Bachelor’s degree in Bioinformatics

Derivatives

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Derivatives and Rates of Change

- The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of **limit**: the limit of the **increment in a function *f*(*x*) divided by the increment in the variable *x***
- This special type of limit is called a **derivative** and we will see that it can be interpreted as a **rate of change**
- This rate of change is needed in a lot of **practical applications**

Tangent

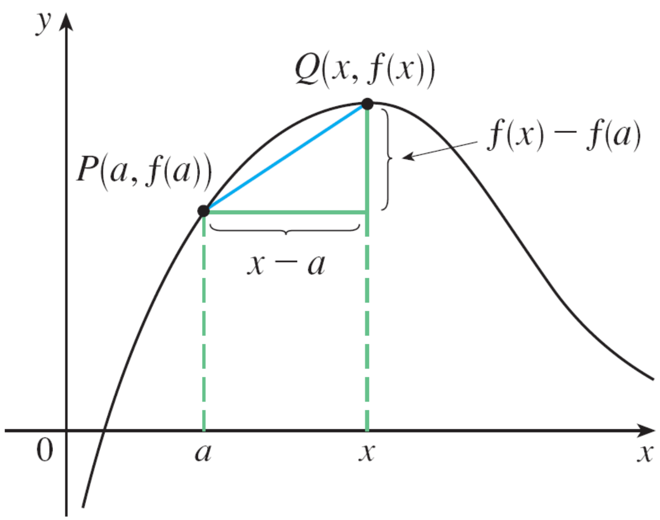
- If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

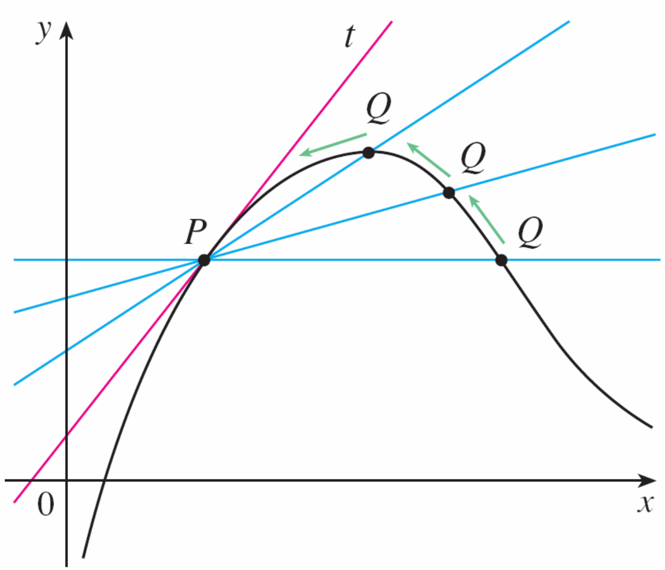
$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

- Then we let Q approach P along the curve C by letting x approach a

Tangent

- If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m
- the tangent line is the “limit” of the secant line PQ as Q approaches P





Tangent

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example

- Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$

- Solution:
- Here we have $a = 1$ and $f(x) = x^2$ so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

Example – Solution

cont'd

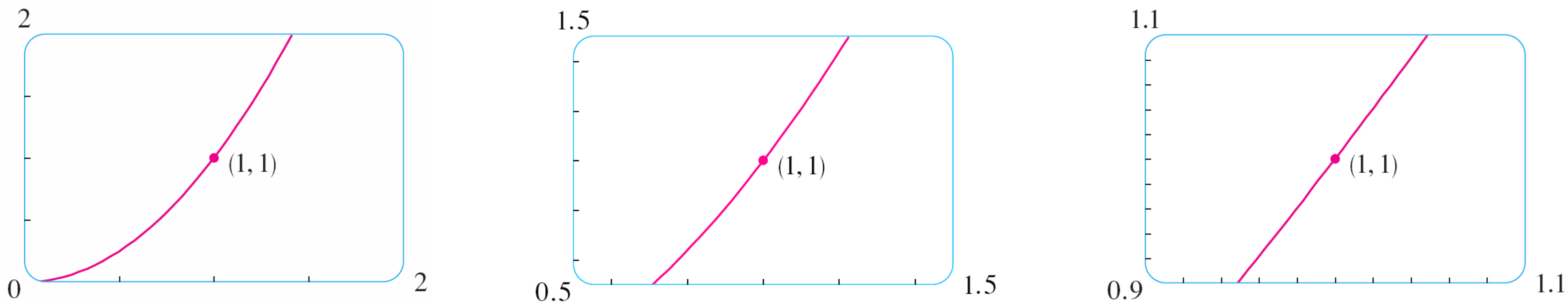
- $\lim_{x \to 1} (x + 1) = 1 + 1 = 2$

- Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) with slope 2 is

- $y - 1 = 2(x - 1)$ or $y = 2x - 1$

Tangent

- We will say that the slope of the tangent line to a curve at a given point is the **slope of the curve** at that point
- The idea is that if we **zoom** in far enough toward the point, the **curve looks almost like a straight line**
- In other words, the curve becomes **almost indistinguishable from its tangent line**



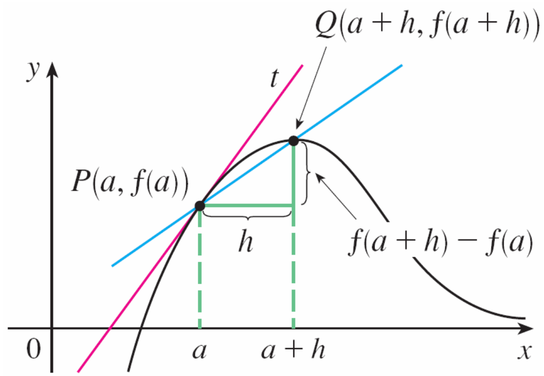
Zooming in toward the point (1, 1) on the parabola $y = x^2$

Tangent

- If we write $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ can also be written

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

- The case for $h > 0$ is illustrated (Q is to the right of P)
- If $h < 0$, Q would be to the left of P, but the concept remains the same



Tangent

- Notice that as *x* approaches *a*, *h* approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

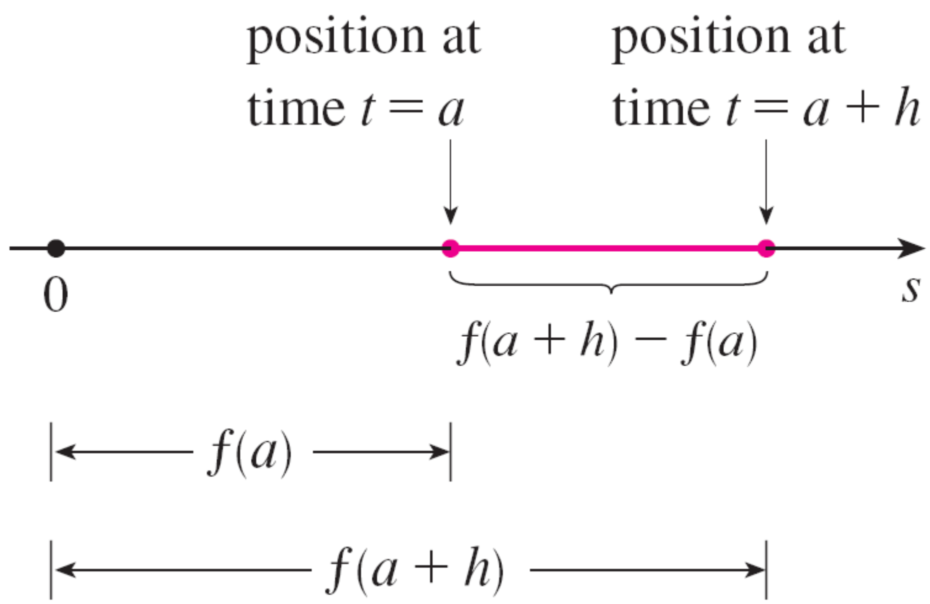
Velocity

- In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t

- The function f that describes the motion is called the **position function** of the object

Velocity

- In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$

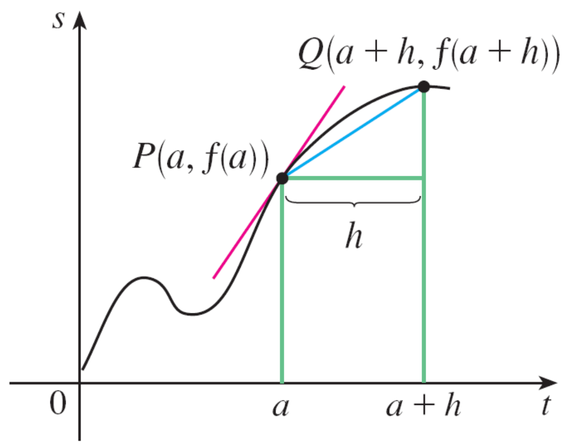


Velocity

- The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

- which is the same formula as the slope of the secant line *PQ* in the previous problem



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

Velocity

- Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*]

- In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (**instantaneous velocity**) *v*(*a*) at time *t* = *a* to be the limit of these average velocities:

3

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- This means that the formula of the velocity at time *t* = *a* is equal to the formula of the slope of the tangent line at *P*

Example

- Consider again a ball dropped from a skyscraper 450 m above the ground
- (a) What is the velocity of the ball after 5 seconds?
- (b) How fast is the ball traveling when it hits the ground?

