Examples of derivatives

- $f(x) = \sqrt{x}$
- $f'(x) = \frac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

- $f(x) = (2x+3)^4$ it is a composite function
- $f'(x) = 4(2x+3)^3\, 2 = 8(2x+3)^3$

- $f(x) = \sqrt{1 + \sin x}$ it is a composite function
- $f'(x) = \dfrac{1}{2\sqrt{1 + \sin x}}(0 + \cos x) = \dfrac{\cos x}{2\sqrt{1 + \sin x}}$

Examples of derivatives

cont'd

- $f(x) = e^{\sqrt{x^2+1}}$

it is a composite of 3 functions: exponential, root and polynomial !

- $f'(x) = e^{\sqrt{x^2+1}} \dfrac{1}{2\sqrt{x^2+1}} (2x) = \dfrac{x\, e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

- $f(x) = (3x^2+x)^2 \ (2x-1)$

a composite function multiplied by another function !

- $f'(x) = 2(3x^2+x)(6x+1)\ (2x-1) + (3x^2+x)^2\ (2) =$
- $2(3x^2+x)\ [(6x+1)\ (2x-1) + (3x^2+x)\ ] =$
- $2x(3x+1)\ [12x^2 +2x -6x -1 +3x^2 +x] =$
- $2x\ (3x+1)\ (15x^2 -3x -1) =$

Examples of derivatives

cont'd

- $f(x) = \dfrac{(\ln x)^2}{(\ln x)+1}$

a composite function divided by another function !

- $f'(x) = \dfrac{2 \ln x \, (1/x) \, (\ln x + 1) - (\ln x)^2 \, (1/x)}{(\ln x + 1)^2} =$

- $\dfrac{(1/x)(\ln x)(2 \ln x + 2 - \ln x)}{(\ln x + 1)^2} =$

- $\dfrac{(\ln x)(\ln x + 2)}{x(\ln x + 1)^2}$

Examples of derivatives

cont'd

- $f(x) = x\,e^{-3x}$

  a product of a function by a composite function !

- $f'(x) = e^{-3x}(1) + x\,e^{-3x}(-3) = e^{-3x}(1 - 3x)$

- $f(x) = x^x$

  a function at the power of another function ! It is neither a simple power nor a simple exponential

- we need to rewrite it as $f(x) = e^{\ln x^x} = e^{x \ln x}$
- $f'(x) = e^{x \ln x}[\ln x\,(1) + (1/x)\,x] =$
- $x^x(\ln x + x/x) =$
- $x^x(\ln x + 1)$

Sample Exam exercises

Study the following function $f(x) = 10x + 5/x$

by finding the domain, the symmetry, the sign of the function over the domain, the behaviour of the function at the extremes of the domain, possible asymptotes, possible maxima and minima, where the function is increasing or decreasing, the concavity.

Solution

1)Find the domain of $f(x)$

It is defined everywhere except in 0, hence D: $\{ x \in \mathbb{R} \mid x \neq 0\}$

2) symmetry

$f(-x) = -10x - 5/x$ is equal to $-f(x)$, so the function is odd, symmetric with respect to the origin

3) intersection with the axes

There is no intersection with the y axis because x=0 is out of the domain of the function.
To find the intersection with the x axis, we search for $0 = 10x + \frac{5}{x}$. Since $x \neq 0$, we write the common denominator $\frac{10x^2+5}{x} = 0$, and we search where the numerator is zero. However, $10\,x^2$ is always positive for $x <> 0$, so there is no intersection with the x axis, too.

4) study the sign of the function

we rewrite the function as $f(x) = \frac{10x^2+5}{x}$ so that we can study the sign of numerator and denominator separately

N: $10x^2 + 5 > 0\ \forall\, x \ni \mathbb{R}$
$D\!: x > 0$ only when x is positive

So the function is positive for $x > 0$ and negative for $x < 0$

5) behaviour of the function at the extremes of the domain

we have that $\lim_{x\to\pm\infty} 10x + \frac{5}{x} = \pm\infty$
Since it goes to infinity there are no horizontal asymptotes
Searching for a slant asymptotes y= mx+q we have that
m= $\lim_{x\to\pm\infty} \frac{(10x+\frac{5}{x})}{x}$ = 10 because $\frac{5}{x}$ goes to 0 and $\frac{10x}{x}$ =10
q is obtained from $\lim_{x\to\pm\infty} [f(x) - mx]$, -> $\lim_{x\to\pm\infty} [10x + \frac{5}{x} - 10x]$.
Since $\lim_{x\to\pm\infty} \frac{5}{x}$ tends to 0, q is 10x-10x, so q= 0.
In conclusion there is a slant asymptotes with equation $y = 10x$

Moreover,
$\lim_{x\to 0-} 10x+\frac{5}{x} = -\infty$ and $\lim_{x\to 0+} 10x+\frac{5}{x} = +\infty$ because $\frac{5}{x}$ tends to ∞ for $x \to 0$;
hence in x = 0 there is a vertical asymptotes

6) we study the first derivative $f'(x) = 10 - \frac{5}{x^2}$, the domain is : $x \neq 0$.

We study the sign of the derivative to find where the function is increasing or decreasing..
We compute the common denominator x^2 . That one is always positive, so we study the sign of the numerator $10x^2 - 5$. That is positive when $10x^2 > 5$ so when $x^2 > 1/2$ hence $x > 1/\sqrt{2}$ and $x < -1/\sqrt{2}$

Since $1/\sqrt{2}$ is normally written as $\sqrt{2}/2$, we have that
the function is increasing in the interval $(-\infty; -\frac{\sqrt{2}}{2})$, and decreasing in $(-\frac{\sqrt{2}}{2}; 0)\ e\ (0; \frac{\sqrt{2}}{2})$ then it is again incresing in the interval $(\frac{\sqrt{2}}{2}; +\infty)$.

Since the derivative passes from positive to negative in $f(-\frac{\sqrt{2}}{2}) = -10\sqrt{2} \approx -14{,}14$ we have a local maximum with coordinates $(-\frac{\sqrt{2}}{2};\ -10\sqrt{2})$

Since the derivative passes from negative to positive in $f(\frac{\sqrt{2}}{2}) = 10\sqrt{2} \approx 14{,}14$. we have a local minimum with coordinates $(\frac{\sqrt{2}}{2}; 10\sqrt{2})$

7) we study the second derivative $f''(x) = \frac{10}{x^3}$,
It is defined for all x except x=0
we see that it is positive for x>0 so the function is concave upward.
It is negative for x<0 so the function is concave downward
Finally, the graph of the function is