Robotics I

June 17, 2019

Exercise 1

Consider the Kawasaki robot S030 shown in Fig. 1, having six revolute joints. The geometric dimensions of the robot workspace are reported in the distributed **data** sheet, together with the joint ranges defined according to the manufacturer's convention and the maximum joint speeds.

Figure 1: The 6R Kawasaki S030 robot.

a. Assign the link frames according to a Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all twist angles α_i, for $i = 1, \ldots, 6$, are either 0 or $+\pi/2$. Draw the frames and fill in the table on the **reply** sheet provided separately. As shown there, frame 0 is on the floor, with $\boldsymbol{z}_0$ pointing upward, while the sixth frame is at the center of the end-effector flange, with the $\boldsymbol{z}_6$ axis in the approach direction. Specify in the table the numerical value of all constant parameters.

b. Determine the actual joint ranges (lower and upper bounds, $\theta_{i,\min}$ and $\theta_{i,\max}$ for $i = 1, \ldots, 6$) according to the DH convention you have defined. Specify also the numerical value $\boldsymbol{\theta}_n \in \mathbb{R}^6$ of the joint variables $\boldsymbol{\theta}$ when the robot is in the configuration shown in the **data** sheet.

c. Compute the symbolic expression of the position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{\theta})$ of point P (center of the spherical wrist) of the robot. Provide its numerical value of $\boldsymbol{p}_n$ when the robot is in the configuration $\boldsymbol{\theta}_n$.

d. For the same value $\boldsymbol{p} = \boldsymbol{p}_n$, determine *all* possible inverse kinematic solutions for the first three joints $\boldsymbol{\theta}_b = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}^T$ of the robot that are feasible with respect to the available joint ranges.

e. Determine the value $\boldsymbol{v}_P \in \mathbb{R}^3$ of the velocity of point P, expressed in the robot base frame, when $\boldsymbol{\theta} = \boldsymbol{\theta}_n$ and the joints have their maximum *positive* speed.

f. Which are the singularities of the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta}_b)$ relating the velocity $\dot{\boldsymbol{\theta}}_b \in \mathbb{R}^3$ of the first three joints ot the linear velocity $\boldsymbol{v}_P \in \mathbb{R}^3$ of point P?

Exercise 2

The desired linear velocity of the end-effector $\boldsymbol{v} \in \mathbb{R}^m$ (with $m = 2$ or 3, in the 2D- or 3D-case) of a robot with n joints is usually defined, at the current configuration, in one of three possible ways: at the joint level, in the base frame B, or in the end-effector/tool frame E. Discuss the pros and cons of these choices and how they relate each to other. Comment also on what may happen when $n < m$, $n = m$, or $n > m$.

With reference to a planar $2R$ robot, with link lengths $l_1 = 0.5$, $l_2 = 0.25$ [m] and in the configuration $\theta_1 = \pi/4$, $\theta_2 = -\pi/2$, provide numerical answers to the following questions:

a. For $\dot{\boldsymbol{\theta}} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ [rad/s], compute ${}^B\boldsymbol{v}$ and ${}^E\boldsymbol{v}$ (both vectors are in $\mathbb{R}^2$);

b. Compute $\dot{\boldsymbol{\theta}} \in \mathbb{R}^2$ when ${}^B\boldsymbol{v}$ or, respectively, ${}^E\boldsymbol{v}$ take the value $\boldsymbol{v} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [m/s].

[180 minutes, open books]

Solution

June 17, 2019

Exercise 1

A possible DH frame assignment for the Kawasaki S030 robot, which satisfies the required conditions on the twist angles α_i, $i = 1, \ldots, 6$, is shown in Fig. 2. The associated parameters are given in Tab. 1, where the numerical values read from the workspace dimensions on the robot data sheet are also reported. The numerical values of the variables q_i refer to the robot configuration shown in Fig. 2. The table gives also the joint ranges obtained from the robot manufacturer's data, once working with the chosen DH convention.