- Solution:
- We will need to find the velocity both when t = 5 and when the ball hits the ground. Let’s compute the velocity at a generic time $t = a$

cont'd

Example – Solution

- Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a + h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$

Example – *Solution*

cont'd

$$= \lim_{h \to 0} 4.9(2a + h)$$

$$= 9.8a$$

- (a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s

Example – Solution

cont'd

- (b) Since the hight is 450 m above the ground, the ball will hit the ground at a time that we call t_1 such that $s(t_1) = 450$, that is,

- $$4.9t_1^2 = 450$$
-
- This gives
- $$t_1^2 = \frac{450}{4.9} \quad \text{and} \quad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

Example – *Solution*

cont'd

- The velocity of the ball as it hits the ground is therefore

- $v(t_1) = 9.8t_1$

  $= 9.8 \times 9.6$

  ≈ 94 m/s

Derivatives

- We have seen that **the same type of limit** arises in two different problems: finding the slope of a tangent line and the velocity of an object
- In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we need a "**rate of change**" in any of the sciences, such as a rate of reaction in chemistry or a marginal cost in economics

- Since this type of limit occurs so **widely**, it is given a **special name** and **notation**

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

- If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example

- Find the derivative of the function $f(x) = x^2 - 8x + 9$ in $x = a$
- Solution:
- From Definition 4 we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

Example – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2a + h - 8)$$

$$= 2a - 8$$

Derivatives

- We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m
- Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say that

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

- If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Rate of Change

- Suppose y is a quantity that **depends on** another quantity x. Thus y is a function of x and we write $y = f(x)$

- If x changes from x_1 to x_2, then the **change** in x (also called the **increment** of x) is

- $$\Delta x = x_2 - x_1$$

and the corresponding **increment** in y is

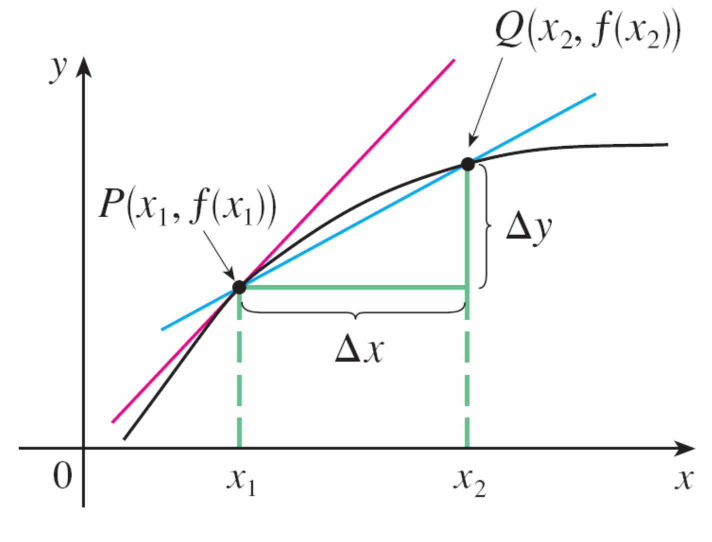
- $$\Delta y = f(x_2) - f(x_1)$$

Rate of Change

- The ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be seen as the slope of the secant line *PQ*



average rate of change = m_{PQ}
instantaneous rate of change = slope of tangent at *P*

Rate of Change

- Consider now the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0
- The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

6 $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- We recognize this limit: it is the derivative $f'(x_1)$

Rate of Change

- We know that one **interpretation** of the derivative $f'(a)$ is as the slope of the tangent line. We now have **a second** interpretation:

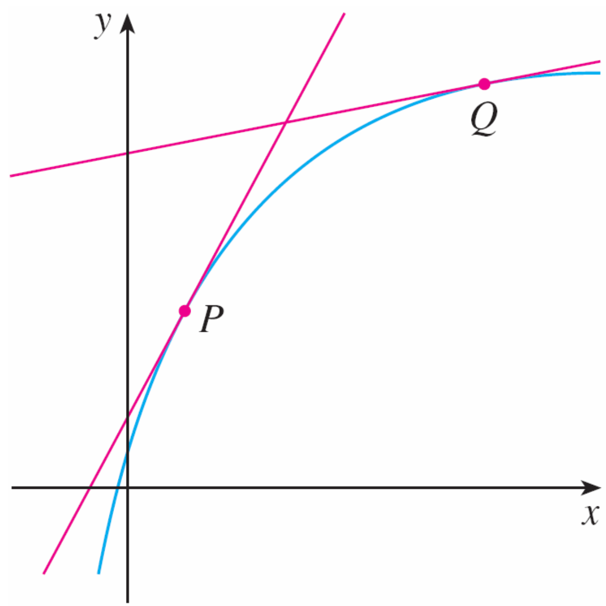
> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

- if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$, so the two interpretations are **equivalent**

Rate of Change

- This means that when the derivative is **large** (and therefore the curve is **steep**, as at the point *P*), the *y*-values change **rapidly**
- When the derivative is **small**, the curve is relatively **flat** (as at point Q) and the y-values change **slowly**

The *y*-values are changing rapidly at *P* and slowly at *Q*



Rate of Change

- So, derivative **large** → the function changes **rapidly**,
- derivative **small** → the function changes **slowly**

- In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the **rate of change** of the displacement s with respect to the time t

- In other words, $f'(a)$ *is the* ***velocity*** *of the particle at time* $t = a$

- and the speed of the particle is the absolute value of the velocity, that is, $|f'(a)|$

Derivative as a function

We have considered the derivative of a function *f* at a fixed number *a*:

1

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we **change** our point of view and let the number *a* vary. If we replace *a* by a **variable** *x*, we obtain

2

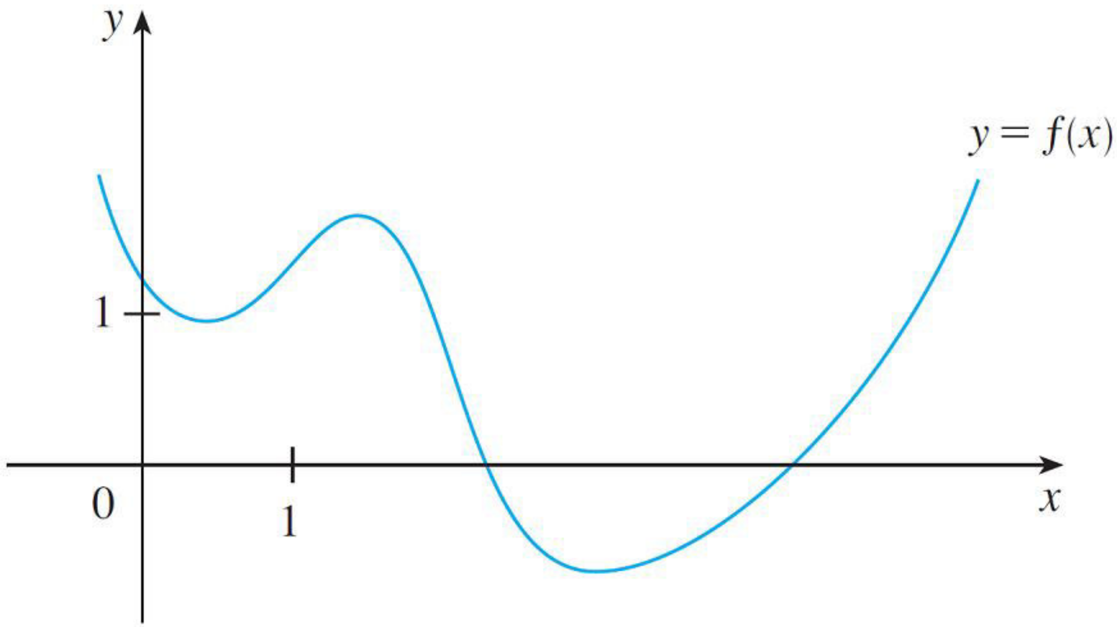
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Derivative as a function

- Given any value x for which this limit exists, we consider the value $f'(x)$ as the value in x of a **function** called f'
- hence f' is a **new function** called the **derivative of *f***
- We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$
- The function f' is called the derivative of f because it has been “derived” from f
- The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and **may be smaller** than the domain of f

