A Proofs

Theorem 1 For any objective sentence about situation $s$, $\phi(s)$,

$$\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \phi(\text{end}[\sigma])$$

if and only if

$$\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{Know}(\phi(\text{now}), \text{end}[\sigma]).$$

Proof Sketch: $\iff$ Follows trivially from the reflexivity of $K$ in the initial situation, and the fact that it is preserved by the successor state axiom for $K$.

$\Rightarrow$ From the successor state axiom for $K$ it follows that:

$$\text{Axioms} \cup \{\text{Sensed}[^{[} \cdot (a, 1)]\} \models \text{Know}\left(\text{SF}_a(\text{now}), \text{end}[\cdot (a, 1)]\right)$$

$$\text{Axioms} \cup \{\text{Sensed}[^{[} \cdot (a, 0)]\} \models \text{Know}\left(\neg\text{SF}_a(\text{now}), \text{end}[\cdot (a, 0)]\right)$$

Suppose not, i.e., there exists a model $M$ of $\text{Axioms} \cup \{\text{Sensed}[\sigma]\}$ such that for some $s'$ such that $M \models K(s', \text{end}[\sigma]), M \models \neg\phi(s')$.

Then take the structure $M'$ obtained from $M$ by intersecting the objects of sort situation with those that in the situation tree rooted in the initial ancestor situation of $s'$, say $s'_0$. $M'$ satisfies all axioms in $\text{Axioms}$ except the reflexivity axiom, the successor state axiom for $K$, and the initial state axiom, which is of the form $\text{Know}(\Psi(\text{now}), S_0)$ (note that the other axioms involve neither $K$ nor $S_0$). Observe that $\text{Trans}$ and $\text{Final}$ for the situation in the tree are defined by considering relations involving only situation in the same tree.

Now consider the $M''$ obtained from $M'$ by adding the constant $S_0$ and making it denote $s'_0$. Although $M'$ and $M''$ does not satisfy $\text{Know}(\Psi(\text{now}), S_0)$, we have that $M'' \models \Psi(S_0)$. Moreover, (*) and (**) and the fact that the successor state axiom for $K$ in $M$ ensure that all predecessor of $s'$ where $K$ alternatives, imply $M'' \models \text{Sensed}[\sigma]$.

Finally let us define $M'''$ by adding to $M''$ the predicate $K$ and making denote the identity relation on situations. Then $M''' \models \text{Axioms} \cup \{\text{Sensed}[\sigma]\}$. On the other hand since $M' \models \neg\phi(s')$ so does $M'''$. Thus getting a contradiction. $\square$

Theorem 2 Let $dp$ be such that $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{EFDP}(dp, \text{end}[\sigma])$.

Then, $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \exists s_f. \text{Do}(dp, \text{end}[\sigma], s_f)$ if and only if all online executions of $(dp, \sigma)$ are terminating.

\footnote{Note that $K$ cannot appear in the $\phi(s)$, however $\text{Trans}$ and $\text{Final}$ can, since they are predicates, although axiomatized using a second-order formula.}
**Proof Sketch:** First of all we observe that \(dp\) is a deterministic program and its possible online executions from \(\sigma\) are completely determined by the sensing outcomes. We also observe that in each model there will be a single execution of \(dp\), since the sensing outcomes are fully determined in the model. Moreover, in all models where with the same sensing outcomes up to a given configuration \((dp_i, s_i)\), the next transition of \(dp\) from \(end[\sigma_i]\) is the same.

\[\Rightarrow\] If \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dp, end[\sigma], s_f)\) then in every model of \(Axioms \cup \{Sensed[\sigma]\}\) the only execution of \(dp\) from \(end[\sigma]\) terminates. Consider an online execution reaching \((dp_i, s_i)\). Then, in all models of \(Axioms \cup \{Sensed[\sigma]\}\) with sensing outcomes as determined by \(\sigma_i\), the next configuration \((dp_{i+1}, s_{i+1})\) is the same, given that \(LEFDP(dp_i, end[\sigma_i])\) requires the next transition to be known in each of these models, and hence by reflexivity of \(K\) we have that such a transition is true as well in each of them. Then, for all a possible online transitions from \((dp_i, end[\sigma_i])\) to \((dp'_i, end[\sigma'_i])\) it must be the case that \(dp'_i = dp_{i+1}\) and \(end[\sigma'_i] = s_{i+1}\), i.e. the next online transitions can differ only wrt the new sensing outcome acquired.

