Bachelor’s degree in Bioinformatics

Using Derivatives

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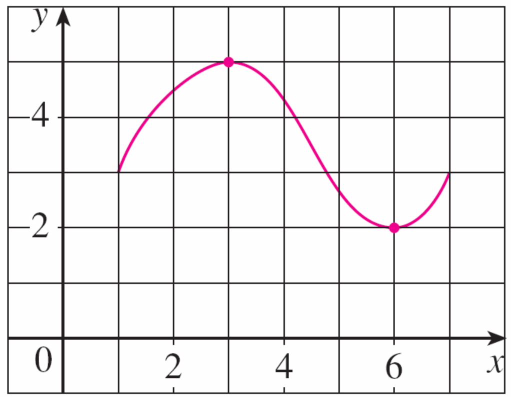
Maximum and Minimum

Some of the most important applications of differential calculus are **optimization problems**, in which we want to find the optimal (best) way of doing something. This is done by finding the **maximum** or **minimum** values of a function

What are maximum and minimum? In this example, the largest value of f is $f(3) = 5$ and the smallest value is $f(6) = 2$

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*



Global Maximum and Minimum

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

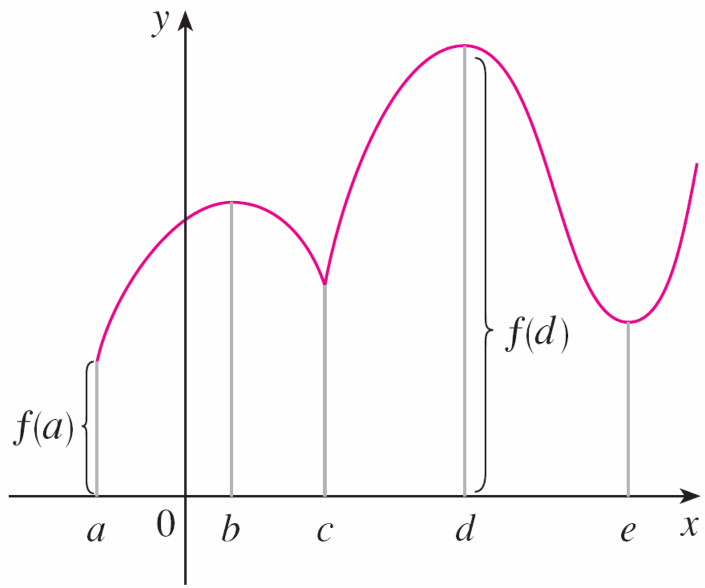
An **absolute** maximum or minimum is also called a **global** maximum or minimum

The maximum and minimum values of f are called **extreme values** of f

Local Maximum and Minimum

Considering this graph, $(d, f(d))$ is the highest point on the graph (global maximum) and $(a, f(a))$ is the lowest point (global minimum)

However, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a **local maximum** of f



min $f(a)$, max $f(d)$

loc min $f(c)$ and $f(e)$, loc max $f(b)$ and $f(d)$

Local Maximum and Minimum

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \le f(x)$ for x near c [in the interval (b, d), for instance]

The function f also has a local minimum at e

In general, we have the following definition

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

In general, if we say that something is true **near** c, we mean that it is true on some open interval containing c

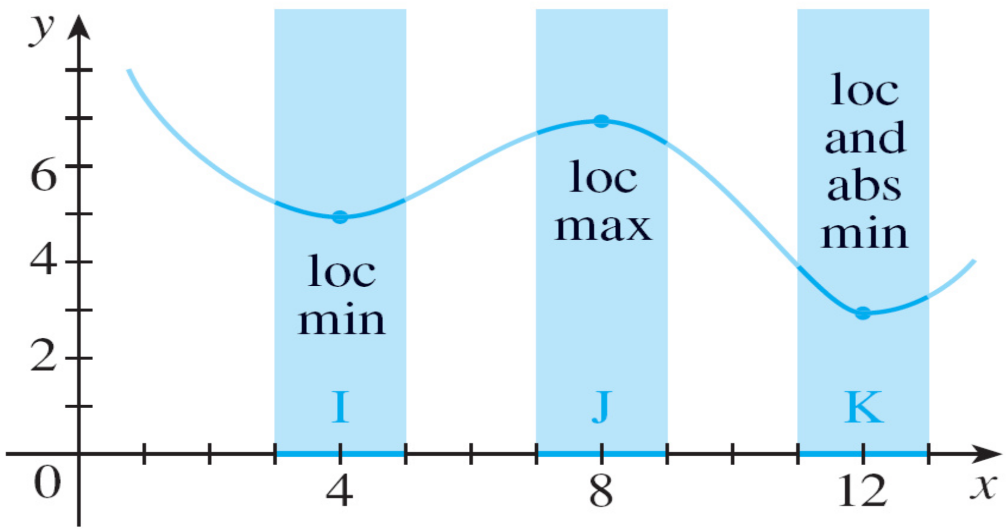
Maximum and Minimum Values

In this example we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I

It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance)

$f(12) = 3$ is both a local minimum and the absolute minimum

$f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1



Example

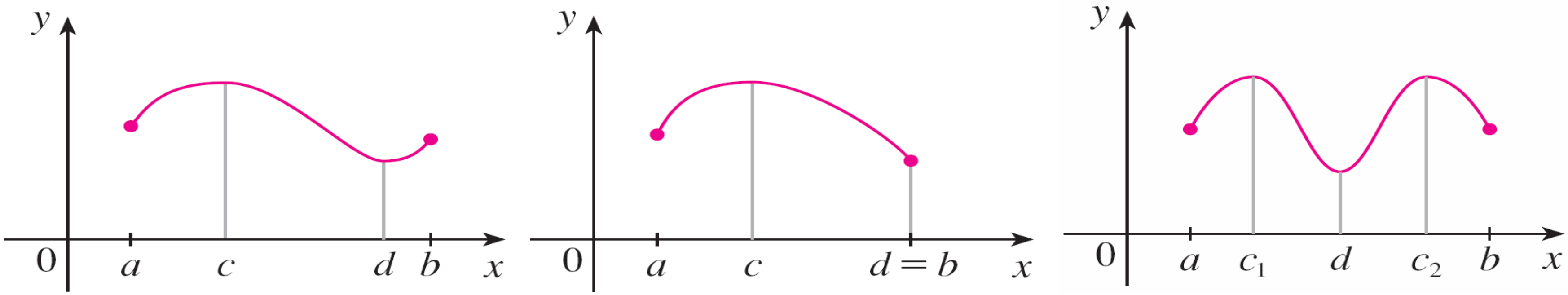
The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x

Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer

Extreme Value Theorem

When is a function guaranteed to possess extreme values?

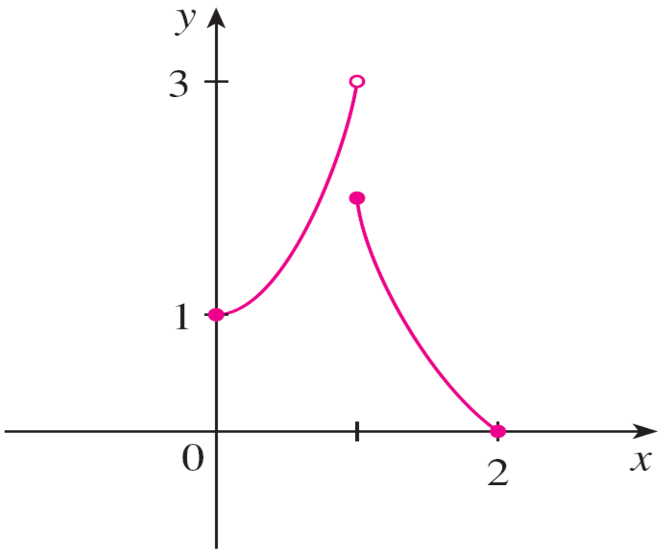
3 **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