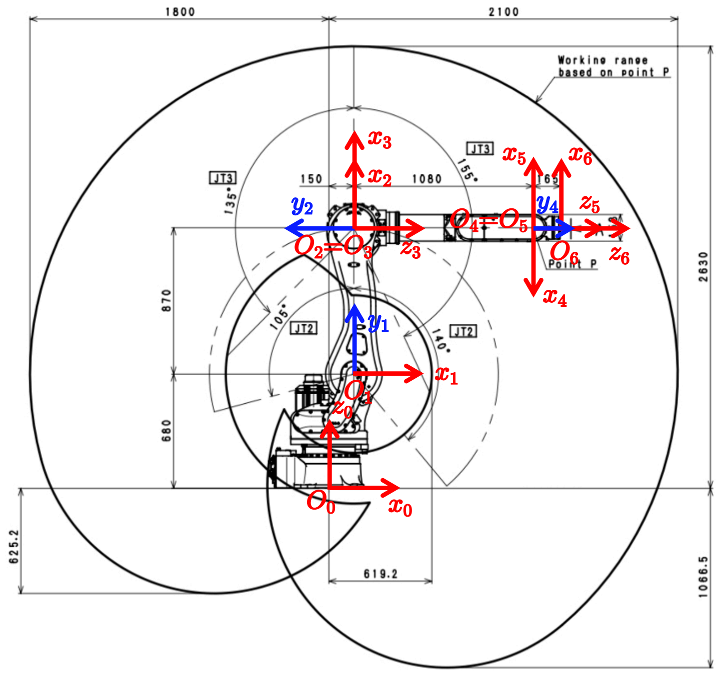

Figure 2: A possible DH frame assignment for the Kawasaki S030 robot.

i	α_i	a_i	d_i	θ_i	$\theta_{i,\min}$	$\theta_{i,\max}$
1	$\pi/2$	$a_1 = 150$	$d_1 = 680$	$q_1 = 0$	$-\pi$	$+\pi$
2	0	$a_2 = 870$	0	$q_2 = \pi/2$	$-50 \cdot (\pi/180)$	$+195 \cdot (\pi/180)$
3	$\pi/2$	0	0	$q_3 = 0$	$-65 \cdot (\pi/180)$	$+225 \cdot (\pi/180)$
4	$\pi/2$	0	$d_4 = 1080$	$q_4 = \pi$	-2π	$+2\pi$
5	$\pi/2$	0	0	$q_5 = \pi$	$+35 \cdot (\pi/180)$	$+325 \cdot (\pi/180)$
6	0	0	$d_4 = 165$	$q_6 = 0$	-2π	$+2\pi$

Table 1: Parameters associated to the DH frames in Fig. 2. Lengths are in [mm], angles in [rad].

Based on Tab. 1, in order to determine the symbolic expression of the position of the center of the spherical wrist (point P), we just need to compute the first four DH homogeneous transformation matrices:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & a_1 \cos q_1 \\ \sin q_1 & 0 & -\cos q_1 & a_1 \sin q_1 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & 0 \\ \sin q_3 & 0 & -\cos q_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

To obtain the position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ (or $= \boldsymbol{f}(\boldsymbol{\theta})$), we make use of the matrix-vector product computations in homogeneous coordinates as

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] \\
&= \begin{pmatrix} \cos q_1 \left(a_1 + a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right) \\ \sin q_1 \left(a_1 + a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right) \\ d_1 + a_2 \sin q_2 - d_4 \cos(q_2 + q_3) \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}.
\end{aligned} \tag{1}
$$

Plugging in (1) the numerical values from Tab. 1 for $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$, i.e., for the first three components of $\boldsymbol{q}_n = \boldsymbol{\theta}_n$, as well as for a_1, a_2, d_1 and d_4, we obtain

$$
\boldsymbol{p}_n = \boldsymbol{f}(\boldsymbol{q}_n) = \begin{pmatrix} 1.23 \\ 0 \\ 1.55 \end{pmatrix} \text{ [m]}. \tag{2}
$$

For a given desired position $\boldsymbol{p} = \boldsymbol{p}_n$ of point P, the inverse kinematics problem requires solving the nonlinear equations of the direct mapping (1) in terms of the unknown $\boldsymbol{q}_b = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$. This is done by inspection.

First, since $p_y/p_x = \tan q_1$, two solutions are found in the four quadrants for the base joint q_1 by choosing

$$
q_{1,[f]} = \text{ATAN2}\{p_y, p_x\} \qquad \text{and} \qquad q_{1,[b]} = \text{ATAN2}\{-p_y, -p_x\}, \tag{3}
$$

corresponding to the robot facing the desired Cartesian position for P with its *front* or with its *back* side.