\[\Leftarrow\] If an online execution of \(dp\) from \(\sigma\) terminates it means that the program \(dp\), from \(end[\sigma]\), terminates in all models of \(Axioms \cup \{Sensed[\sigma]\}\) with the sensing outcome as in the online execution. Since by hypothesis all online executions terminate, thus covering all possible sensing outcome, then \(dp\), from \(end[\sigma]\), terminates in all models. \(\blacksquare\)

**Theorem 3** If \(Axioms \cup \{Sensed[\sigma]\} \models Trans(\Sigma_e(p), end[\sigma], p', s')\), then

1. \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(p, end[\sigma], s_f)\)
2. \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(\Sigma_e(p), end[\sigma], s_f)\)
3. All online executions from \((\Sigma_e(p), \sigma)\) terminate.

**Proof Sketch:** (1) and (2) follow immediately from the definition of \(Trans\) for \(\Sigma_e\).

(3) By the definition of \(Trans\) for \(\Sigma_e\), there exists a \(dp\) and such that \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(dp, end[\sigma]) \land \exists s_f. Trans(dp, end[\sigma], p', s') \land Do(p', s', s_f)\). The conditions of Theorem 2 are satisfied, thus we have that all online executions from \((dp, \sigma)\) are terminating. Since these include all online executions from \((p', \sigma')\) with \(s' = end[\sigma']\), all online executions from \((p', \sigma')\) must also be terminating. Hence the thesis follows. \(\blacksquare\)

**Theorem 4** Let \(dpt\) be a tree program, i.e., \(dpt \in TREE\). Then, for all histories \(\sigma\),
if \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt, end[\sigma], s_f)\),
then \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpt, end[\sigma])\).
Proof Sketch: By induction on the structure of \(dpt\).

Base cases: for \(nil\), it is known that \(nil\) is Final, so \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(nil, end[\sigma])\) holds; for \(False'\), the antecedent is false, so the thesis holds.

Inductive cases: Assume that the thesis holds for \(dpt_1\) and \(dpt_2\). Assume that \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt, end[\sigma], s_f)\).

For \(dpt = a; dpt_1\): \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(a; dpt_1, end[\sigma], s_f)\) implies that \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt_1, do(a, end[\sigma]), s_f)\). Since \(a\) is a non-sensing action, \(Sensed[\sigma \cdot (a, 1)] = Sensed[\sigma]\), so we also have \(Axioms \cup Sensed[\sigma \cdot (a, 1)] \models \exists s_f. Do(dpt_1, end[\sigma \cdot (a, 1)], s_f)\). Thus, by the induction hypothesis we have \(Axioms \cup \{Sensed[\sigma \cdot (a, 1)]\} \models EFDP(dpt_1, end[\sigma \cdot (a, 1)])\). It follows that \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpt_1, do(a, end[\sigma]))\). The assumption \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(a; dpt_1, end[\sigma], s_f)\) also implies that \(Axioms \cup \{Sensed[\sigma]\} \models Poss(a, end[\sigma])\) and this must be known by Theorem 1, i.e., \(Axioms \cup \{Sensed[\sigma]\} \models \text{Know}(Poss(a, now), end[\sigma])\). Thus, we have that

\[
Axioms \cup \{Sensed[\sigma]\} \models \text{Know}(Trans(a; dpt_1, now, dpt_1, do(a, now)), end[\sigma]).
\]

It is also known that this is the only transition possible for \(a; dpt_1\), so \(Axioms \cup \{Sensed[\sigma]\} \models LEFDP(a; dpt_1, end[\sigma]). Therefore, \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(a; dpt_1, end[\sigma])\).

For \(dpt = True'; dpt_1\): the argument is similar, but simpler since the test does not change the situation.