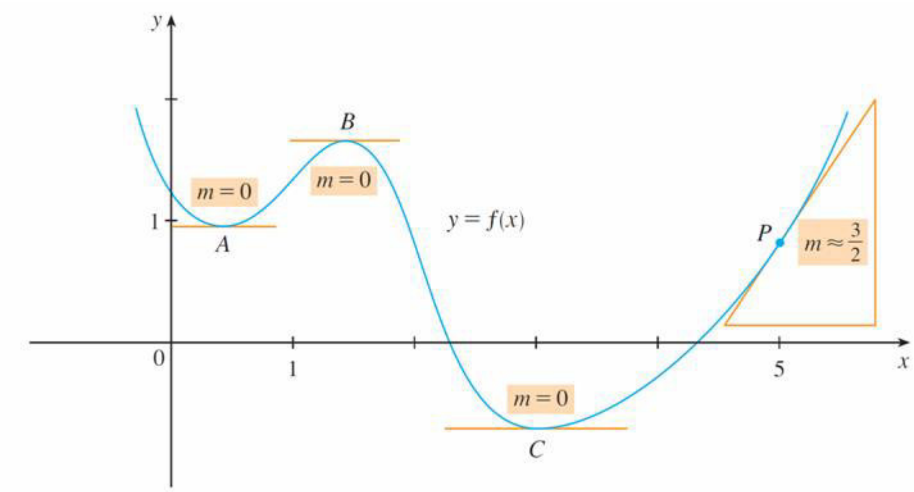
Example

The graph of a function *f* is given. Use it to sketch the graph of the derivative *f′*



Example – Solution

We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P and estimate its slope to be about $3/2 = 1.5$, so $f'(5) \approx 1.5$



Example – *Solution*

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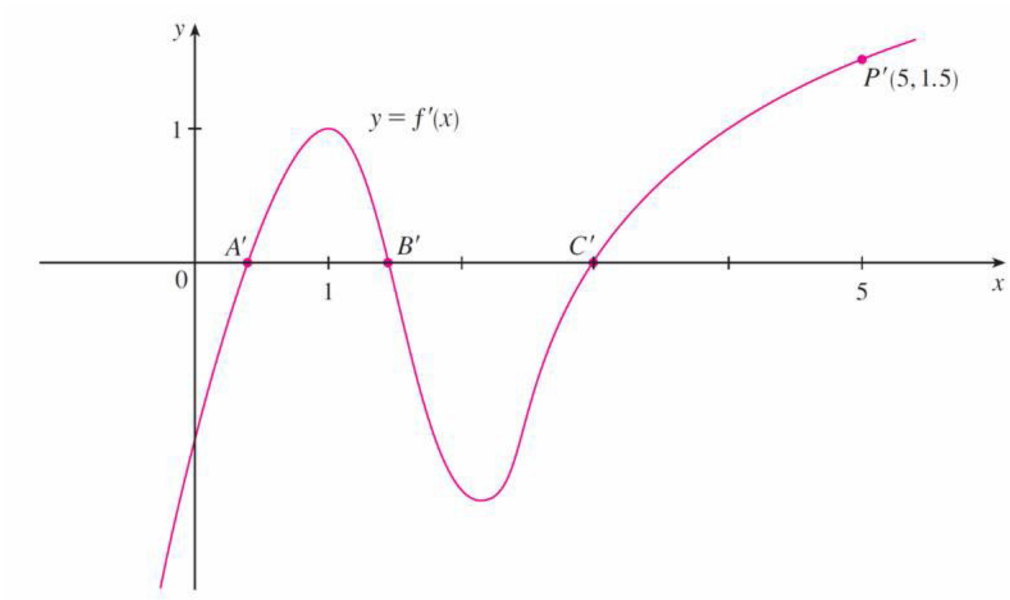
This allows us to plot the point *P'*(5, 1.5) on the graph of *f'* directly beneath *P*. Repeating this procedure at several points, we get the graph shown here



Example – *Solution*

cont'd

Notice that the tangents at *A*, *B*, and *C* are horizontal, so the **derivative is 0** there and the graph of f' crosses the *x*-axis at the points A', B', and C', directly beneath *A*, *B*, and *C*

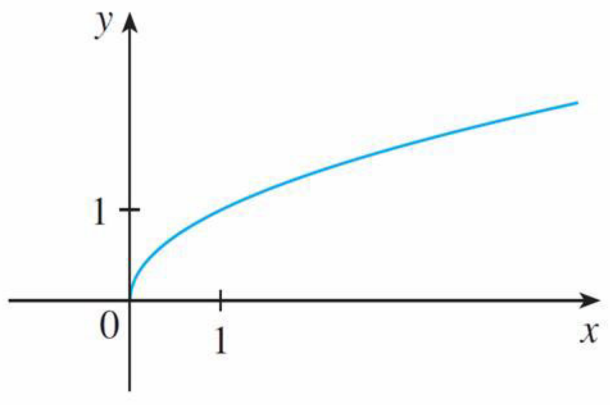
Between *A* and *B* the tangents have **positive** slope, so $f'(x)$ is positive there. But between *B* and *C* the tangents have **negative** slope (the increment in the function is negative as *x* increases), so $f'(x)$ is negative there

Other Example

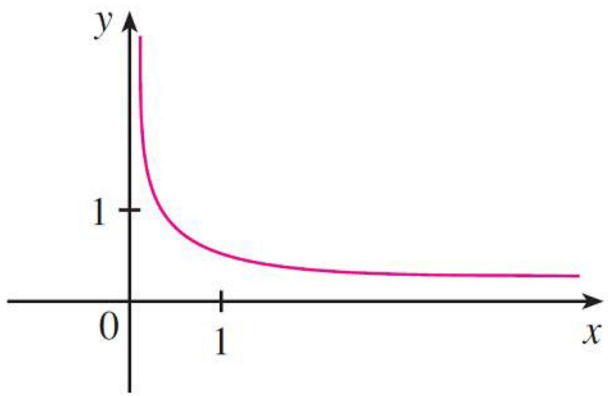
Consider $f(x) = \sqrt{x}$

When x is close to 0, $f(x)$ is also close to 0 but rapidly increasing, so $f'(x)$, which will have expression $1/(2\sqrt{x})$, is very large

When x is large, $f(x)$ is flattening (the tangent tend to be horizontal on the far right), so $f'(x)$ becomes small and has an horizontal asymptote



(a) $f(x) = \sqrt{x}$



(b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative

Other Notations

The symbol *dy*/*dx*, which was introduced by Leibniz, should not be regarded precisely as a division; it simply refers to the fact that the derivative $f'(x)$, of which it is synonym, is obtained as the limit of the incremental ratio $\Delta y/\Delta x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Other Notations

If we want to indicate the value of a derivative *dy*/*dx* in Leibniz notation at a specific value *a*, we use the notation

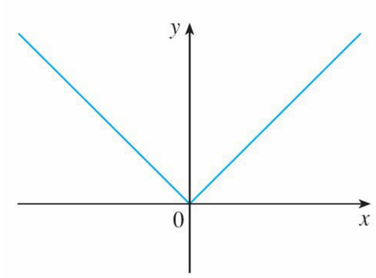
$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$

3 Definition A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example

Where is the function $f(x) = |x|$ differentiable?



Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$

Example – *Solution*

cont'd

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$

Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$

Example – Solution

cont'd

For x = 0 we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Example – Solution

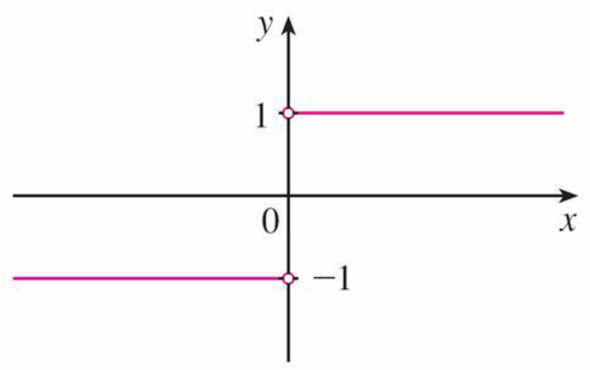
cont'd

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is

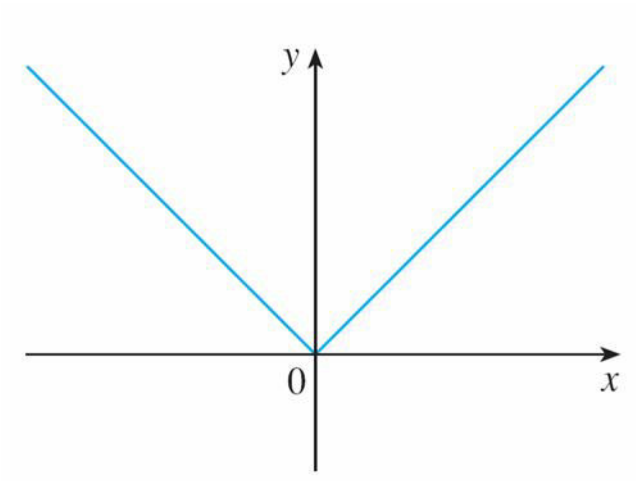


$y = f'(x)$

Example – *Solution*

cont'd

The slope of the tangent is indeed 1 for $x > 0$ and -1 for $x < 0$. The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0)



