Bisimulation and Simulation

Bisimilarity

Intuition:

Two (states of two) transition systems are bisimilar if they have the same behavior.

In the sense that:

• Locally they (the two states) look indistinguishable
• Every action that can be done on one of them can also be done on the other remaining indistinguishable
**Bisimilarity**

- A binary relation $R$ is a **bisimulation** iff:

$$ (s,t) \in R \text{ implies that } $$
- $s$ is **final** iff $t$ is **final**
- for all actions $a$:
  - if $s \xrightarrow{a} s'$ then $\exists t'. t \xrightarrow{a} t'$ and $(s',t') \in R$
  - if $t \xrightarrow{a} t'$ then $\exists s'. s \xrightarrow{a} s'$ and $(s',t') \in R$

- A state $s_0$ of transition system $S$ is **bisimilar**, or simply **equivalent**, to a state $t_0$ of transition system $T$ iff there exists a **bisimulation** between the initial states $s_0$ and $t_0$.

- Notably:
  - **bisimilarity** is a bisimulation
  - **bisimilarity** is the **largest** bisimulation

*Note it is a co-inductive definition!*

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**Computing Bisimilarity on Finite Transition Systems**

**Algorithm** ComputingBisimulation

**Input:** transition system $TS_S = \langle A, S_0, S, \delta_S, F_S \rangle$ and transition system $TS_T = \langle A, T_0, T, \delta_T, F_T \rangle$

**Output:** the **bisimilarity** relation (the largest bisimulation)

**Body**

$$ R = S \times T $$
$$ R' = R - \{(s,t) \mid \neg(s \in F_S \equiv t \in F_T)\} $$

while ($R \neq R'$) {

$$ R' := R' - \{(s,t) \mid \exists s',a. s \xrightarrow{a} s' \land \neg \exists t'. t \xrightarrow{a} t' \land (s',t') \in R' \} $$
$$ \{ (s,t) \mid \exists t',a. t \xrightarrow{a} t' \land \neg \exists s'. s \xrightarrow{a} s' \land (s',t') \in R' \} $$

$$ \} $$

return $R'$

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**Service Integration**
Example of Bisimilarity

Are $S$ and $T$ bisimilar?

Computing Bisimilarity

We need to compute the greatest fixpoint (gfp): we do it by computing approximates starting from the Cartesian product:

- $R_0=\{(t_0,s_0), (t_0,s_1), (t_0,s_2), (t_1,s_0), (t_1,s_1),(t_1,s_2)\}$ – Cartesian product
- $R_1=\{(t_0,s_0),(t_1,s_1),(t_1,s_2)\}$ – removed those pairs that violate local condition on final (final iff final)
- $R_2=\{(t_0,s_0),(t_1,s_1),(t_1,s_2)\}$ – removed those pairs where one can do action and other cannot copy remaining in the relation.

$R_1=R_2$ greatest fixpoint reached!

S and T are bisimilar

$((t_0,s_0) \text{ do belong to gfp})$
Example of NON Bisimilarity

Are S and T bisimilar?

Computing Bisimilarity

We need to compute the greatest fixpoint: we do it by computing approximates starting from the cartesian product:

- R0=\{(t0,s0), (t0,s1), (t0,s2), (t1,s0), (t1,s1),(t1,s2)\} – cartesian product
- R1=\{(t0,s0),(t1,s1),(t1,s2)\} – removed those pairs that violate local condition on final (final iff final)
- R2=\{(t0,s0),(t1,s1)\} – removed (t1,s2) since t1 can do c but s2 cannot.
- R3=\{(t1,s1)\} – removed (t0,s0) since t0 can do b, s2 can do b as well, but then the resulting states (t1,s2) are NOT in R2.
- R4 = {} – removed (t1,s1) since t1 can do c, s1 can do c as well, but then the resulting states (t0,s0) are NOT in R3.
- R5 = {}

R4=R5 greatest fixpoint reached!

S and T are NOT bisimilar
((t0,s0) do not belong to gfp)
**Simulation**

**Intuition:**

One (state of a) transition system can be mimicked (or copied) by another (state of another) transition system.

In the sense that:

- Locally the property that hold on the state of the "to be copied" transitions systems, holds also in the state of the "coping" transition
- Every action that the "to be copied" transition system can do in the current state, can be copied by the "coping" transition system (in the current state) and the same thing holds in the resulting states.

**Simulation**

- A binary relation $R$ is a simulation iff:

  $(s,t) \in R$ implies that
  - $s$ is final implies that $t$ is final
  - for all actions $a$
    - if $s \xrightarrow{a} s'$ then $\exists t' . t \xrightarrow{a} t'$ and $(s',t') \in R$

- A state $s_0$ of transition system $S$ is simulated by a state $t_0$ of transition system $T$ iff there exists a simulation between the initial states $s_0$ and $t_0$.