For \(dpt = sense_\phi; \text{if } \phi \text{ then } dpt_1 \text{ else } dpt_2\): Suppose that the sensing action returns \(1\) and let \(\sigma_1 = \sigma \cdot (sense_\phi, 1)\). Next we show that \(Axioms \cup \{Sensed[\sigma]\} \models LEFDP(dpt, end[\sigma])\). The assumption that \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt, end[\sigma], s_f)\) implies that \(Axioms \cup \{Sensed[\sigma_1]\} \models \exists s_f. Do(dpt_1, end[\sigma_1], s_f)\). Thus, by the induction hypothesis we have \(Axioms \cup \{Sensed[\sigma_1]\} \models EFDP(dpt_1, end[\sigma_1])\). It follows that \(Axioms \cup \{Sensed[\sigma]\} \models \phi(do(sense_\phi, end[\sigma]) \supset EFDP(dpt_1, do(sense_\phi, end[\sigma]))\). By a similar argument, it also follows that we must have that \(Axioms \cup \{Sensed[\sigma]\} \models \neg \phi(do(sense_\phi, end[\sigma]) \supset EFDP(dpt_2, do(sense_\phi, end[\sigma]))\). The assumption \(Axioms \cup \{Sensed[\sigma]\} \models \exists s_f. Do(dpt, end[\sigma], s_f)\) also implies that \(Axioms \cup \{Sensed[\sigma]\} \models Poss(sense_\phi, end[\sigma])\) and this must be known by Theorem 1, i.e., \(Axioms \cup \{Sensed[\sigma]\} \models \text{Know}(Poss(sense_\phi, now), end[\sigma])\). Thus, we have that

\[
Axioms \cup \{Sensed[\sigma]\} \models \text{Know}(Trans(dpt, now, if \phi \text{ then } dpt_1 \text{ else } dpt_2, do(sense_\phi, now)), end[\sigma]).
\]

It is also known that this is the only transition possible for \(dpt\), so \(Axioms \cup \{Sensed[\sigma]\} \models LEFDP(dpt, end[\sigma])\). Thus, \(Axioms \cup \{Sensed[\sigma]\} \models EFDP(dpt, end[\sigma])\).
EFDP(dpt, end[σ]). ■

**Theorem 5** For any program dp that is

1. an epistemically feasible deterministic program, i.e.,
   Axioms ∪ {Sensed[σ]} ⊨ EFDP(dp, end[σ]) and
2. such that there is a known bound on the number of steps it needs to terminate, i.e., where there is an n such that Axioms ∪ {Sensed[σ]} ⊨ \( \exists p', s', k.k \leq n \land \text{Trans}^k(dp, end[σ], p', s') \land \text{Final}(p', s') \),

there exists a tree program dpt ∈ TREE such that Axioms ∪ {Sensed[σ]} ⊨ \( \forall s_f. \text{Do}(dp, end[σ], s_f) \equiv \text{Do}(dpt, end[σ], s_f) \).

**Proof Sketch:** We construct the tree program dpt = m(dp, σ) from dp using the following rules:

- \( m(dp, σ) = \text{False} \) iff Axioms ∪ {Sensed[σ]} is inconsistent, otherwise
- \( m(dp, σ) = \text{nil} \) iff
  Axioms ∪ {Sensed[σ]} ⊨ Final(dp, end[σ]), otherwise
- \( m(dp, σ) = a; m(dp', σ \cdot (a, 1)) \) iff
  Axioms ∪ {Sensed[σ]} ⊨ Trans(dp, end[σ], dp', do(a, end[σ])) for some non-sensing action a,
- \( m(dp, σ) = \text{sense}_ϕ; \text{if } φ \text{ then } m(dp_1, σ \cdot (\text{sense}_ϕ, 1)) \text{ else } m(dp_2, σ \cdot (\text{sense}_ϕ, 0)) \) iff
  Axioms ∪ {Sensed[σ]} ⊨ Trans(dp, end[σ], dp', do(\text{sense}_ϕ, end[σ])) for some sensing action \text{sense}_ϕ,
- \( m(dp, σ) = \text{True} \); \( m(dp', σ) \) iff
  Axioms ∪ {Sensed[σ]} ⊨ Trans(dp, end[σ], dp', end[σ]).

Let us show that
Axioms ∪ {Sensed[σ]} ⊨ Do(dp, end[σ], s_f) \equiv Do(m(dp, σ), end[σ], s_f).