$y = f(x) = |x|$

Continuity vs Differentiability

Both continuity and differentiability are desirable properties for a function to have. These properties are related:

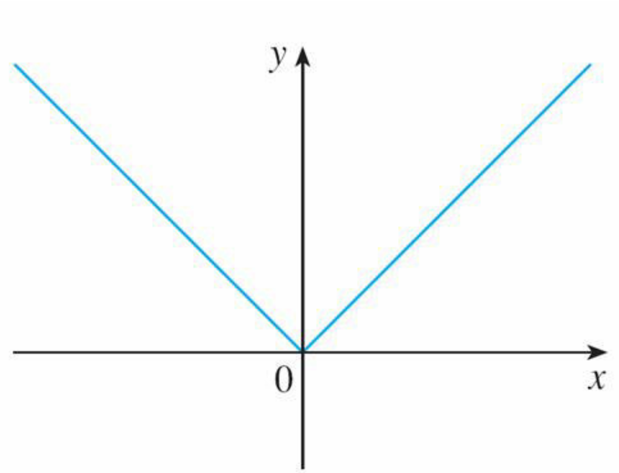
4 Theorem If f is differentiable at a, then f is continuous at a.

The converse is false; there are functions that are continuous but not differentiable!

When is a function not differentiable?

How Can a Function Fail to Be Differentiable?
We saw that the function $y = |x|$ is not differentiable at 0 because its graph changes direction abruptly when $x = 0$

In general, even if a function f is continuous, if its graph has a "corner", it has no tangent there, so f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different]



$y = f(x) = |x|$

When is a function not differentiable?

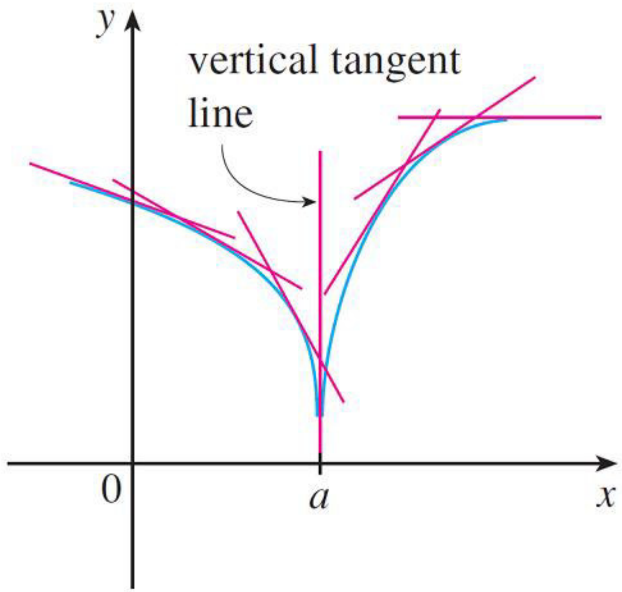
Moreover, if *f* is not continuous at *a*, then *f* is clearly not differentiable at *a*. So at any discontinuity *f* fails to be differentiable

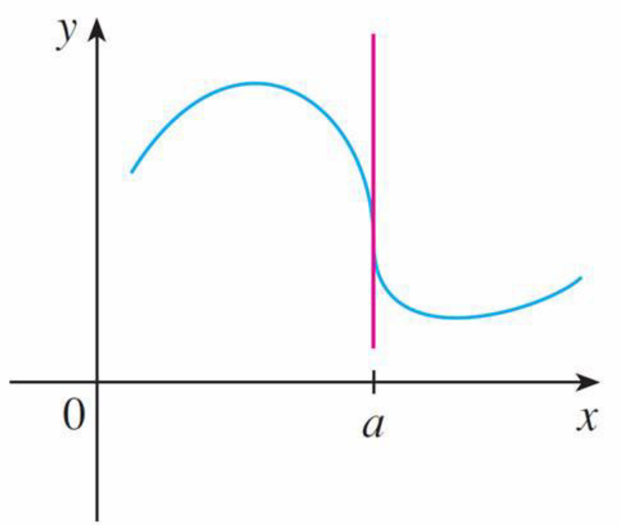
A third possibility of non-differentiability is that the curve has a **vertical tangent line** when $x = a$; that is, *f* is continuous at *a* but

$$\lim_{x \to a} |f'(x)| = \infty$$

Vertical tangent line

This means that the tangent lines become steeper and steeper as $x \to a$. Here are some examples of how this can happen

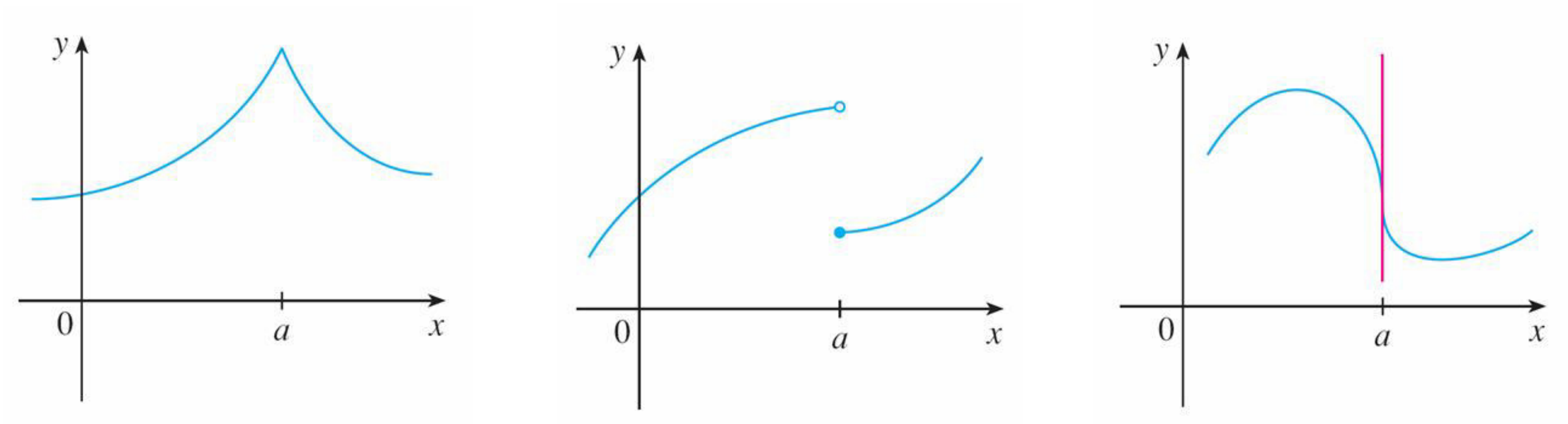




A vertical tangent

Non-differentiability

To summarize, here are 3 sample graphs representing the 3 possible causes of not differentiability



(a) A corner (b) A discontinuity (c) A vertical tangent

Three ways for *f* not to be differentiable at *a*

Second Derivative

If *f* is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of *f* because it is the derivative of the derivative of *f*

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example

If $f(x) = x^3 - x$, find and interpret $f''(x)$

Solution:

The first derivative of $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$

So the second derivative is

$$f''(x) = (f')'(x)$$

$$= \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x + h)^2 - 1] - [3x^2 - 1]}{h}$$

Example – *Solution*

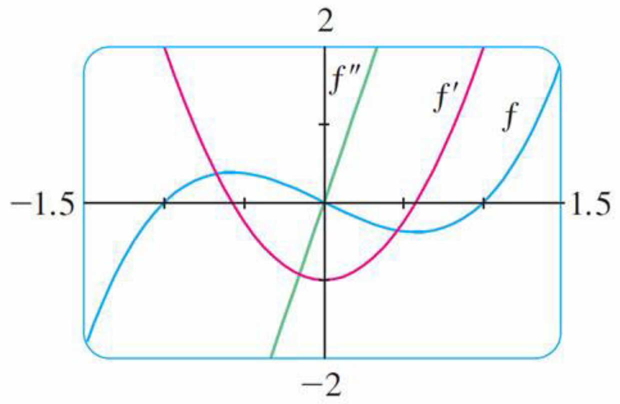
cont'd

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$= 6x$$

The graphs of *f*, *f′*, and *f″* are as follows



Example – *Solution*

cont'd

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$

Notice that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$

In general, the n-th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is sometimes called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

