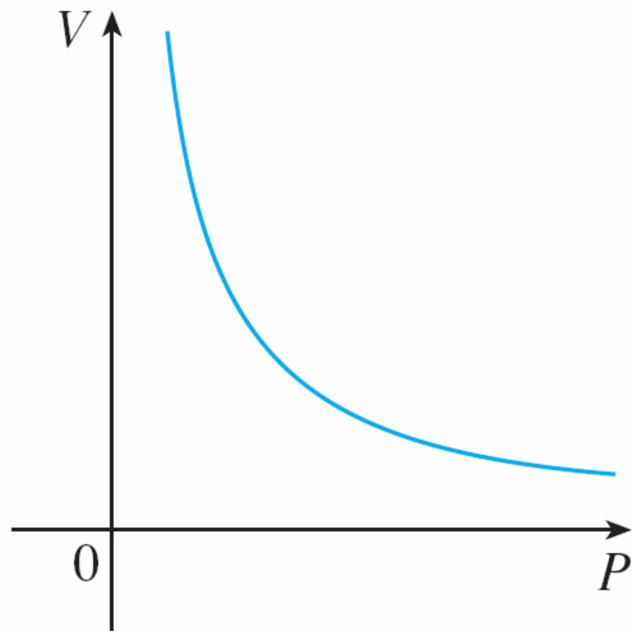
The reciprocal function

Example of reciprocal function

This function arises for example in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume *V* of a gas is inversely proportional to the pressure *P*:

$$V = \frac{C}{P}$$

where *C* is a constant



Volume as a function of pressure
at constant temperature

Algebraic functions

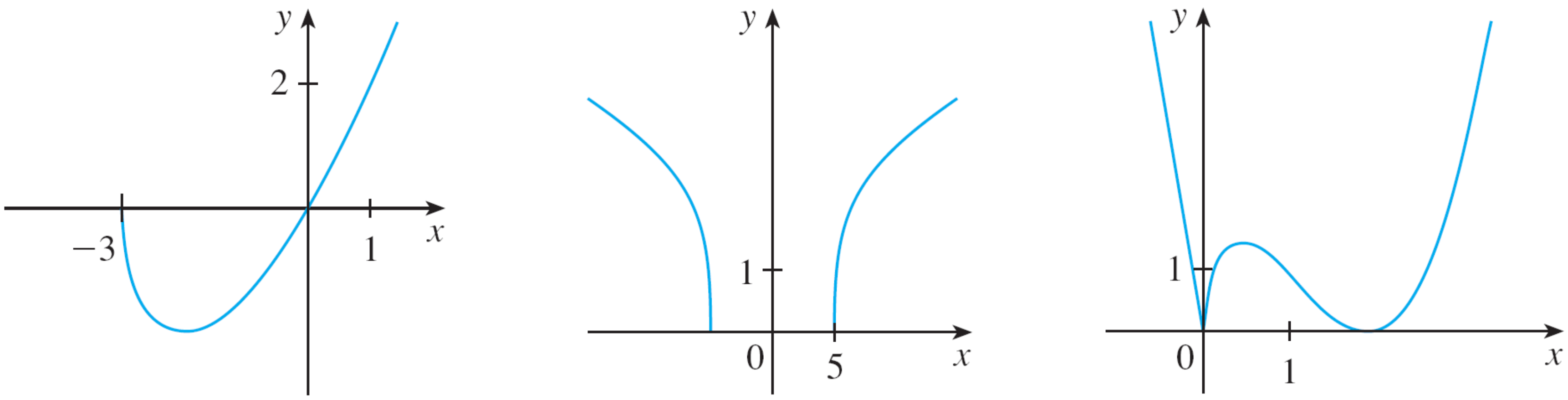
A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials

Here are two examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic functions

The graphs of algebraic functions can assume a variety of shapes. Here are some of the possibilities.