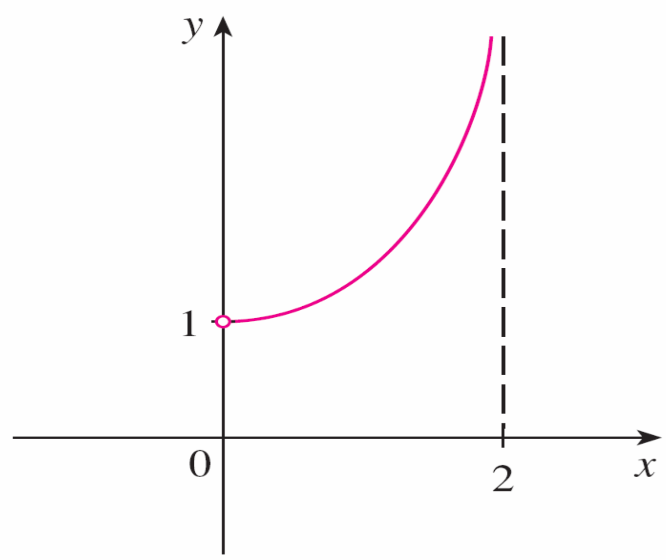
Note that an extreme value can be taken on more than once

Absence of Maximum or Minimum

If the function is not continuous or the interval is not closed, then it may not have minimum and/or maximum



This non continuous function has minimum value $f(2) = 0$, but no maximum value (the value 3 is never reached)

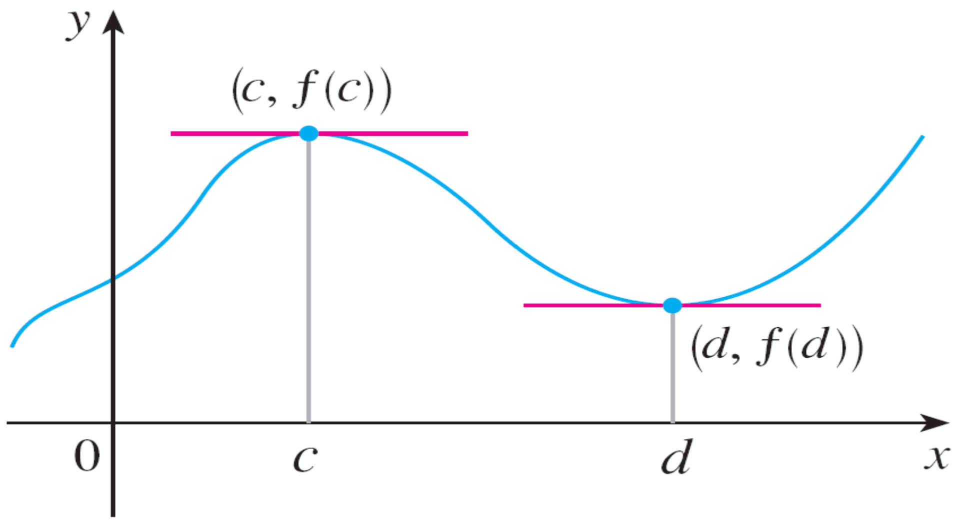


This function g is continuous but on the open interval $(0, 2)$: it has no maximum or minimum, the range is $(1, \infty)$ but it never reaches these two values

Finding Maximum and Minimum

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not say how to find these extreme values. We start by looking for local extreme values

For example, we want to find the local maximum at *c* and the local minimum at *d*



Using horizontal tangent

At the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0, so it appears that $f'(c) = 0$ and $f'(d) = 0$

This is always true for differentiable functions – as long as *c* or *d* are **not endpoints of the domain**

However, horizontal tangent does not always imply maximum or minimum

Fermat's Theorem:

Consider a function *f* defined on an interval *I*, and *c* not an endpoint of *I*.

If f has a local maximum or minimum at *c*, and if $f'(c)$ exists, then

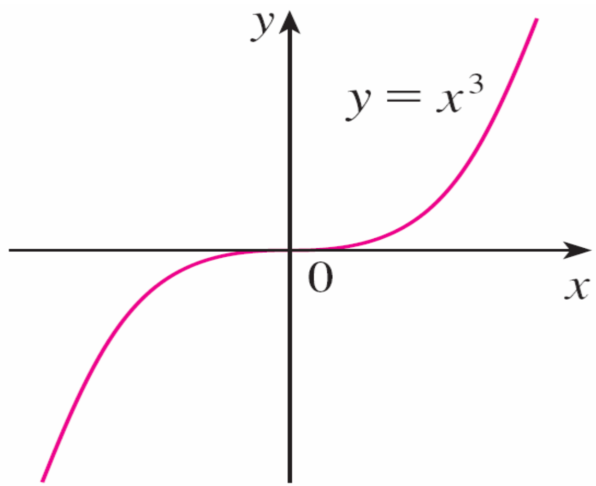
$$f'(c)=0$$

Example 1

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$

But f has no maximum or minimum at 0

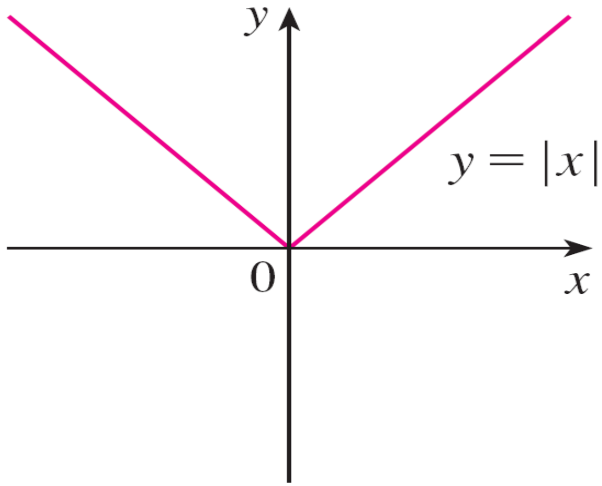
In this case, $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent in the point (0, 0)



If $f(x) = x^3$, then $f'(0) = 0$ but it is not maximum or minimum
Note that this f has no maximum or minimum

Example 2

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by searching for $f'(x) = 0$ because, $f'(0)$ does not exist



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Finding Maximum and Minimum

Examples 1 and 2 show that we must be careful when using Fermat's Theorem

As in Example 1, even if **$f'(c) = 0$, f doesn't necessarily have a maximum or minimum at c**

In other words, the converse of Fermat's Theorem is false in general

As in Example 2, **there may be an extreme value even when $f'(c)$ does not exist**

Finding Maximum and Minimum

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such values are called critical numbers or critical values

$\boxed{6}$ **Definition** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

So, Fermat's Theorem can be rephrased as follows

7 Consider a function f defined on an interval I. If f has a local maximum or minimum at c, and c is not an endpoint of I, then

c is a critical number of f

Finding Maximum and Minimum

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval

Thus the following three-step procedure always works

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

From experiments, we find that the average blood alcohol concentration (BAC) after consumption of 1 drink (15 mL of ethanol) is given by this function, where t is in minutes after consumption and C is in mg/mL

$$C(t) = 0.0225t\, e^{-0.0467t}$$

Find when is the maximum value of BAC in the first hour

We begin by finding the derivative using the product rule and the chain rule

$$C'(t) = 0.0225t\; e^{-0.0467t}(-0.0467) + 0.0225\; e^{-0.0467t}$$

$$= 0.0225\; e^{-0.0467t}(-0.0467t + 1)$$

The derivative is 0 when $0.0467\,t = 1$ $\quad t = 1 / 0.0467 \approx 21.4$

Example

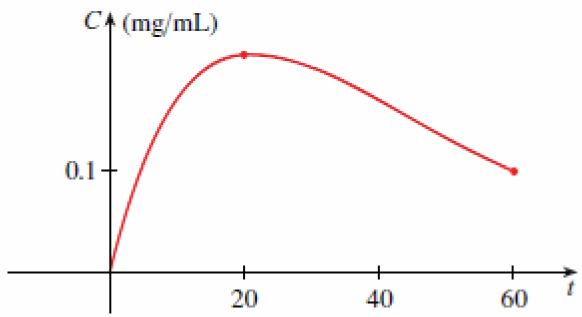
Then we compute the value of C at this critical value and at the endpoints 0 and 60 of the interval

$$C(21.4) \approx 0.177$$

$$C(0) = 0$$

$$C(60) \approx 0.0819$$

So the maximum of BAC in the first hour is 0.177 mg/mL, and it happens after about 21 minutes