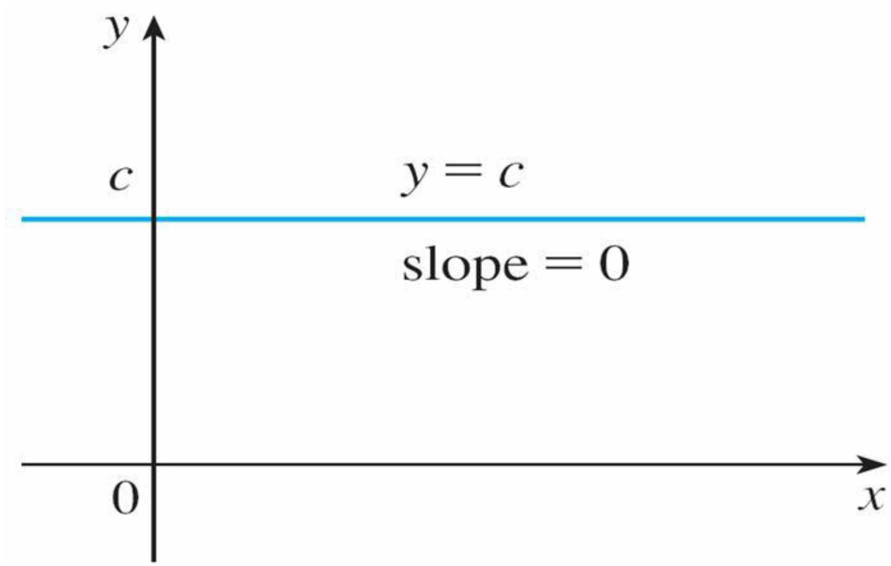
Thus the jerk j is the rate of change of acceleration

Computing the Derivatives

- In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions

- Let's start with the simplest of all functions, the constant function $f(x) = c$

- The graph of this function is the horizontal line $y = c$, which has slope 0, so we simply have $f'(x) = 0$



- The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$

Computing the Derivatives

- A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

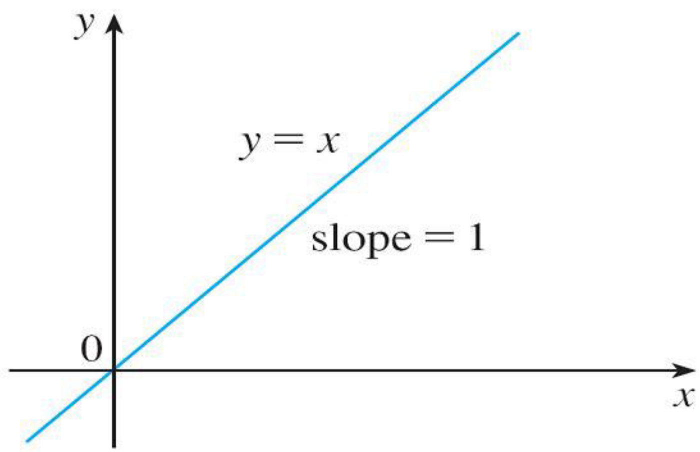
- In Leibniz notation, we write this rule as follows

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

- We next look at the functions $f(x) = x^n$, where n is a positive integer
- If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1



- The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$

Power Functions

- So **1**

$$\frac{d}{dx}(x) = 1$$

- For $n = 2$ we find the derivative of $f(x) = x^2$ as follows:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$\lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - x^2)}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

- So **2**

$$\frac{d}{dx}(x^2) = 2x$$

Power Functions

- For $n = 3$ we find the derivative of $f(x) = x^3$ as follows:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$\lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - x^3)}{h} =$$

$$\lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} =$$

$$\lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Power Functions

- Thus $\frac{d}{dx}x^3 = 3x^2$

- If we continue, we see that:

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example

- (a) If $f(x) = x^6$, then $f'(x) = 6x^5$
- (b) If $y = x^{1000}$, then $y' = 1000x^{999}$
- (c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$
- (d) $\frac{d}{dr}(r^3) = 3r^2$

Power Functions

- Even for *n* not a positive integer, we still have:

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- Example: $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2\sqrt{x}}$

- Example: $\frac{d}{dx}\frac{1}{x} = \frac{d}{dx} x^{-1} = \frac{-1}{x^2}$

Derivation rules

- When functions are formed from basic functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the basic functions

- For example, *the derivative of a constant times a function is the constant times the derivative of the function*

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example

(a) $$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4)$$

$$= 3(4x^3)$$

$$= 12x^3$$

(b) $$\frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x]$$

$$= (-1)\frac{d}{dx}(x)$$

$$= -1(1)$$

$$= -1$$

Derivation rules

- *the derivative of a sum of functions is the sum of the derivatives*

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

- The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h)]' = (f + g)' + h' = f' + g' + h'$$

- Hence, we also have the constant multiplication rule:

$$(c f)' = c f'$$

Derivation rules

- By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiplication Rule, we obtain

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

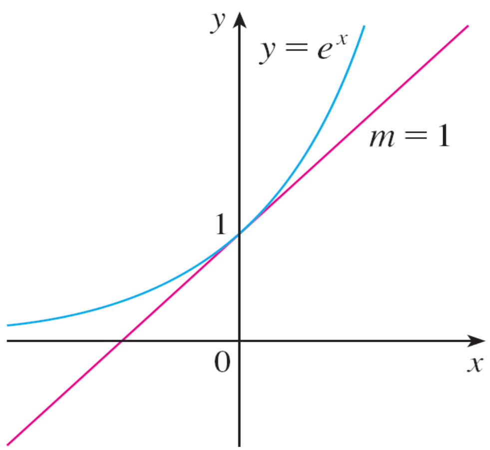
- Example: find $$\frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)$$
- We obtain $8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

Exponential function

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

- Thus the exponential function $f(x) = e^x$ has the important property that it is its own derivative: the **slope** of a tangent line to the curve $y = e^x$ in a point p is always **equal to the y-coordinate** of the point p
- We have already seen that in point (0,1) the tangent has slope 1



Example

- Given $f(x) = e^x - x$, find the derivative

- Solution:
- Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x)$$

$$= \frac{d}{dx}(e^x) - \frac{d}{dx}(x)$$

$$= e^x - 1$$

Derivative of a product

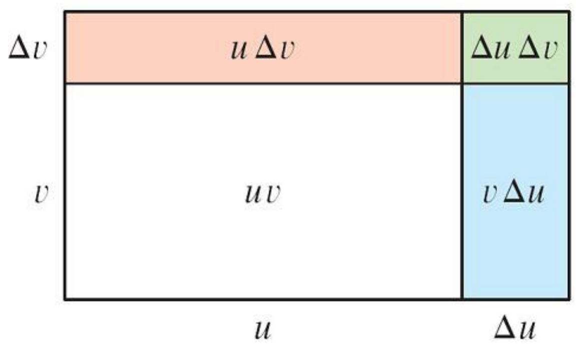
- By analogy with the Sum and Difference Rules, one might think that the derivative of a product is the product of the derivatives
- however, this can be proved wrong by looking at an example

- Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$
- but $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$, while $f'g' = 2x$
- Thus $(fg)' \neq f'g'$

Derivative of a product

- The correct Product Rule can be discover as follows

- Assume that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as the white area of this rectangle



- The geometry of the Product Rule

Derivative of a product

- If x changes by an amount Δx, then the corresponding changes in u and v are

  $\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$

- and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle (provided that Δu and Δv happen to be positive)

- The change in the area of the rectangle is

  $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$

  = the sum of the three colored areas

Derivative of a product

- If we divide by Δx, we get

$$\frac{\Delta(uv)}{\Delta x} = u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x}$$

- If we now let $\Delta x \to 0$, we get the derivative of uv :

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \left(u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x} \right)$$

$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \Delta u \right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right)$$

$$= u\,\frac{dv}{dx} + v\,\frac{du}{dx} + 0 \cdot \frac{dv}{dx}$$

Derivative of a product

- Hence, we have:

$$\boxed{2} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since f is differentiable and therefore continuous

- Although we started by assuming (for the geometric interpretation) that all the quantities are positive, the rule is always true whether u, v, Δu, Δv are positive or negative

Derivative of a product

- In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function*

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example

If $f(x) = xe^x$, find

- (a) the derivative $f'(x)$
- (b) the n-th derivative $f^{(n)}(x)$

- Solution:
- (a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x+1)e^x$$

Example – *Solution*

- cont'd

- (b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx}[(x + 1)e^x]$$

$$= (x + 1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x + 1)$$

$$= (x + 1)e^x + e^x \cdot 1$$

$$= (x + 2)e^x$$

Example – *Solution*

- cont'd

- Further applications of the Product Rule give
- $f'''(x) = (x + 3)e^x \qquad f^{(4)}(x) = (x + 4)e^x$
- In fact, each successive differentiation adds another term e^x, so
- $f^{(n)}(x) = (x + n)e^x$