Next, we sum the first two scalar equations in (1) multiplied by $\sin q_1$ and $\cos q_1$ respectively, reorganize terms and square, and then add this to the third equation, also reorganized and squared, obtaining

$$
\begin{aligned}
\left(p_x \cos q_1 + p_y \sin q_1 - a_1\right)^2 + \left(p_z - d_1\right)^2 &= \left(a_2 \cos q_2 + d_4 \sin(q_2 + q_3)\right)^2 + \left(a_2 \sin q_2 - d_4 \cos(q_2 + q_3)\right)^2 \\
&= a_2^2 + d_4^2 + 2a_2 d_4 \left(\sin(q_2 + q_3) \cos q_2 - \cos(q_2 + q_3) \sin q_2\right) \\
&= a_2^2 + d_4^2 + 2a_2 d_4 \sin q_3.
\end{aligned}
$$

From this, for each value substituted from (3), we compute the quantities

$$s_{3,[f,b]} = \frac{(p_x \cos q_{1,[f,b]} + p_y \sin q_{1,[f,b]} - a_1)^2 + (p_z - d_1)^2 - a_2^2 + d_4^2}{2a_2 d_4}$$

and

$$c_{3,[f,b]} = \pm\sqrt{1 - s_{3,[f,b]}^2}\,.$$

Combining these two expressions, once with the sign + and the other with the sign − before the square root (and, in each case, using the two values labeled f or b in the evaluation of q_1), four different solutions are found for the elbow joint q_3, i.e.,

$$q_{3,[f,b;+]} = \text{ATAN2}\left\{s_{3,[f,b]}, +\left|c_{3,[f,b]}\right|\right\} \qquad \text{and} \qquad q_{3,[f,b;-]} = \text{ATAN2}\left\{s_{3,[f,b]}, -\left|c_{3,[f,b]}\right|\right\}. \tag{4}$$

Finally, we consider again the sum of the first two scalar equations in (1) multiplied by $\sin q_1$ and $\cos q_1$, respectively, and the third one rearranged, and expand then $\sin(q_2 + q_3)$ and $\cos(q_2 + q_3)$. We isolate in this way the yet unknown trigonometric expressions $\sin q_2$ and $\cos q_2$ in the two resulting (linear) equations

$$\begin{aligned} a_2 \cos q_2 + d_4(\sin q_2 \cos q_3 + \cos q_2 \sin q_3) &= p_x \cos q_1 + p_y \sin q_1 - a_1 \\ a_2 \sin q_2 - d_4(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) &= p_z - d_1, \end{aligned}$$

or, in matrix form

$$\begin{pmatrix} a_2 + d_4 \sin q_3 & d_4 \cos q_3 \\ -d_4 \cos q_3 & a_2 + d_4 \sin q_3 \end{pmatrix} \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} = \begin{pmatrix} p_x \cos q_1 + p_y \sin q_1 - a_1 \\ p_z - d_1 \end{pmatrix}. \tag{5}$$

Unless $\det \boldsymbol{A} = a_2^2 + d_4^2 + 2a_2 d_4 \sin q_3 = 0$, which happens if and only if $\sin q_3 = -1$ and $a_2 = d_4$ (thus, never in our case), there is a unique solution to (5) given by

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos q_2 \\ \sin q_2 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b}. \tag{6}$$

When evaluating the linear system (5) using the previously obtained results for q_1 and q_3, from (6) we obtain also the associated set of four solutions for the elbow joint variable

$$q_{2,[f,b;+,-]} = \text{ATAN2}\left\{x_{2,[f,b;+,-]}, x_{1,[f,b;+,-]}\right\}. \tag{7}$$