Rolle's Theorem

We will see an important result called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need some other steps

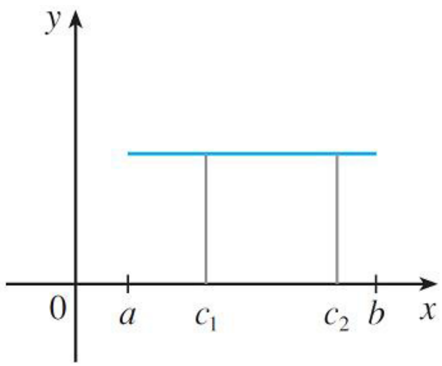
Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

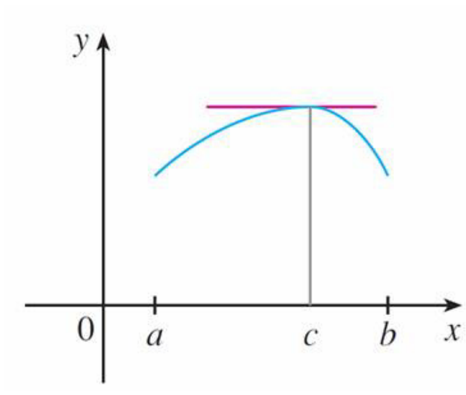
Then there is a number c in (a, b) such that $f'(c) = 0$.

Rolle’s Theorem

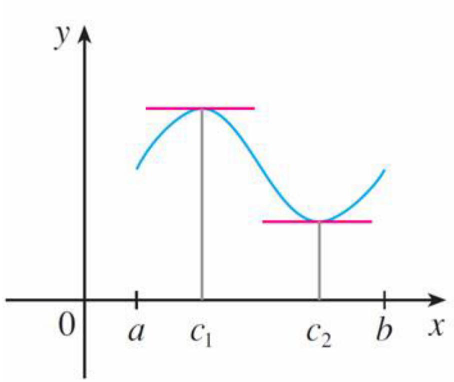
In each example there is at least one point where the tangent is horizontal and therefore $f'(c) = 0$



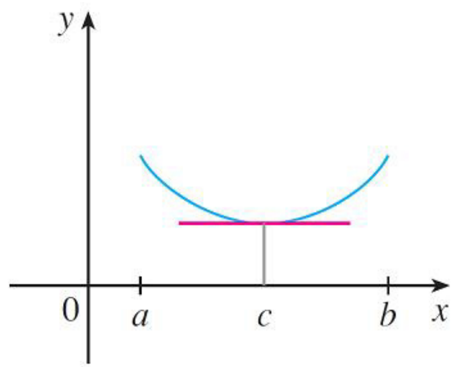
(a)



(b)



(c)



(d)

Example

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root (or solution)

Solution:
First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$

Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a value c between 0 and 1 such that $f(c) = 0$

Thus the given equation has at least one solution

Example – Solution

cont'd

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction

Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$

Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$

Example – Solution

cont'd

But

$$f'(x) = 3x^2 + 1 \geq 1 \qquad \text{for all } x$$

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction

Therefore the equation can't have two real roots

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

The slope of the secant line *AB* is

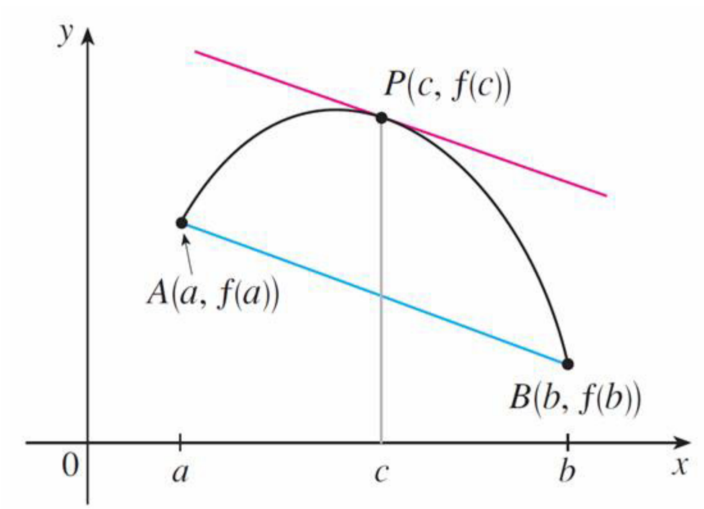
$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

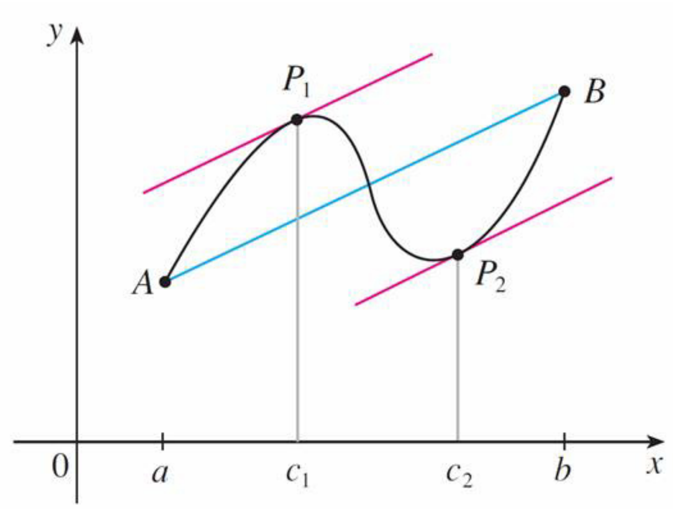
Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem says that there is at least one point $P(c, f(c))$ on the graph where the **slope of the tangent line** is the same as **the slope of the secant line** *AB*

In other words, there is a point *P* where the tangent line is **parallel** to the secant line *AB*

The Mean Value Theorem

So, there is at least one point where the derivative has slope parallel to the secant line AB. We can see that it is evident by considering the points *A*(*a*, *f*(*a*)) and *B*(*b*, *f*(*b*)) on the graphs of two differentiable functions





Example

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f(x) = x^3 - x \qquad a = 0, \; b = 2$$

Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on [0, 2] and differentiable on (0, 2)

Therefore, by the Mean Value Theorem, there is a number c in (0, 2) such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example

cont'd

Now $f(2) = 6$,

$f(0) = 0$, and

$f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2$$

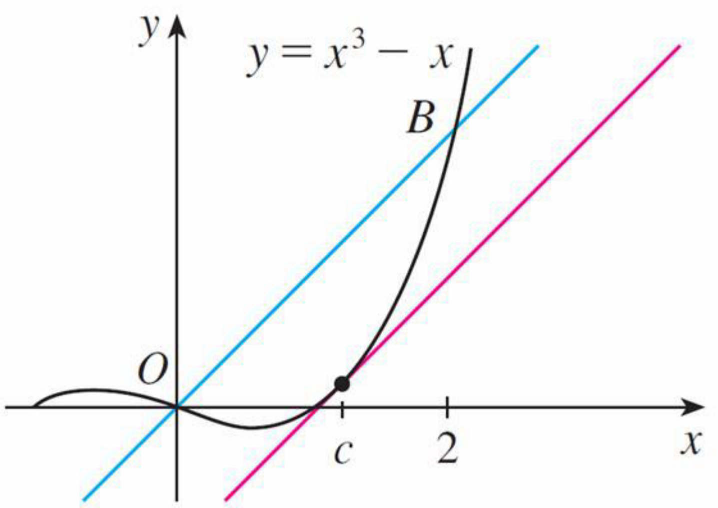
$$= 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in (0, 2), so $c = 2/\sqrt{3}$.