Derivative of a quotient

- We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in the same way that we found the Product Rule

- If x, u, and v change by amounts Δx, Δu, and Δv, then the corresponding change in the quotient u/v is

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

Derivative of a quotient

- so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\,\dfrac{\Delta u}{\Delta x} - u\,\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

- As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous

- Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \dfrac{\Delta v}{\Delta x}}{v \lim\limits_{\Delta x \to 0} (v + \Delta v)} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

Derivative of a quotient

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

- In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator*

Example

- Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Differentiation rules

- We can summarize all in this Table of Differentiation Formulas

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f + g)' = f' + g'$	$(f - g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	

Derivatives of trigonometric functions

- These are the differentiation formulas for trigonometric functions. Remember that they are valid only when *x* is measured in radians

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

Derivatives of composite functions

- If we want to differentiate the function $F(x) = \sqrt{x^2 + 1}$

the differentiation formulas seen up to now are not enough

- Observe that F is a composite function. In fact, if we let
  $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$,
  then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$

- We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ using the derivatives of f and g

The Chain Rule

- The derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This is called the *Chain Rule*

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Example

- Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$

- Solution :
- We have expressed F as
  $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$

- Since $f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$ and $g'(x) = 2x$

- we have $F'(x) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

Chain Rule with Power Rule

- Let's see the special case of the Chain Rule where the outer function *f* is a power function
- If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example

- Differentiate $y = (x^3 - 1)^{100}$

- Solution:
- Taking $u = g(x) = x^3 - 1$ and $y = f(u) = u^{100}$, we have
- $= 100(x^3 - 1)^{99} \dfrac{d}{dx}(x^3 - 1)$
- $= 100(x^3 - 1)^{99} \cdot 3x^2$
- $= 300x^2(x^3 - 1)^{99}$

Chain Rule for Exp. functions

- We can use the Chain Rule to differentiate an exponential function with any base $a > 0$. Recall that $a = e^{\ln a}$. So
- $$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$
- and the Chain Rule gives
- $$\frac{d}{dx}\left(e^{(\ln a)x}\right) = e^{(\ln a)x} \frac{d}{dx}(\ln a)x$$
- $$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$
- because $\ln a$ is a constant. So we have the formula

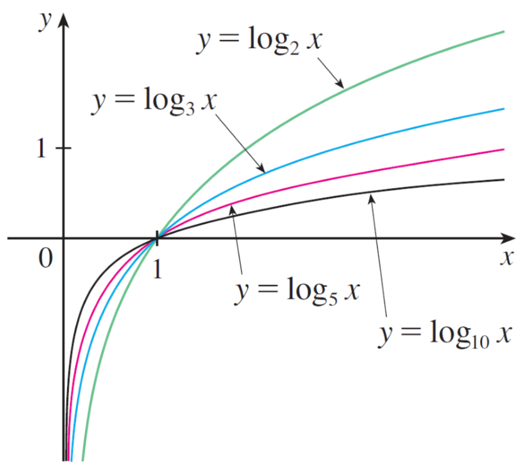
$$\frac{d}{dx}(a^x) = a^x \ln a$$

Example

- For example, if a = 2, we get
- $$\frac{d}{dx}(2^x) = 2^x \ln 2$$

Derivatives of logarithmic functions

- We now see the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$
- It can be proved that logarithmic functions are differentiable; this is also visible from their graphs



Derivatives of logarithmic functions

- We have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

- and d/dx ($\log_a x$) = $\dfrac{1}{x \ln a}$

- If the function is composite we can use the Chain Rule

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

Example

- Find $\dfrac{d}{dx}\ln(\sin x)$
- Solution: we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x$$

$$= \cot x$$

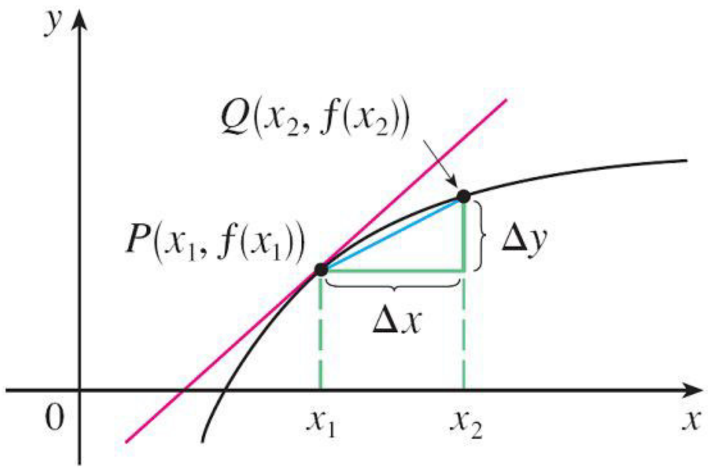
Derivatives in Sciences

- We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x
- We have seen that, if x changes from x_1 to x_2, then the change in x is
- $$\Delta x = x_2 - x_1$$
- and the corresponding change in y is
- $$\Delta y = f(x_2) - f(x_1)$$

Derivatives in Sciences

- The difference quotient $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
- is the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ (the slope of the secant line *PQ*)
- Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, the **instantaneous rate of change of *y* with respect to *x*** or the slope of the tangent line at $P(x_1, f(x_1))$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

Derivatives in Physics

- If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time)

- The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$

Example 1

- The position of a particle is given by the equation
- $s = f(t) = t^3 - 6t^2 + 9t$
- where t is measured in seconds and s in meters
- (a) Find the velocity at time t
- (b) What is the velocity after 2 s? After 4 s?
- (c) When is the particle at rest?
- (d) When is the particle moving forward (that is, in the positive direction)?

Example 1

cont'd

- (e) Draw a diagram to represent the motion of the particle
- (f) Find the total distance traveled by the particle during the first five seconds
- (g) Find the acceleration at time t and after 4 s
- (h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$
- (i) When is the particle speeding up? When is it slowing down?

Example 1 – *Solution*

- (a) The velocity function is the derivative of the position function

- $$s = f(t) = t^3 - 6t^2 + 9t$$

- $$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

Example 1 – *Solution*

cont'd

- (b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,
- $$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9$$
- $$= -3 \text{ m/s}$$
- The velocity after 4 s is
- $$v(4) = 3(4)^2 - 12(4) + 9$$
- $$= 9 \text{ m/s}$$

Example 1 – *Solution*

cont'd

- (c) The particle is at rest when $v(t) = 0$, that is,
- $3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$
- $= 3(t - 1)(t - 3)$
- $= 0$
- and this is true when $t = 1$ or $t = 3$
- Thus the particle is at rest after 1 s and after 3 s

Example 1 – *Solution*

cont'd

- (d) The particle moves in the positive direction when $v(t) > 0$, that is

- $$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

- This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$)

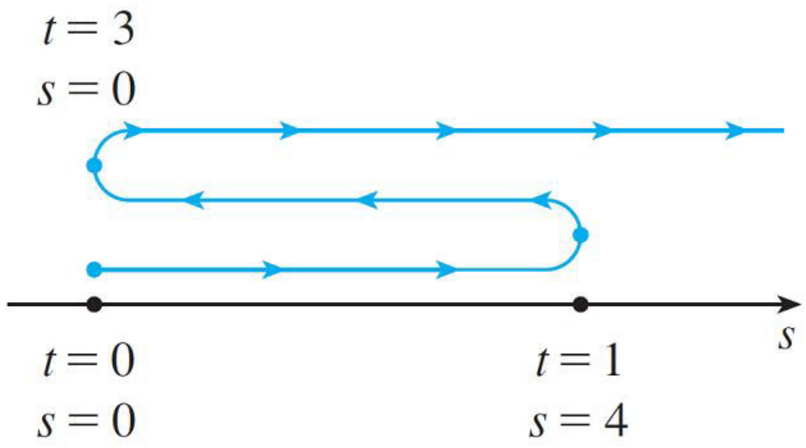
- Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$

- It moves backward (in the negative direction) when $1 < t < 3$

Example 1 – *Solution*

cont'd

- (e) Using the information from part (d) we make a schematic sketch of the motion of the particle back and forth along a line (the *s*-axis)



Example 1 – *Solution*

cont'd

- (f) Since it moves back and forth, we need to find separately the distances traveled in the time intervals [0, 1], [1, 3], [3, 5]
- The distance traveled in the first second is
- $$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$
- From $t = 1$ to $t = 3$ the distance traveled is
- $$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$
- From $t = 3$ to $t = 5$ the distance traveled is
- $$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$
- The total distance is 4 + 4 + 20 = 28 m