When using the numerical value of $\boldsymbol{p}_n$ in (2), application of the above closed-form formulas (3), (4), and (7) for the inverse kinematics problem yields the four joint configurations

$$\boldsymbol{q}_{f,+} = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 0 \\ 90^\circ \\ 0 \end{pmatrix}, \qquad \boldsymbol{q}_{f,-} = \begin{pmatrix} 0 \\ -0.2146 \\ \pi \end{pmatrix} [\text{rad}] = \begin{pmatrix} 0 \\ -12.29^\circ \\ 180^\circ \end{pmatrix},$$

$$\boldsymbol{q}_{b,+} = \begin{pmatrix} \pi \\ -3.0495 \\ 0.4036 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 180^\circ \\ -174.72^\circ \\ 23.12^\circ \end{pmatrix}, \qquad \boldsymbol{q}_{b,-} = \begin{pmatrix} \pi \\ 1.9245 \\ 2.7380 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 180^\circ \\ 110.27^\circ \\ 156.88^\circ \end{pmatrix}.$$

The first solution $\boldsymbol{q}_{f,+}$ is the one shown in Fig. 2, which was used for computing $\boldsymbol{p}_n$. It is indeed a feasible solution with respect to the available joint ranges of the robot. Two out of the three other inverse kinematic solutions, namely $\boldsymbol{q}_{f,-}$ and $\boldsymbol{q}_{b,-}$, are also feasible. On the other hand, $\boldsymbol{q}_{b,+}$ is unfeasible because it is below the lower limit for joint 2.

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}_b)$ (or $\boldsymbol{J}(\boldsymbol{\theta}_b)$) of interest in the forward differential mapping

$$\boldsymbol{v}_P = \boldsymbol{J}(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_b, \qquad \boldsymbol{q}_b = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$$

is obtained from eq. (1) as

$$\boldsymbol{J}(\boldsymbol{q}_b) = \frac{\partial \boldsymbol{f}(\boldsymbol{q}_b)}{\partial \boldsymbol{q}_b} =$$

$$\begin{pmatrix} -\sin q_1\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & \cos q_1\,(d_4\cos(q_2+q_3) - a_2\sin q_2) & d_4\cos(q_2+q_3)\cos q_1 \\ \cos q_1\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & \sin q_1\,(d_4\cos(q_2+q_3) - a_2\sin q_2) & d_4\cos(q_2+q_3)\sin q_1 \\ 0 & a_2\cos q_2 + d_4\sin(q_2+q_3) & d_4\sin(q_2+q_3) \end{pmatrix}. \quad (8)$$

This matrix can be equivalently expressed in the rotated reference frame 1 as

$$\begin{aligned} {}^1\boldsymbol{J}(\boldsymbol{q}_b) &= {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}_b) = \\ & \begin{pmatrix} 0 & d_4\cos(q_2+q_3) - a_2\sin q_2 & d_4\cos(q_2+q_3) \\ 0 & d_4\sin(q_2+q_3) + a_2\cos q_2 & d_4\sin(q_2+q_3) \\ -(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)) & 0 & 0 \end{pmatrix}, \end{aligned} \quad (9)$$

which is simpler for the investigation of its singularities. In fact, the determinant factorizes as

$$\det \boldsymbol{J}(\boldsymbol{q}_b) = \det {}^1\boldsymbol{J}(\boldsymbol{q}_b) = a_2 d_4 \cos q_3\,(a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3)),$$

and the singularities are as follows:

$\cos q_3 = 0 \iff q_3 = \pm\frac{\pi}{2} \iff$ arm stretched (+) or folded (−, not in feasible range of joint 3!);

$a_1 + a_2\cos q_2 + d_4\sin(q_2+q_3) = 0 \iff p_x = p_y = 0 \iff$ point P is on the axis of joint 1.

When the robot is in the configuration $\boldsymbol{q}_n$, the Jacobian (8) becomes

$$\boldsymbol{J}_n = \boldsymbol{J}(\boldsymbol{q}_n) = \begin{pmatrix} 0 & -0.87 & 0 \\ 1.23 & 0 & 0 \\ 0 & 1.08 & 1.08 \end{pmatrix}.$$

When applying the maximum positive (according to the counterclockwise convention) speed at the first three joints, we obtain

$$\boldsymbol{v}_P = \boldsymbol{J}_n \cdot \begin{pmatrix} 3.1416 \\ 3.1416 \\ 3.2289 \end{pmatrix} \text{[rad/s]} = \begin{pmatrix} -2.7332 \\ 3.8642 \\ 6.8801 \end{pmatrix} \text{[m/s]}.$$