- Notably
  - simulated-by is a simulation
  - simulated-by is the largest simulation

  *Note it is a co-inductive definition!*

- NB: A simulation is just one of the two directions of a bisimulation
Computing Simulation on Finite Transition Systems

**Algorithm** ComputingSimulation

**Input:** transition system $TS_S = < A, S, S^0, \delta_S, F_S >$ and transition system $TS_T = < A, T, T^0, \delta_T, F_T >$

**Output:** the **simulated-by** relation (the largest simulation)

**Body**

$R = S \times T$

$R' = S \times T - \{(s,t) | s \in F_S \land \neg(t \in F_T)\}$

while ($R \neq R'$) {

$R := R'$

$R' := R' - \{(s,t) | \exists s', a. s \rightarrow_a s' \land \neg \exists t'. t \rightarrow_a t' \land (s', t') \in R' \}$

}

return $R'$

Ydob

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**Example of Simulation**

Are $S$ and $T$ similar?

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Service Integration

Giuseppe De Giacomo
Computing Simulation

We need to compute the greatest fixpoint (gfp): we do it by computing approximates starting from the Cartesian product:

• $R_0 = \{(t_0, s_0), (t_0, s_1), (t_0, s_2), (t_1, s_0), (t_1, s_1), (t_1, s_2)\}$ – Cartesian product
• $R_1 = \{(t_0, s_0), (t_1, s_0), (t_1, s_1), (t_1, s_2)\}$ – removed those pairs that violate local condition on final (if T final then S final)
• $R_2 = \{(t_0, s_0), (t_1, s_1), (t_1, s_2)\}$ – removed $(t_1, s_0)$, since $t_1$ can do $c$ and $s_0$ cannot
• $R_3 = \{(t_0, s_0), (t_1, s_1), (t_1, s_2)\}$ – nothing is further removed
• $R_2 = R_3$ greatest fixpoint reached!

S simulates T: S’ s behavior “includes” T’s ((t0,s0) do belong to gfp)

Example of Simulation

Does S simulate T?
Computing Simulation

We need to compute a greatest fixpoint: we do it by computing approximates starting from the Cartesian product:

- $R_0 = \{(t_0,s_0), (t_0,s_1), (t_0,s_2), (t_1,s_0), (t_1,s_1),(t_1,s_2)\}$ – Cartesian product
- $R_1 = \{(t_0,s_0), (t_1,s_0), (t_1,s_1),(t_1,s_2)\}$ – removed those pairs that violate local condition on final (if $T$ final then $S$ final)
- $R_2 = \{(t_0,s_0),(t_1,s_1)\}$ – removed $(t_1,s_0)$ since $t_1$ can do $c$ but $s_0$ cannot; removed $(t_1,s_2)$ since $t_1$ can do $c$ but $s_2$ cannot
- $R_3 = \{(t_1,s_1)\}$ – removed $(t_0,s_0)$ since $t_0$ can do $b$, $s_2$ can do $b$ as well, but then the resulting states $(t_1,s_2)$ are NOT in $R_2$.
- $R_4 = \{}$ – removed $(t_1,s_1)$ since $t_1$ can do $c$, $s_1$ can do $c$ as well, but then the resulting states $(t_0,s_0)$ are NOT in $R_3$.
- $R_5 = \{}$

$R_4=R_5$ greatest fixpoint reached!

S does **not simulate** T  
($(t_0,s_0)$ do not belong to gfp)

Example of Simulation

Does T **simulate** S?
Computing Simulation

We need to compute the greatest fixpoint: we do it by computing approximates starting from the Cartesian product:

- \( R_0 = \{(s_0,t_0), (s_0,t_1), (s_1,t_0), (s_1,t_1), (s_2,t_0)\} \) – Cartesian product
- \( R_1 = \{(s_0,t_0), (s_1,t_0), (s_2,t_0)\} \) – removed those pairs that violate local condition on final (if \( S \) final then \( T \) final)
- \( R_2 = \{(s_0,t_0), (s_1,t_1), (s_2,t_0)\} \) – removed \((s_1,t_0)\) since \( s_1 \) can do c but \( t_0 \) cannot; removed \((t_1,s_2)\) since \( t_1 \) can do c but \( s_2 \) cannot
- \( R_3 = \{(s_0,t_0), (s_1,t_1), (s_2,t_0)\} \) – nothing is removed

\( R_2 = R_3 \) greatest fixpoint reached!

\[ T: \begin{array}{ccc} & a & \\ t_0 & b & t_1 \end{array} \]

\[ S: \begin{array}{ccc} c & \\ s_0 & a & s_1 \\ & b & s_2 \end{array} \]

T does simulate S ((s0,t0) do belong to gfp)

Exercises

Consider the following transition systems.

\[ T: \begin{array}{ccc} s_1 & a & s_2 \\ & a & \\ & a & \end{array} \]

\[ S: \begin{array}{ccc} t_1 & a & t_2 & a & t_3 \end{array} \]

\[ Q: \begin{array}{ccc} q_1 & a & q_2 & a & q_3 \end{array} \]

- **Which simulations hold between the three?** If simulation holds, write a simulation relation, otherwise show where simulation breaks.

- **Which bisimulations hold between the three?** If bisimulation holds, write a bisimulation relation, otherwise show where simulation breaks.
Exercises
Consider the following transition systems.

- Does $T$ simulates $S$? If so, write a simulation relation. If not, show where simulation breaks.
- Does $S$ simulates $T$? If so, write a simulation relation. If not, show where simulation breaks.
- Are they bisimilar? If so, write a bisimulation relation. If not, show where bisimulation breaks.