(a) $f(x) = x\sqrt{x+3}$

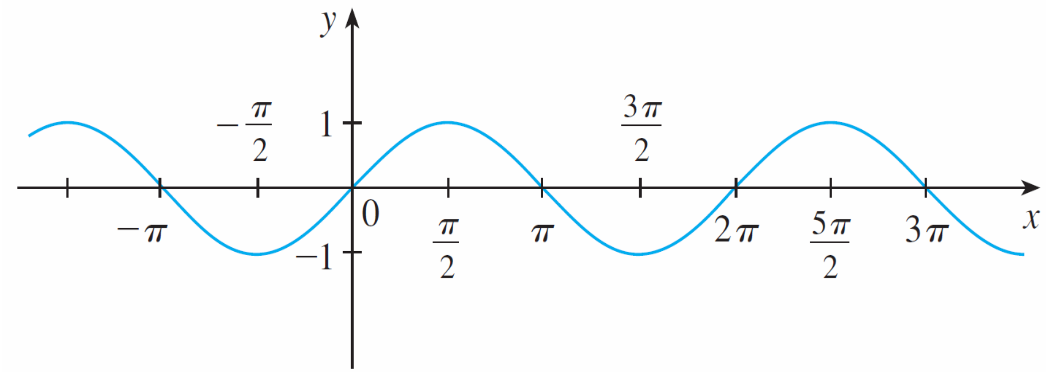
(b) $g(x) = \sqrt[4]{x^2 - 25}$

(c) $h(x) = x^{2/3}(x-2)^2$

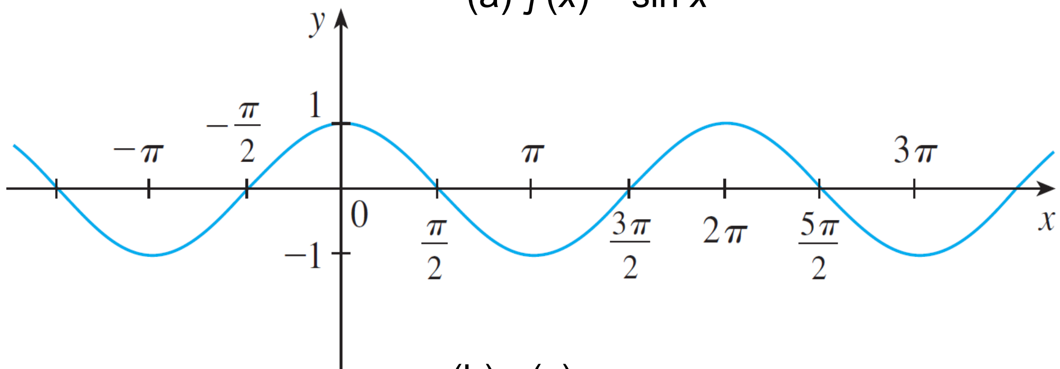
Trigonometric functions

Trigonometric functions are those using sin(x), cos(x), tan(x), etc

For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

Trigonometric functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1]

Thus, for all values of *x*, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1$$

Trigonometric functions

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \qquad \text{when} \qquad x = n\pi \qquad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are **periodic functions** and have period 2π

This means that, for all values of *x*,

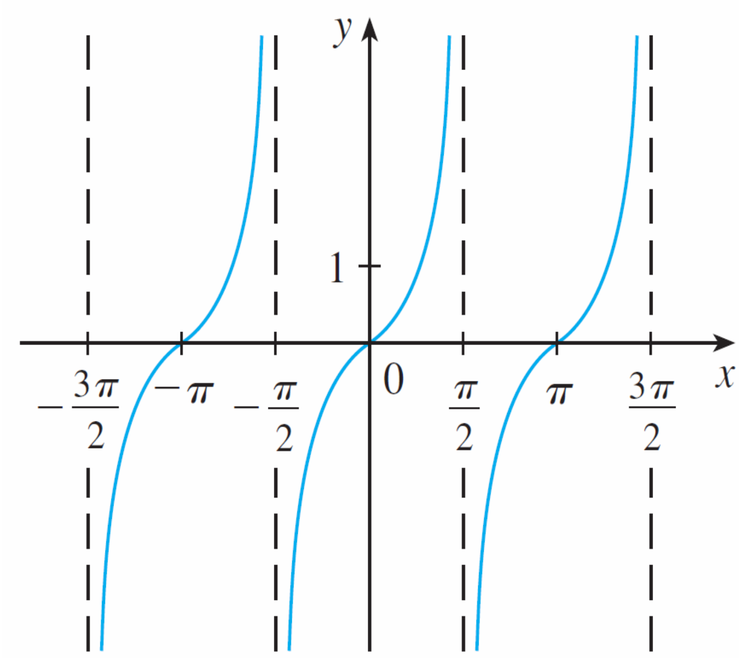
$$\sin(x + 2\pi) = \sin x \qquad\qquad \cos(x + 2\pi) = \cos x$$

Trigonometric functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

Note that it is undefined whenever cos x = 0, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$