Example

cont'd

The tangent line at this value of c is parallel to the secant line OB



Detecting constant functions

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus

One of these basic facts is the following theorem

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Detecting constant functions

Note:

Care must be taken in applying Theorem 5. Let

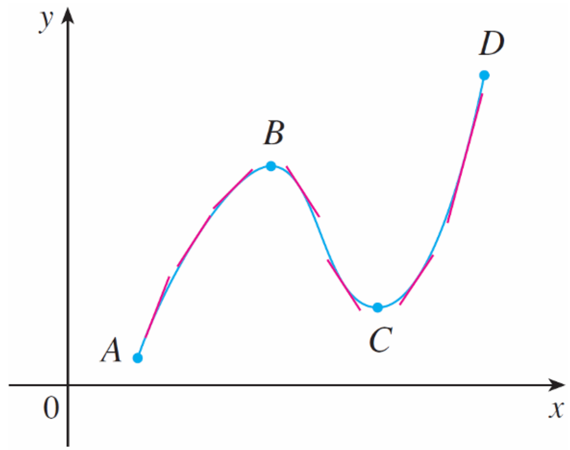
$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function

This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$

What Does f' Say About f?

As seen, the derivative of f tells where a function is increasing or decreasing



Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$

Between B and C the tangent lines have negative slope and so $f'(x) < 0$

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing

Solution:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

To use the **I/D Test** we have to know where $f'(x) > 0$ and where $f'(x) < 0$

This depends on the signs of the three factors of $f'(x)$:
$12x$, $x - 2$, and $x + 1$

Example

cont'd

We divide the real line into **intervals** whose endpoints are the **critical numbers** where the derivative is zero: –1, 0, 2 and arrange our work in a chart

A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test

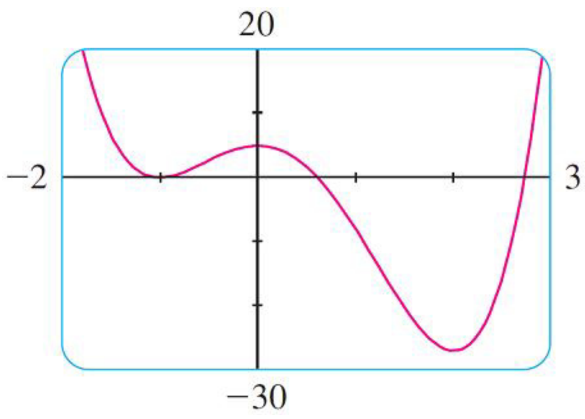
For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on (0, 2) (and also on the closed interval [0, 2])

Example

cont'd

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

The graph confirms the information in the chart



The First Derivative Test

In this example, $f(0) = 5$ is a local maximum of f because f increases on $(–1, 0)$ and decreases on $(0, 2)$

In derivatives, $f'(x)>0$ for $–1<x< 0$ and $f'(x)<0$ for $0<x< 2$

In other words, the sign of $f'(x)$ changes from positive to negative at 0. In general:

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

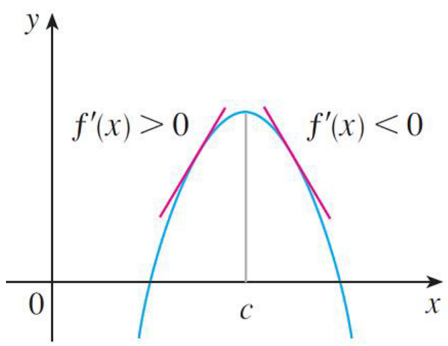
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

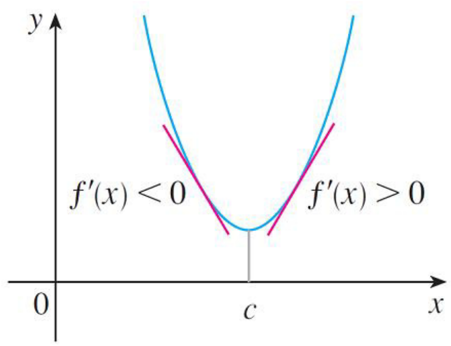
The First Derivative Test

In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c

It is easy to remember the First Derivative Test by visualizing these diagrams

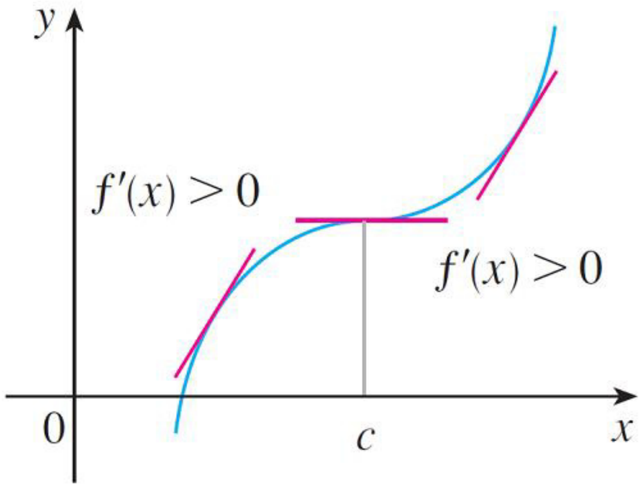


Local maximum

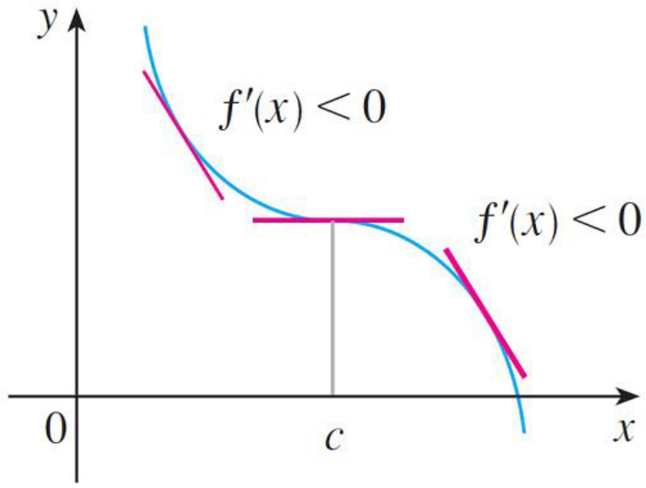


Local minimum

The First Derivative Test



No maximum or minimum

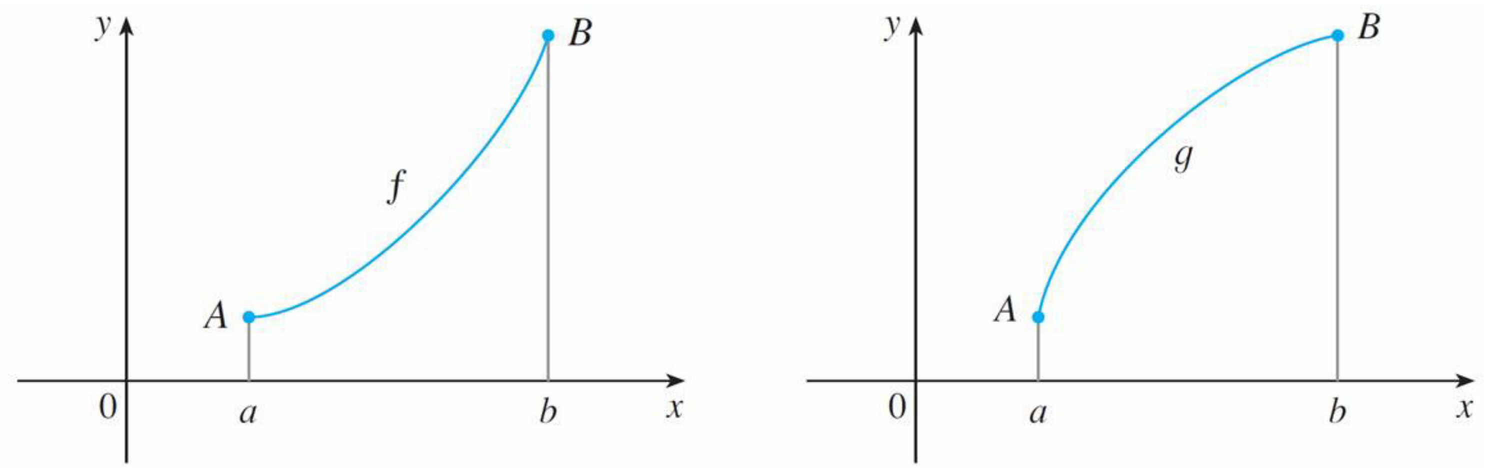


No maximum or minimum

What Does *f″* Say About *f* ?