It turns out that, under the hypothesis of the theorem, for all dp and all σ, 
\((dp, σ)\) is bisimilar to \((m(dp, σ), σ)\) with respect to online executions. Indeed, it is easy to check that the relation \([(dp, σ), (m(dp, σ), σ)]\) is a bisimulation, i.e., for all dp and σ, \([(dp, σ), (m(dp, σ), σ)]\) implies that

- Axioms ∪ {Sensed[σ]} ⊨ Final(dp, end[σ]) iff Axioms ∪ {Sensed[σ]} ⊨ Final(m(dp, σ), end[σ]),
- for all dp', σ' if Axioms ∪ {Sensed[σ]} ⊨ Trans(dp, end[σ], dp', end[σ']) with Axioms ∪ {Sensed[σ']} consistent, then Axioms ∪ {Sensed[σ']} ⊨ Trans(m(dp, σ), end[σ], m(dp', σ'), end[σ']) and \([(dp', σ'), (m(dp', σ'), σ')]\),
for all $dp'$, $\sigma'$ if $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{Trans}(m(dp, \sigma), \text{end}[\sigma], m(dp', \sigma'), \text{end}[\sigma'])$ with $\text{Axioms} \cup \{\text{Sensed}[\sigma']\}$ consistent, then $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{Trans}(dp, \text{end}[\sigma], dp', \text{end}[\sigma'])$ and $[(dp', \sigma'), (m(dp', \sigma'), \sigma')]$.

Now, assume that $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \exists s_f.\text{Do}(dp, \text{end}[\sigma], s_f)$, then since $dp$ is an EFDP, by Theorem 2 all online execution from $(dp, \sigma)$ terminate. Hence since $(dp, \sigma$ and $(m(dp, \sigma), \sigma)$ are bisimilar, $(m(dp, \sigma), \sigma)$ has the same online execution (apart from the program appearing in the configurations).

Next, observe that given an online execution of $(dp, \sigma)$ terminating in $(dp_f, \sigma_f)$, in all models of $\text{Axioms} \cup \{\text{Sensed}[\sigma]\}$ with sensing outcomes as in $\sigma_f$ both the program $dp$ and $m(dp, \sigma)$ reach the same situation $\text{end}[\sigma_f]$. Since there are terminating online executions for all possible sensing outcomes, the thesis follows.

**Theorem 6** Let $dpl$ be a linear program, i.e., $dpl \in \text{LINE}$. Then, for all histories $\sigma$, if $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \exists s_f.\text{Do}(dpl, \text{end}[\sigma], s_f)$, then $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{EFDP}(dpl, \text{end}[\sigma])$.

*Proof Sketch:* This is a corollary of Theorem 4 for tree programs. Since linear programs are tree programs, the thesis follows immediately from this theorem.

**Theorem 7** For any $dp$ that does not include sensing actions, such that $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{EFDP}(dp, \text{end}[\sigma])$, there exists a linear program $dpl$ such that $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \forall s_f.\text{Do}(dp, \text{end}[\sigma], s_f) \equiv \text{Do}(dpl, \text{end}[\sigma], s_f)$.

*Proof Sketch:* We show this using the same approach as for Theorem 5 for tree programs. Since $dp$ cannot contain sensing actions, the construction method used in the proof of Theorem 5 produces a tree program that contains no branching and is in fact a linear program. Then, by the same argument as used there, the thesis follows.

**Theorem 8** $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{Trans}(\Sigma(p), \text{end}[\sigma], dpl, s')$ if and only if there exists a situation $s_f$ such that $\text{Axioms} \cup \{\text{Sensed}[\sigma]\} \models \text{Do}(p, \text{end}[\sigma], s_f)$. 

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Proof Sketch: \( \Leftarrow \) If for same \( s_f \) we have \( Axioms \cup \{ Sensed[\sigma] \} \models Do(p, end[\sigma], s_f) \) then the sequence of actions from \( end[\sigma] \) to \( s_f \) is an \( LINE \) program, which trivially satisfies the left-hand-side of the axiom for \( \Sigma_4 \). Observe that if \( s' = end[\sigma] \) then the linear program can be simply \( True ? \).

\( \Rightarrow \) By hypothesis there exists a \( dpl \) that is a \( LINE \). If \( s' = s \) and then \( dpl = true ? ; dpl' \) and if \( s' = do(a, s) \), for same action \( a \), and then \( dpl = a ; dpl' \). In both cases \( dpl' \) must be an \( LINE \). In every model \( dpl' \) reaches from \( s' \) a final situation of the original program \( p \). Observe that such situation will be the same in every model since the sequence of actions \( a \) starting from \( s' \) is fixed by \( dpl' \). It follows that the sequence of action done by \( dpl \) starting from \( s \) reaches a situation \( s_f \) such that \( Axioms \cup \{ Sensed[\sigma] \} \models Do(p, end[\sigma], s_f) \). ■