Example 1 – *Solution*

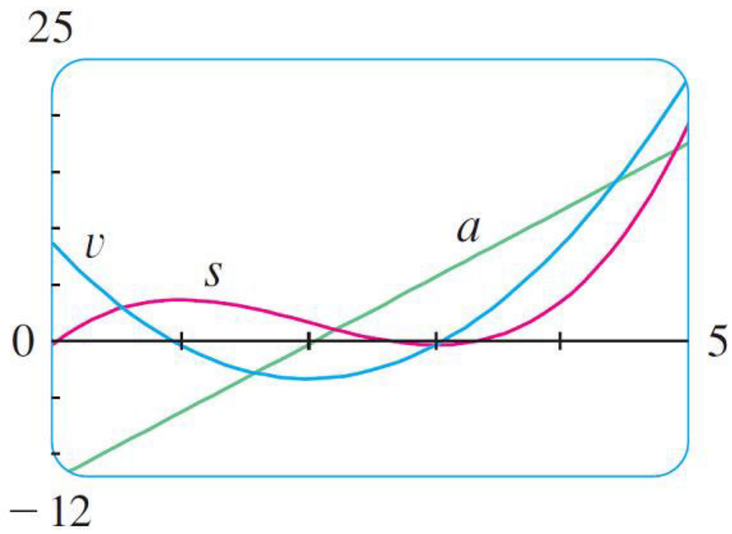
cont'd

- (g) The acceleration is the derivative of the velocity:
- $$a(t) = \frac{d^2s}{dt^2}$$
- $$= \frac{dv}{dt}$$
- $$= 6t - 12$$
- $$a(4) = 6(4) - 12$$
- $$= 12 \text{ m/s}^2$$

Example 1 – *Solution*

cont'd

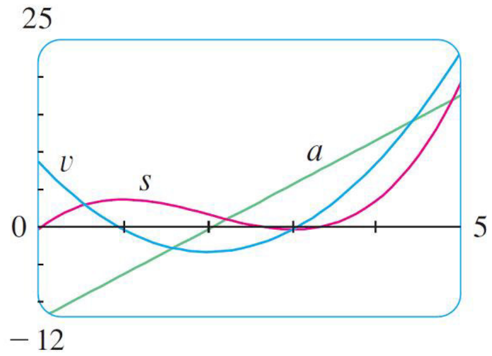
- (h) the graphs of *s*, *v*, and *a*



Example 1 – *Solution*

cont'd

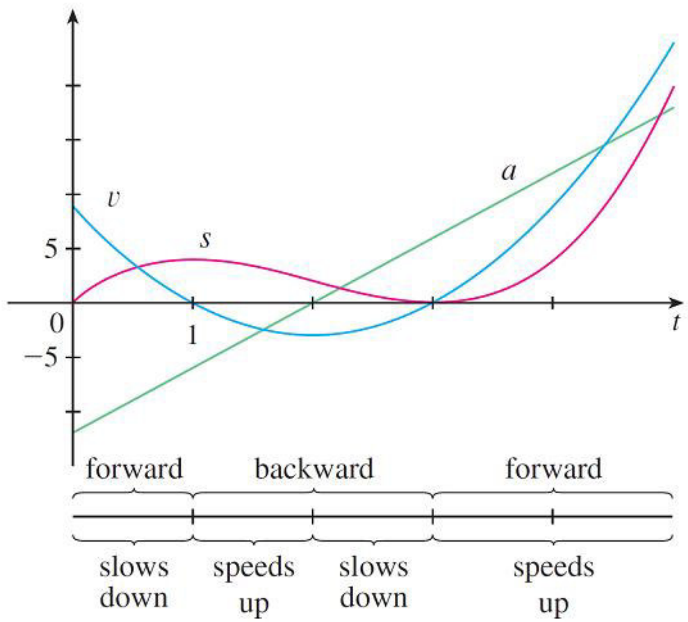
- (i) The particle speeds up when the velocity is positive and increasing (*v* and *a* are both positive) and also when the velocity is negative and decreasing (*v* and *a* are both negative)
- In other words, the particle speeds up when the velocity and acceleration have the same sign
- From the previous figure (blue and green lines) we see that this happens when $1 < t < 2$ and when $t > 3$



Example 1 – *Solution*

cont'd

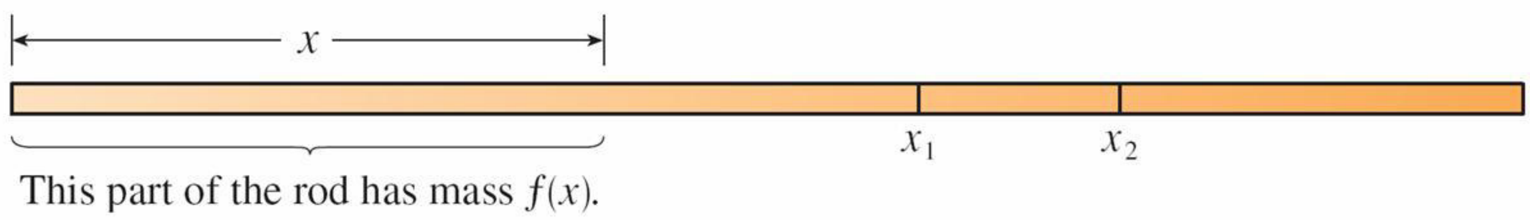
- The particle slows down when v and a have opposite signs, that is, when $0 \leq t < 1$ and when $2 < t < 3$



Example 2

- If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m / l$) and measured in kilograms per meter

- Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$



Example 2

cont'd

- The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals
- The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$; that is, the linear density is the rate of change (= the derivative) of mass with respect to length

Example 2

cont'd

- Symbolically, $$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

- For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leq x \leq 1.2$ is

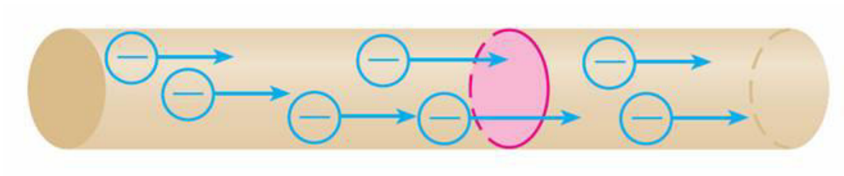
- $$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

- while the linear density right at $x = 1$ is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = 0.50 \text{ kg/m}$$

Example 3

- A current exists whenever electric charges move. Consider electrons moving through a plane surface, shaded red



- If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

Example 3

cont'd

- If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current *I*** at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

- Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes)

Example 4

- A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the "equation"

$$2H_2 + O_2 \rightarrow 2H_2O$$

  means that two molecules of hydrogen and one molecule of oxygen form two molecules of water

- Let's consider a generic reaction $A + B \rightarrow C$
  where A and B are the reactants and C is the product

Example 4

cont'd

- The **concentration** of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A]
- The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t)
- The average rate of reaction of the product C over a time interval $t_1 \le t \le t_2$ is

$$\frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{[\mathrm{C}](t_2) - [\mathrm{C}](t_1)}{t_2 - t_1}$$

Example 4

cont'd

- But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{d[\mathrm{C}]}{dt}$$

- Since the concentration of the product increases as the reaction proceeds, the derivative $d\,[\mathrm{C}]\,/dt$ will be positive, and so the rate of reaction of C is positive

Example 4

cont'd

- The concentrations of the reactants, however, decrease during the reaction, so their rates of reaction (= the derivatives) $d[A]/dt$ and $d[B]/dt$ are negative
- We can put minus signs in front of these derivatives to make them positive numbers
- [A] and [B] each decrease at the same rate that [C] increases, so we have

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = -\frac{d[\mathrm{A}]}{dt} = -\frac{d[\mathrm{B}]}{dt}$$

Example 4

cont'd

- More generally, it turns out that for a reaction of the form

$$aA + bB \rightarrow cC + dD$$

- we have

$$-\frac{1}{a}\frac{d[A]}{dt} = -\frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} = \frac{1}{d}\frac{d[D]}{dt}$$

- The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction

Example 5

- One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P
- We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$
- The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Example 5

cont'd

- Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature

- For instance, the volume V (in cubic meters) of a sample of air at 25°C is related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

Example 5

cont'd

- The rate of change of V with respect to P when P = 50 kPa is given by the derivative of the function $V = \dfrac{5.3}{P}$

$$\left.\frac{dV}{dP}\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50}$$

$$= -\frac{5.3}{2500}$$

$$= -0.00212 \text{ m}^3/\text{kPa}$$

Example 5

cont'd

- The compressibility at that pressure is

$$\beta = -\frac{1}{V}\frac{dV}{dP}\bigg|_{P=50}$$

$$= \frac{0.00212}{\frac{5.3}{50}}$$

$$= 0.02\ (\mathrm{m}^3/\mathrm{kPa})/\mathrm{m}^3$$

Example 6

- Let $n = f(t)$ be the number of individuals in an animal or plant population at time t
- The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \le t \le t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