Consider these two increasing functions on (a, b)

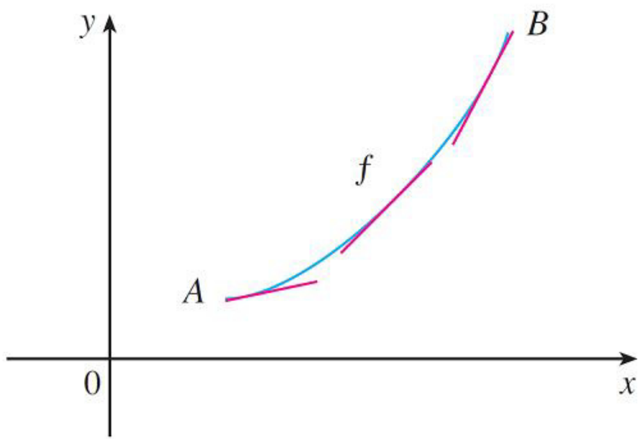
Both graphs join point A to point B but they look different because they bend in different directions



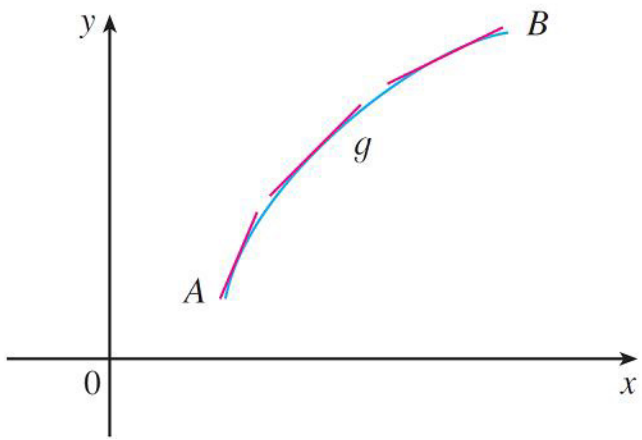
What Does *f*″ Say About *f* ?

In the first case the curve lies above the tangents and *f* is called *concave upward* on (a, b)

In the second case the curve lies below the tangents and *g* is called *concave downward* on (a, b)



Concave upward

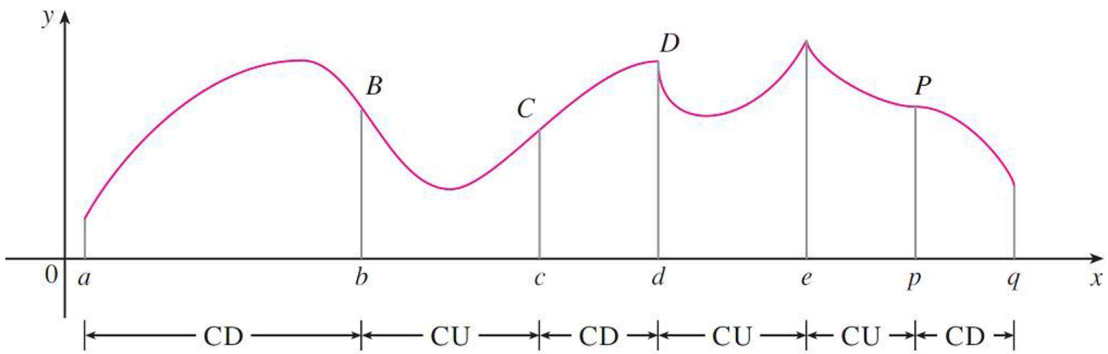


Concave downward

What Does f'' Say About f ?

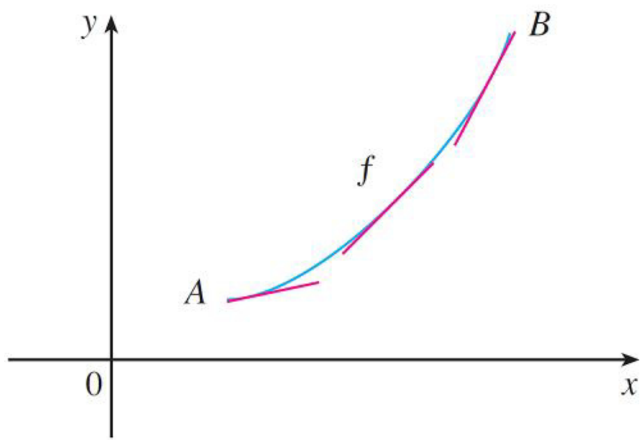
Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).



What Does *f″* Say About *f* ?

The second derivative helps determine the intervals of concavity. In this case, going from left to right, the slope of the tangent increases

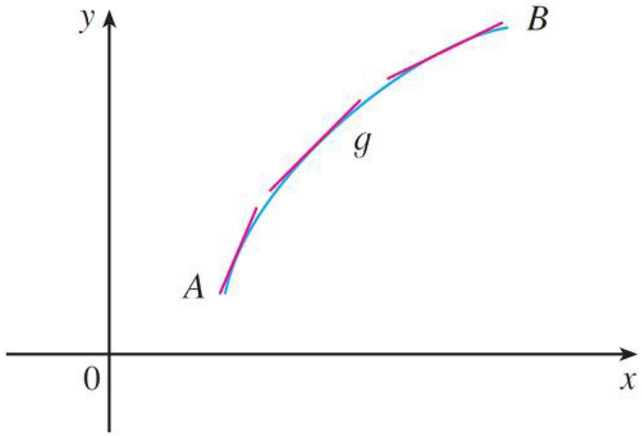


Concave upward

This means that the derivative *f′* is an increasing function and therefore its derivative *f″* is positive

What Does f'' Say About f ?

Here the slope of the tangent decreases from left to right, so f' decreases and therefore f'' is negative



Concave downward

What Does f'' Say About f ?

This reasoning can be reversed and suggests that:

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve

Solution:

If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Example

cont'd

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$

Since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0

Instead, $f'(x) > 0$ for $x > 3$, so for the First Derivative Test we have a local minimum at 3

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

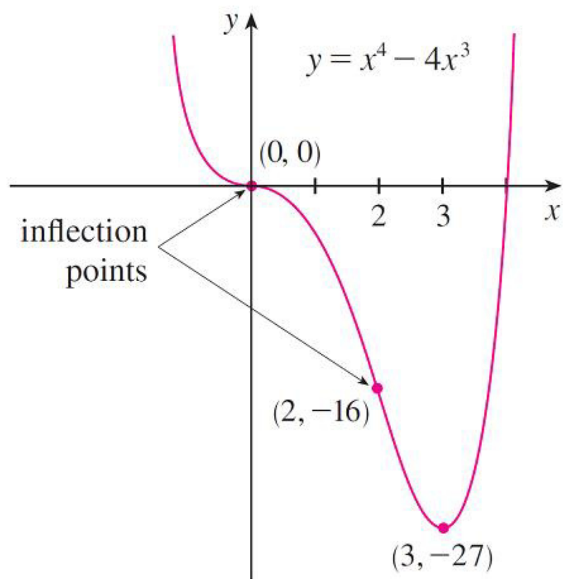
Example

cont'd

The point (0, 0) is an inflection point since the curve changes from concave upward to concave downward

Also (2, –16) is an inflection point since the curve changes from concave downward to concave upward there

Using the local minimum, the intervals of concavity, and the inflection points, we can sketch the curve



The Second Derivative Test

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Note: The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or neither

This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use

Example -- continued

Use the second derivative test to find the local extreme values of $f(x) = x^4 - 4x^3$

We already know $f'(0) = f'(3) = 0$. We now evaluate f'' there

$f''(0) = 0$ $f''(3) = 36 > 0$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0

Other Example

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$

Solution:

Calculation of the first two derivatives gives

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4 and 6

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 4$	$+$	$+$	$+$	$+$	increasing on $(0, 4)$
$4 < x < 6$	$-$	$+$	$+$	$-$	decreasing on $(4, 6)$
$x > 6$	$-$	$+$	$+$	$-$	decreasing on $(6, \infty)$