Exercise 2

The linear velocity $\boldsymbol{v} \in \mathbb{R}^m$ (with $m = 2$ in 2D or $m = 3$ in 3D) of the end-effector of a robot with n joints is uniquely specified, at a given configuration $\boldsymbol{q}$, by the joint velocity vector $\dot{\boldsymbol{q}}$, no matter if n is larger, equal, or smaller than m. Indeed, given a desired $\boldsymbol{v}$, there exists no solution for $\dot{\boldsymbol{q}}$ if $\boldsymbol{v} \notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q})\}$, being $\boldsymbol{J}$ the $m \times n$ (analytic = geometric) robot Jacobian related to the linear motion of the end-effector. This Jacobian is usually (i.e., by default) expressed in the base frame. If $\boldsymbol{v} \in \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q})\}$, the solution if unique for $n \leq m$, or there is an infinity of joint velocity solutions when $n > m$. The (pseudo-)inversion of the Jacobian matrix may run into trouble around or at a singular configuration. Moreover, the end-effector velocity expressions in the base frame B and in the end-effector/tool frame E are related by

$${}^E\boldsymbol{v} = {}^E\boldsymbol{R}_B\,{}^B\boldsymbol{v} = {}^E\boldsymbol{R}_B\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = {}^E\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with matrix ${}^E\boldsymbol{R}_B \in SO(m)$ and where the robot Jacobian ${}^E\boldsymbol{J}$ is expressed now in the end-effector frame. Representing the vector $\boldsymbol{v}$ in the local end-effector frame E is more useful for visualizing the instantaneous direction of the commanded motion, typically in response to a sensory input at the end-effector level. Other than for this, the two representations are fully equivalent.

For a planar $2R$ robot with link lengths $l_1 = 0.5$, $l_2 = 0.25$ [m], the Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -0.5\sin\theta_1 - 0.25\sin(\theta_1+\theta_2) & -0.25\sin(\theta_1+\theta_2) \\ 0.5\cos\theta_1 + 0.25\cos(\theta_1+\theta_2) & 0.25\cos(\theta_1+\theta_2) \end{pmatrix}.$$

In the configuration $\theta_1 = \pi/4$, $\theta_2 = -\pi/2$, the Jacobian becomes

$$\boldsymbol{J} = \begin{pmatrix} -\sqrt{2}/8 & \sqrt{2}/8 \\ 3\sqrt{2}/8 & \sqrt{2}/8 \end{pmatrix} = \begin{pmatrix} -0.1768 & 0.1768 \\ 0.5303 & 0.1768 \end{pmatrix},$$

which is clearly nonsingular ($\det \boldsymbol{J} = -1/8$). The (planar) rotation matrix from the base to the end-effector frame, once evaluated at the desired configuration, is

$$^{B}\boldsymbol{R}_E = \left.\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}\right|_{\substack{\theta_1=\pi/4,\\ \theta_2=-\pi/2}} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{pmatrix}.$$

Therefore, we have the following numerical results.

a. For $\dot{\boldsymbol{\theta}} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ [rad/s], the end-effector velocities (in [m/s]) are

$$^{B}\boldsymbol{v} = \boldsymbol{J}\dot{\boldsymbol{\theta}} = \begin{pmatrix} -\sqrt{2}/4 \\ -\sqrt{2}/4 \end{pmatrix} = \begin{pmatrix} -0.3536 \\ -0.3536 \end{pmatrix}, \quad {}^{E}\boldsymbol{v} = {}^{B}\boldsymbol{R}_E^T\,\boldsymbol{J}\dot{\boldsymbol{\theta}} = {}^{E}\boldsymbol{J}\,\dot{\boldsymbol{\theta}} = \begin{pmatrix} -0.5 & 0 \\ 0.25 & 0.25 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}.$$

b. For the inverse differential problem, the two requested joint velocities (in [rad/s]) are

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\,{}^{B}\boldsymbol{v} = \begin{pmatrix} -1.4142 & 1.4142 \\ 4.2426 & 1.4142 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}, \quad \dot{\boldsymbol{\theta}} = {}^{E}\boldsymbol{J}^{-1}\,{}^{E}\boldsymbol{v} = \begin{pmatrix} -2 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

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