$y = \tan x$

Its range is $(-\infty, \infty)$

Trigonometric functions

Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions

Exponential functions

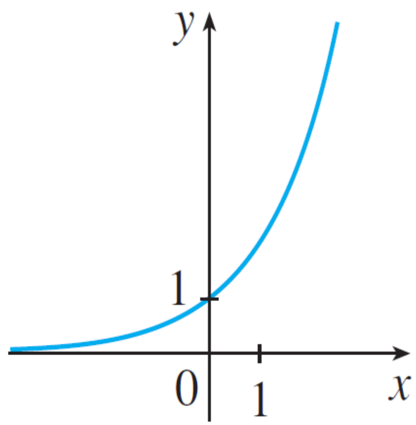
The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant and the independent variable x is its power

Note the difference with power functions: here x is the exponent, not the base!

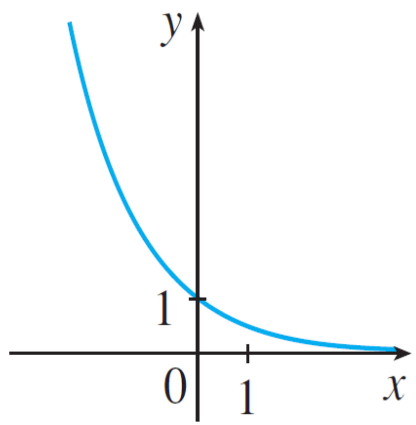
Consider the graphs of $y = 2^x$ and $y = (0.5)^x$

In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$

They all pass by the point (0,1) since any base with exponent 0 gives 1



(a) $y = 2^x$



(b) $y = (o.5)^x$

Exponential functions

Exponential functions are useful for modeling many natural phenomena

If $a > 1$ they can model something **increasing**, where the more we have, the faster it increases, such as population growth

If $a < 1$ they can model something **decreasing**, where the less we have, the slower it decreases, such as radioactive decay

Example:
Applications of Exponential Functions

- The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth.

- First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

Example:

Applications of Exponential Functions

If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

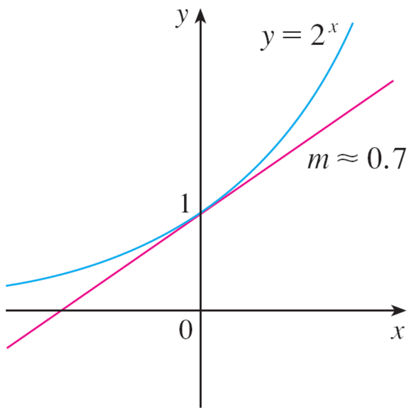
- $p(1) = 2p(0) = 2 \times 1000$
- $p(2) = 2p(1) = 2^2 \times 1000$
- $p(3) = 2p(2) = 2^3 \times 1000$

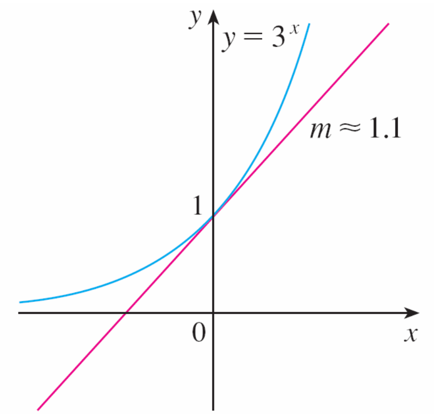
From this pattern, we infer that the population is a constant multiple of the exponential function $y = 2^t$

- $p(t) = 2^t \times 1000 = (1000)2^t$

The number e

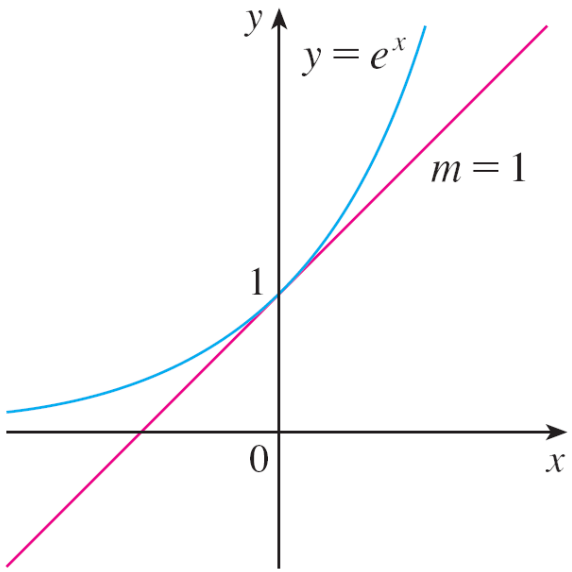
- Of all possible bases for an exponential function, there is one that is particularly convenient. The choice of a base a is influenced by the way the graph of $y = a^x$ crosses the y-axis. Figures show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point (0, 1) and the slope m