Other Example

cont'd

To find the local extreme values we use the First Derivative Test

Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum

Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum

The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could also be used at 4, but not at 0 or 6 since f'' does not exist at either of these numbers)

Other Example

cont'd

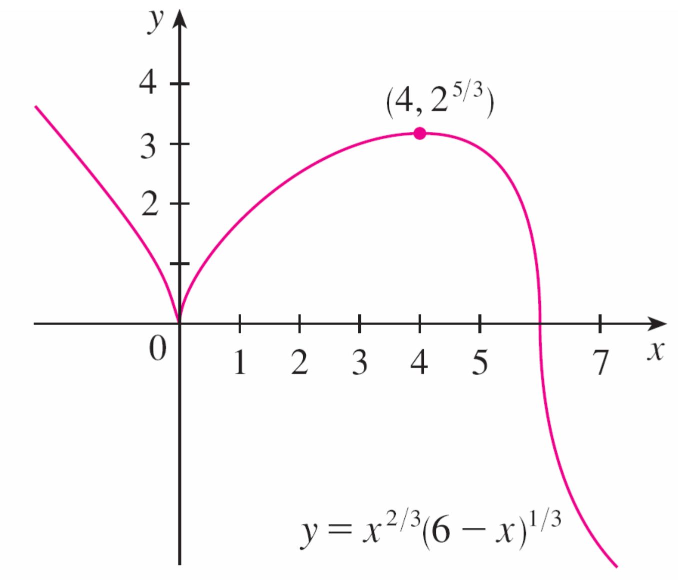
Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geq 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$, but $f''(x) > 0$ for $x > 6$

So f is concave downward on $(-\infty, 0)$ and $(0, 6)$, but it is concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$

Other Example

cont'd

The graph is



Note that the curve has vertical tangents at (0,0) and (6,0) because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$

Indeterminate Forms

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x* = 1, we need to know how *F* behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Indeterminate Forms

In computing this limit we can't apply the quotient low for limits because the limit of the denominator is 0. In fact, although the limit exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type zero over zero**

Indeterminate Forms

We will see a systematic method, known as *l'Hospital's Rule* (or *l'Hôpital,* or *Bernoulli-l'Hôpital*)*,* for the evaluation of indeterminate forms

Another indeterminate situation occurs when we want to evaluate the limit of *F* at infinity:

$$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$

Indeterminate Forms

There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite number

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞**

l'Hospital's Rule

L'Hospital's Rule applies to this type of indeterminate form

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

l’Hospital’s Rule

Note 1:

L’Hospital’s Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l’Hospital’s Rule

Note 2:

L’Hospital’s Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, “$x \to a$” can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$

Example

Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

Solution:

Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

Example – *Solution*

cont'd

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)}$$

$$= \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x}$$

$$= 1$$

Dealing with Indeterminate Forms

Clearly, l'Hospital rule does not apply always.
For example, not to this limit, because is not an indetrminate form: it is $1^+/0^+$

$$\lim_{x \to 1^+} \frac{x}{\ln x}$$

Another important limit is

$$\lim_{x \to 0} \frac{\sin x}{x}$$

In this case, it is often said that l'Hospital rule cannot be applied since this limit actually corresponds to the derivative of sin x

sin x/ x = (sin$x$ - sin0)/(x-0)

The derivative of sinx is cosx, and its limit for x→0 is 1

Dealing with Indeterminate Forms

L'Hospital's rule can also be applied by computing the derivative more than once

$$\lim_{x\to\infty}\frac{e^x}{x^2} \longrightarrow \lim_{x\to\infty}\frac{e^x}{2x}$$

However it is still indeterminate!
We need to apply again the rule, that means to derive again

$$\longrightarrow \lim_{x\to\infty}\frac{e^x}{2}=\infty$$

This also shows that the exponential function approaches infinity faster than any power of *x*

Dealing with Indeterminate Forms

Another interesting case is this, with $p > 0$

$$\lim_{x\to\infty}\frac{\ln x}{x^p} \longrightarrow \lim_{x\to\infty}\frac{1/x}{px^{p-1}} = \frac{1}{px^p} = 0$$

This shows that the logarithmic function approaches infinity more slowly than any power of *x*

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$(or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} [f(x)\, g(x)]$, if any, will be. There is a struggle between f and g

If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$)

Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**

Indeterminate Products

We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule

Example

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution:

The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$

Example – Solution

cont'd

Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} (-x)$$

$$= 0$$

Example – Solution

Note:
In solving this example another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type 0/0, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with

In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit!

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$. Again there is a contest between the two

Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Indeterminate Powers

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

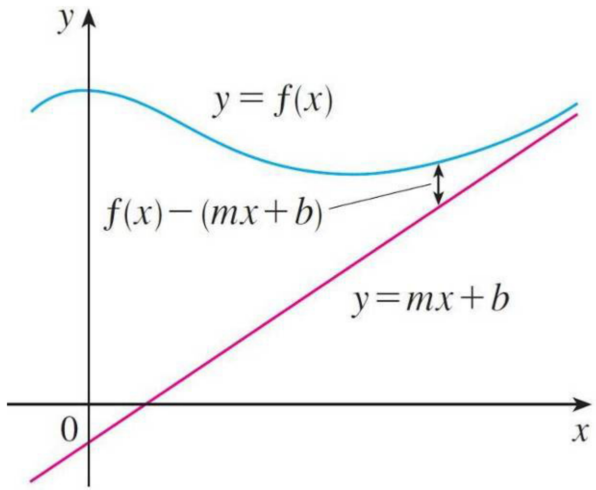
In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$

Slant Asymptotes

Some curves have asymptotes that are *oblique,* that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0



A similar situation may exist if we let $x \to -\infty$

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Clearly, some item may be not applicable (for instance, a given curve might not have an asymptote or possess symmetry)

But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined

Guidelines for Sketching a Curve

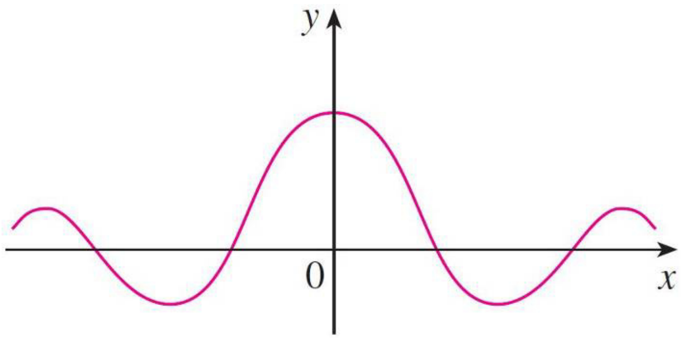
B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (sometimes omitted if the equation is difficult to solve)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis

Guidelines for Sketching a Curve

This means that our work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect about the y-axis to obtain the complete curve

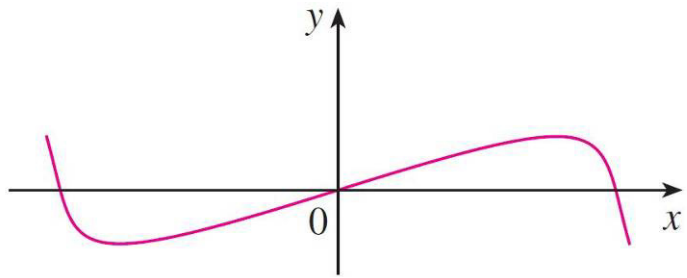


Even function: reflectional symmetry

Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$

Guidelines for Sketching a Curve

(ii) If $f(-x) = -f(x)$ for all in x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$ by rotating 180° about the origin