- The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

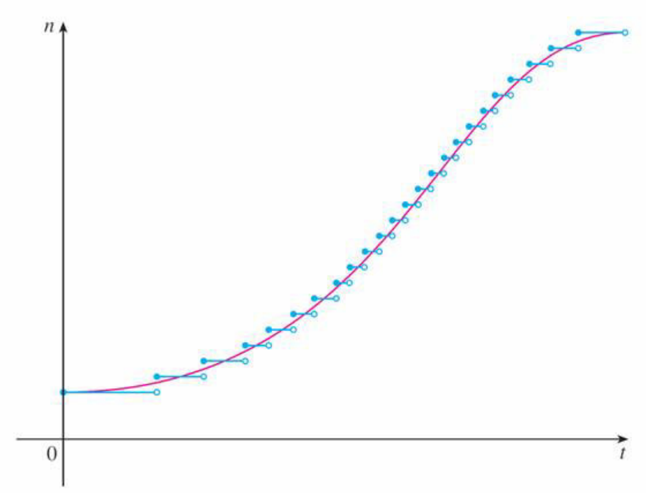
$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Example 6

cont'd

- Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable

- However, for a large enough population, we can replace the graph by a smooth approximating



A smooth curve approximating a growth function

Example 6

cont'd

- To be more specific, consider a population of bacteria in a homogeneous nutrient medium
- Suppose that, by sampling the population at certain intervals, it is determined that the population doubles every hour
- If the initial population is n_0 and the time t is measured in hours, then
- $$f(1) = 2f(0) = 2n_0$$
- $$f(2) = 2f(1) = 2^2 n_0$$

Example 6

cont'd

- $$f(3) = 2f(2) = 2^3 n_0$$
- and, in general,
- $$f(t) = 2^t n_0$$
- The population function is $n = n_0 2^t$
- We know that the derivative of an exponential function is

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Example 6

cont'd

- So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t) \qquad = n_0 2^t \ln 2$$

- For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth after 4 hours is

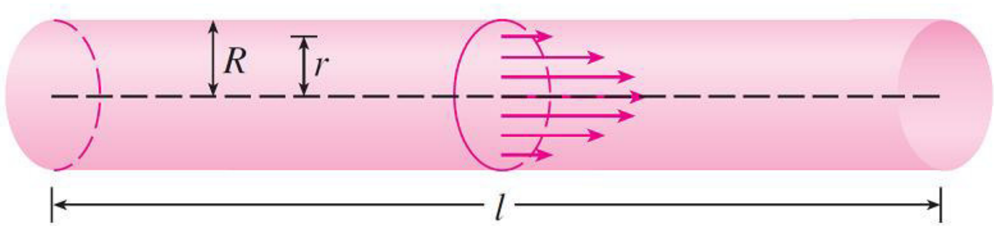
$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \ln 2$$

- $= 1600 \ln 2 \quad \approx 1109$

- This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour

Example 7

- Consider the flow of blood through a blood vessel, such as a vein or artery. We can model the shape of the blood vessel by a cylindrical tube with radius *R* and length *l*



Blood flow in an artery

- Because of friction at the walls of the tube, the velocity *v* of the blood is greatest along the central axis of the tube and decreases as the distance *r* from the axis increases until *v* becomes 0 at the wall

Example 7

cont'd

- The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840
- This law states that $$v = \frac{P}{4\eta l}(R^2 - r^2)$$
- where η is the viscosity of the blood and P is the pressure difference between the ends of the tube
- If P and l are constant, then v is a function of r with domain $[0, R]$

Example 7

cont'd

- The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

- and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r :

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Example 7

cont'd

- Using the law, we obtain $\dfrac{dv}{dr} = \dfrac{P}{4\eta l}(0 - 2r) \quad = -\dfrac{Pr}{2\eta l}$

- For one of the smaller human arteries we can take η = 0.027, R = 0.008 cm, l = 2 cm, and P = 4000 dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$

Example 7

cont'd

- $\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$
- At $r = 0.02$ cm the blood is flowing at a speed of
- $v\,(0.002) \approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6})$
- $= 1.11$ cm/s
- and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74\ (\text{cm/s})/\text{cm}$$

Example 7

cont'd

- To better view what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm

- The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of r = 20 μm

- The fact that dv/dr = –74 (μm/s)/μm means that, when r = 20 μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center

Example 8

- Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity

- The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

$$= \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Example 8

cont'd

- The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

- Note that, when *x* takes on only integer values (for example we produce cars), it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6

Example 8

cont'd

- It is often appropriate to represent a total cost function by a polynomial
- $$C(x) = a + bx + cx^2 + \ldots$$

- where *a* represents the overhead fixed costs (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to *x*, but labor costs might sometimes depend partly on higher powers of *x* because of possible overtime costs and inefficiencies involved in large-scale operations)

Example 8

cont'd

- For instance, suppose a company has estimated that the cost (in dollars) of producing x items is
- $$C(x) = 10{,}000 + 5x + 0.01x^2$$
- Then the marginal cost function is
- $$C'(x) = 5 + 0.02x$$
- The marginal cost at the production level of 500 items is
- $$C'(500) = 5 + 0.02(500)$$
- $$= \$15/\text{ item}$$

Example 8

- This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item

- The actual cost of producing the 501st item is
- $C(501) - C(500) = [10,000 + 5(501) + 0.01(501)^2]$
- $\qquad - [10,000 + 5(500) + 0.01(500)^2]$
- $\qquad = \$15.01$

- Notice that $C'(500) \approx C(501) - C(500)$
- Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n+1)$-st unit]

Derivatives in Other Sciences

- Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks
- An engineer wants to know the rate at which water flows into or out of a reservoir
- An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases
- A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height

A Single Idea, Many uses

- Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative

A Single Idea, Many uses

- A single abstract mathematical concept (such as the derivative) can have different uses and interpretations in each of the sciences

- When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences

- This is much more efficient than developing properties of special concepts in each separate science

Examples of derivatives

- $f(x) = \sqrt{x}$
- $f'(x) = \frac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

- $f(x) = (2x+3)^4$ it is a composite function
- $f'(x) = 4(2x+3)^3\, 2 = 8(2x+3)^3$

- $f(x) = \sqrt{1 + \sin x}$ it is a composite function
- $f'(x) = \dfrac{1}{2\sqrt{1 + \sin x}} (0 + \cos x) = \dfrac{\cos x}{2\sqrt{1 + \sin x}}$

Examples of derivatives

cont'd

- $f(x) = e^{\sqrt{x^2+1}}$

it is a composite of 3 functions: exponential, root and polynomial !

- $f'(x) = e^{\sqrt{x^2+1}} \dfrac{1}{2\sqrt{x^2+1}} (2x) = \dfrac{x\, e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

- $f(x) = (3x^2+x)^2 \; (2x-1)$

a composite function multiplied by another function !

- $f'(x) = 2(3x^2+x)(6x+1)\,(2x-1) + (3x^2+x)^2\,(2) =$
- $2(3x^2+x)\,[(6x+1)\,(2x-1) + (3x^2+x)\,] =$
- $2x(3x+1)\,[12x^2 + 2x - 6x - 1 + 3x^2 + x] =$
- $2x\,(3x+1)\,(15x^2 - 3x - 1) =$

Examples of derivatives

cont'd

- $f(x) = \dfrac{(\ln x)^2}{(\ln x)+1}$ — a composite function divided by another function !

- $f'(x) = \dfrac{2 \ln x \,(1/x)\,(\ln x + 1) - (\ln x)^2 \,(1/x)}{(\ln x + 1)^2} =$

- $\dfrac{(1/x)(\ln x)(2 \ln x + 2 - \ln x)}{(\ln x + 1)^2} =$

- $\dfrac{(\ln x)(\ln x + 2)}{x(\ln x + 1)^2}$

Examples of derivatives

cont'd

- $f(x) = x e^{-3x}$ a product of a function by a composite function !
- $f'(x) = e^{-3x}(1) + x e^{-3x}(-3) = e^{-3x}(1 - 3x)$

- $f(x) = x^x$ a function at the power of another function ! It is neither a simple power nor a simple exponential
- we need to rewrite it as $f(x) = e^{\ln x^x} = e^{x \ln x}$
- $f'(x) = e^{x \ln x} [\ln x (1) + (1/x) x] =$
- $x^x(\ln x + x/x) =$
- $x^x(\ln x + 1)$