The number *e*

- It turns out that some of the formulas of calculus will be greatly simplified if we choose the base *a* so that the slope of the tangent line to $y = a^x$ at (0, 1) is *exactly* 1



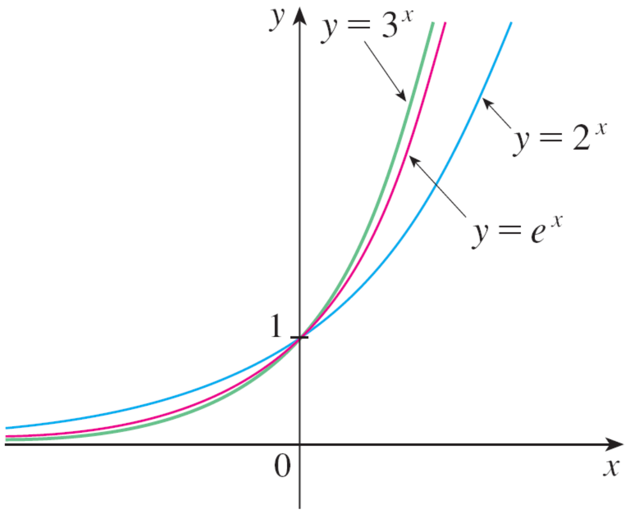
The natural exponential function crosses the *y*-axis with a slope of 1

The number *e*

- This number exists, and it is called *e*. It lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$

- The value of *e*, correct to 5 decimal places, is $e \approx 2.71828$

- We call the function $f(x) = e^x$ the **natural exponential function**

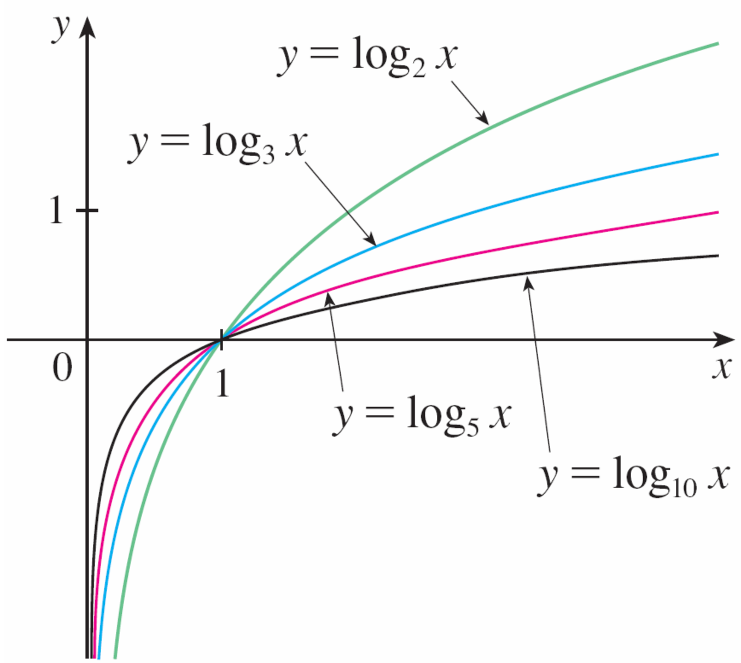


Logarithmic functions

A **logarithmic function** is $f(x) = \log_a x$, where the base a is a positive constant, and is the inverse functions of the exponential functions. Recall that $\log_a x$ means: the exponent that we must give to the base a to obtain x

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$

They all pass by the point (1,0) since to obtain 1 we give to any base exponent 0



Example

Classify the following functions as one of the types of functions that we have discussed

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Example – *Solution*

(a) $f(x) = 5^x$ is an exponential function (x is the exponent)

(b) $g(x) = x^5$ is a power function (x is the base)
We could also consider it to be a polynomial of degree 5

(c) $h(x) = \frac{1 + x}{1 - \sqrt{x}}$ is an algebraic function

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4

Combinations of functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$

Combinations of functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$

Since *y* is a function of *u* and *u* is, in turn, a function of *x*, it follows that *y* is ultimately a function of *x*. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions *f* and *g*.

Combinations of functions

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g *and* is denoted by $f \circ g$ ("f circle g")

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined

Example

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$