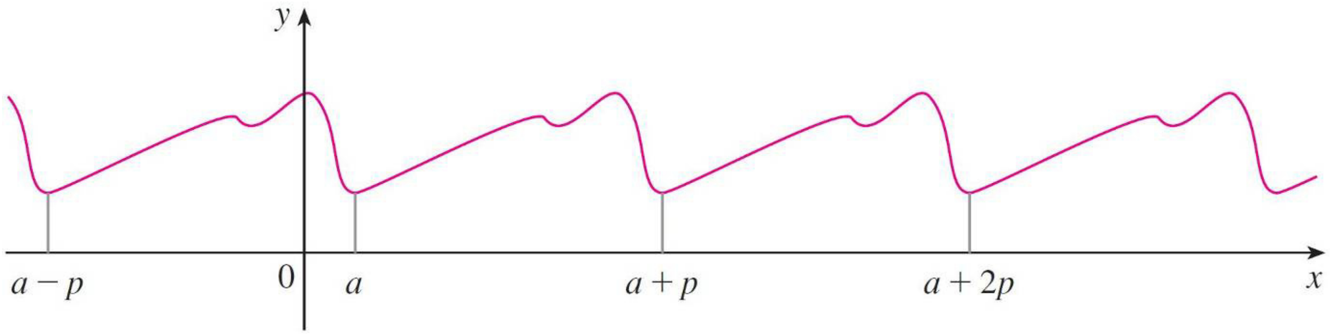
Odd function: rotational symmetry

Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

Guidelines for Sketching a Curve

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**

For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph



Periodic function: translational symmetry

Guidelines for Sketching a Curve

D. Asymptotes

(i) *Horizontal Asymptotes.* If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$

If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve

The same applies when considering $\lim_{x\to -\infty} f(x)$

Guidelines for Sketching a Curve

(ii) *Vertical Asymptotes.* The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

For rational functions you can also locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors, but for other functions this method does not apply

Guidelines for Sketching a Curve

If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite

(iii) *Slant Asymptotes.* Search for $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing)

Guidelines for Sketching a Curve

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum

If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$

Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum

Guidelines for Sketching a Curve

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points

Guidelines for Sketching a Curve

Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes

If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds

Example

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$$

$$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The *x*- and *y*-intercepts are both 0

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the *y*-axis

Example

cont'd

D. $\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$

Therefore the line $y = 2$ is a horizontal asymptote

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

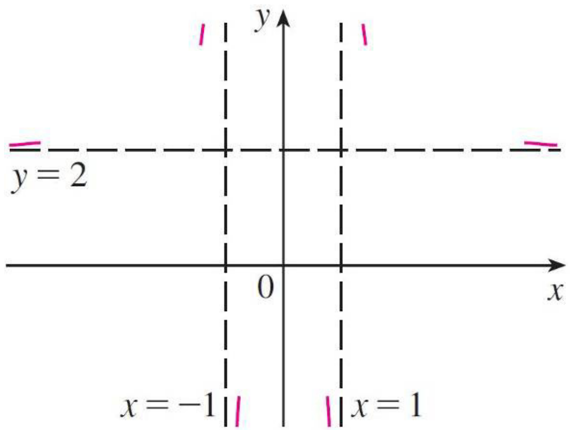
$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Example

cont'd

Therefore the lines x = 1 and x = –1 are vertical asymptotes

This information about limits and asymptotes enables us to draw this preliminary sketch, showing the parts of the curve near the asymptotes



Preliminary sketch

Example

cont'd

E. $f'(x) = \dfrac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \dfrac{-4x}{(x^2 - 1)^2}$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$

F. The only critical number is $x = 0$

Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test

Example

cont'd

G. $f''(x) = \dfrac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \dfrac{12x^2 + 4}{(x^2 - 1)^3}$

Since $12x^2 + 4 > 0$ for all x, we have

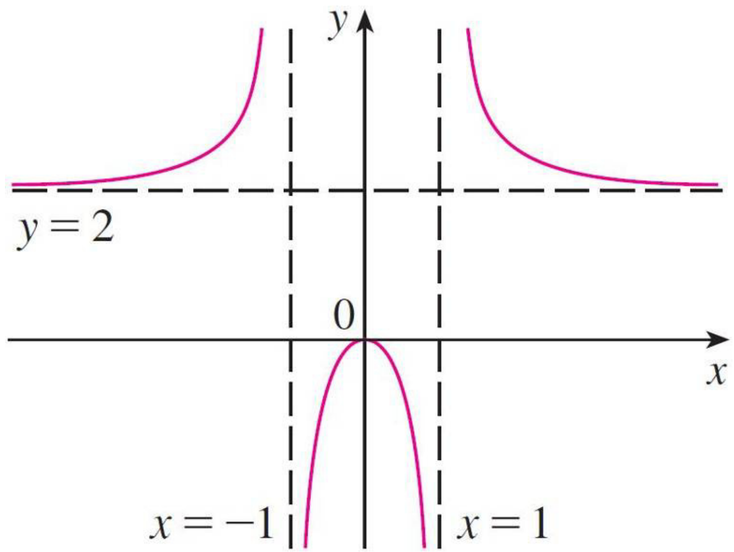
$$f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow |x| > 1$$

and $f''(x) < 0 \Leftrightarrow |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and –1 are not in the domain of f

Example

cont'd

H. Using the information in E–G, we finish the sketch



Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

Optimization Problems

In solving many practical problems the challenge is often to convert them into mathematical optimization by setting up a function that is to be maximized or minimized (=**optimized**) and possible constraints

When facing one of these problems, we need to:

1. understand the real-world problem and the relationships among the quantities

2. understand which quantities we need to decide (**variables**) and which are just data of the problem

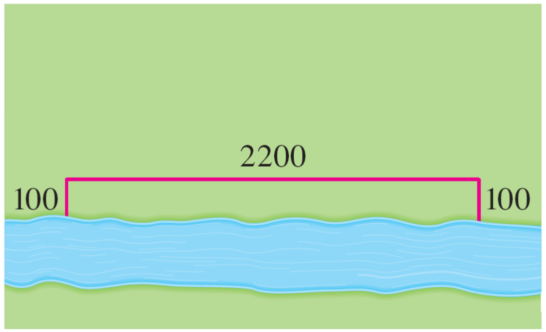
3. write an optimization model, with an **objective function** that we want to maximize or minimize, and possibly some **constraints** to specify which **solutions** (= values for the variables) are **feasible**

4. Find the **optimal solution** (= the one maximizing or minimizing the objective function) among the feasible solutions

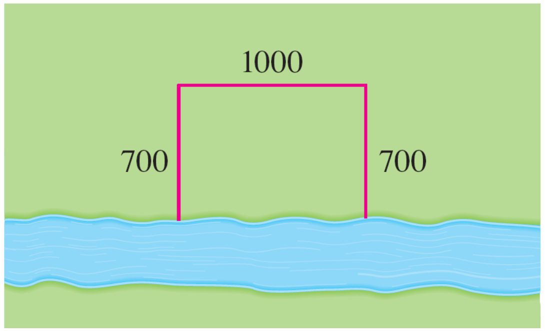
Example

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

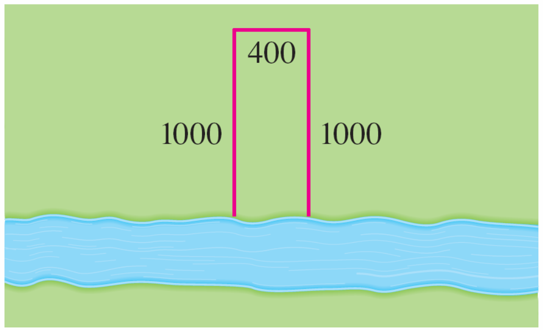
To better understand this problem, let's see some possible ways of laying out the fencing. These are 3 feasible solutions, we want the optimal solution



Area = $100 \cdot 2200 = 220{,}000$ ft^2



Area = $700 \cdot 1000 = 700{,}000$ ft^2



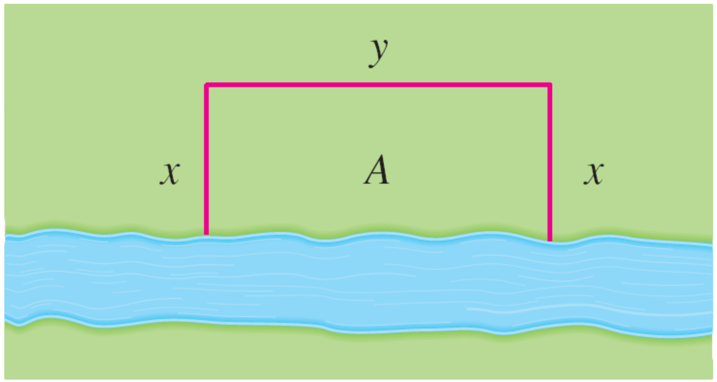
Area = $1000 \cdot 400 = 400{,}000$ ft^2

Example – *Solution*

cont'd

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

We wish to maximize the area *A* of the rectangle, and we need to decide x and y



Example – *Solution*

cont'd

Let x and y be the depth and width of the rectangle (in m). Then we express A in terms of x and y:

$$A = xy$$

We can express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 m.

Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Example – *Solution*

cont'd

Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$

The maximum value of A must occur either at this critical number or at an endpoint of the interval

Example – *Solution*

cont'd

Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, so the maximum value is $A(600) = 720{,}000$

Alternatively, we could observe that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum

Thus the rectangular field should be 600 m deep and 1200 m wide

Global maximum or minimum

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Optimization in Economics

Consider the cost of producing x units of a certain product. We can write it as a function $C(x)$ called **cost function**. Then, the cost of producing one additional unit, called the **marginal cost**, is the rate of change of C with respect to x, hence, it is the derivative $C'(x)$ of the cost function

Now let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**) and it is generally a decreasing function of x

Optimization in Economics

If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

and R is called the **revenue function**

The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold

Optimization in Economics

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**

The **marginal profit function** is P', the derivative of the profit function

Example

A store has been selling 200 Blu-ray disc players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution:

If x is the number of Blu-ray players sold per week, then the weekly increase in sales is $x - 200$

If the price is decreased by \$10 there is an increase of 20 units sold

Example – *Solution*

cont'd

So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$

This value of x gives an absolute maximum by the First Derivative Test because R' changes from positive to negative (or simply by observing that the graph of R is a parabola that opens downward)

Example – *Solution*

cont'd

The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is 350 – 225 = 125

Therefore, to maximize revenue, the store should offer a rebate of $125

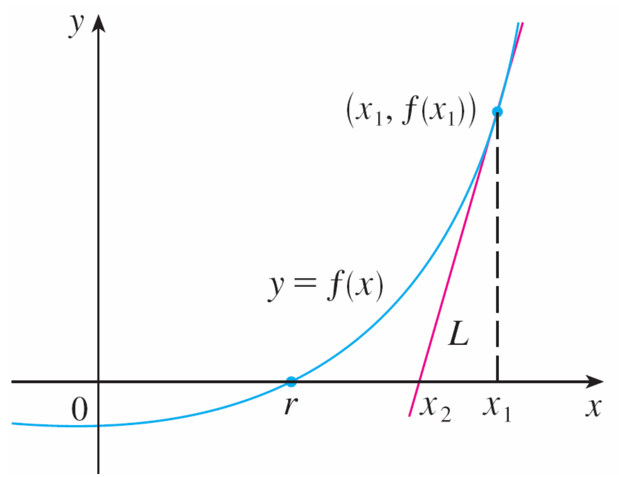
Indeed, for 200 units sold at 350$ the total revenue is 70,000$

For 450 units sold at 225$ the total revenue is 101,250$

Newton's Method

To find the roots of an equation $f(x)=0$ we can use the **Newton's method**, a.k.a. **Newton-Raphson method**

Imagine we want to find the solution corresponding to point r



We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of f, or from a computer-generated graph of f

Newton's Method

Then we take the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2

The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x-intercept

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Newton's Method

Since the x-intercept of L is x_2, we set $y = 0$ and we search for the value of x_2

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r

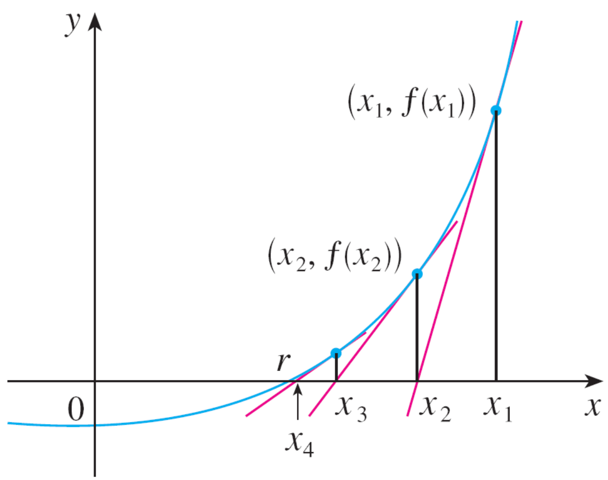
Next we repeat this procedure with x_1 replaced by the second approximation x_2, using the tangent line at $(x_2, f(x_2))$

Newton's Method

This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations x_1, x_2, x_3, x_4, . . .



Newton's Method

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

2

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the numbers x_n become closer and closer to r as n becomes large, then we can find the root, and we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Example

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$

Solution:

We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

(Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$)

Example – *Solution*

cont'd

Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With n = 1 we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$= 2.1$$

Example – *Solution*

cont'd

Then with n = 2 we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$$

$$\approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places

Antiderivatives

In some case, we may need to find a function *F* whose derivative is a given function *f*. If such a function *F* exists, it is called an *antiderivative* of *f*

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$

But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$
Therefore both F and G are antiderivatives of f

Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f

Antiderivatives

The following theorem says that f has no other antiderivative

$\boxed{1}$ **Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

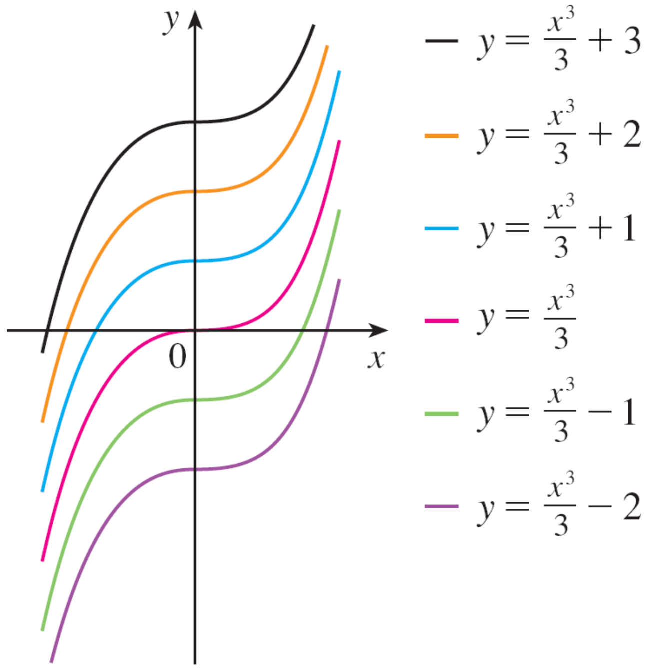
where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $x^3/3 + C$

Antiderivatives

By assigning specific values to the constant *C*, we obtain a family of functions whose graphs are vertical translates of one another

This makes sense because each curve must have the same slope at any given value of *x*



Members of the family of antiderivatives of $f(x) = x^2$

Example

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ **(b)** $f(x) = 1/x$ **(c)** $f(x) = x^n, \; n \neq -1$

Solution:

(a) Recall that d/dx (cos(x)) = −sen(x)

So, if $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$

By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$

Example – *Solution*

cont'd

(b) Recall

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of $1/x$ is $\ln x + C$. We also learned that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln |x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$

Example – *Solution*

cont'd

So the general antiderivative of f is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Example – *Solution*

cont'd

This is valid for $n \geq 0$ since then $f(x) = x^n$ is defined on any interval. If n is negative (but $n \neq -1$, we have already seen that case), it is valid on any interval that doesn't contain 0

Table of Antiderivatives

By considering in the reverse way the basic derivatives seen, we can write a **Table of Antiderivatives**

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1 - x^2}}$	$\sin^{-1} x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{1 + x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

To obtain the most general anti derivative from the particular ones in this Table, we have to add a constant (or constants)