Robotics 1

Remote Exam – July 15, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, made by a 3R planar arm mounted on a rail. The world coordinate frame $(\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$ is also shown. Assign the Denavit-Hartenberg (D-H) frames to the robot and provide the associated table of parameters. Place the last D-H frame on the robot gripper with its origin in P. Draw the frames on the robot, together with the joint variables and the non-zero constant parameters. In the configuration shown, specify the signs assumed by the four joint variables (angles are defined as always in the interval $(-\pi, +\pi]$). Finally, find the homogeneous transformation between the world frame and the first D-H frame.

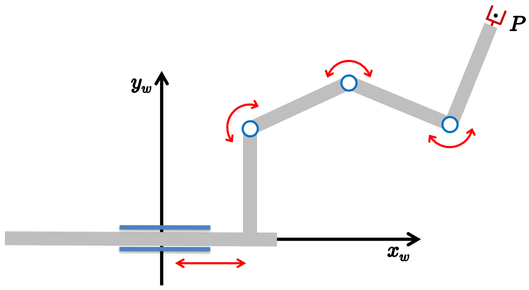

Figure 1: A 4-dof planar robot with the world coordinate frame $(\boldsymbol{x}_w, \boldsymbol{y}_w, \boldsymbol{z}_w)$.

Exercise #2

The 2-dof Cartesian robot in Fig. 2 should execute with its end-effector the following desired eight-shaped periodic trajectory

$$\boldsymbol{p}_d(t) = \begin{pmatrix} c + a\sin 2\omega t \\ c + b\sin \omega t \end{pmatrix}, \qquad \text{with } a, b, c, \omega > 0, \text{ for } t \in \left[0, \frac{2\pi}{\omega}\right]. \tag{1}$$

The robot joint velocities and accelerations are bounded as

$$|\dot{q}_i| \leq V_i > 0, \qquad |\ddot{q}_i| \leq A_i > 0, \qquad i = 1, 2,$$

while the velocity along the Cartesian path is bounded in norm as $\|\dot{\boldsymbol{p}}_d(t)\| \leq V_{c,max} > 0$. The robot is commanded by joint accelerations.

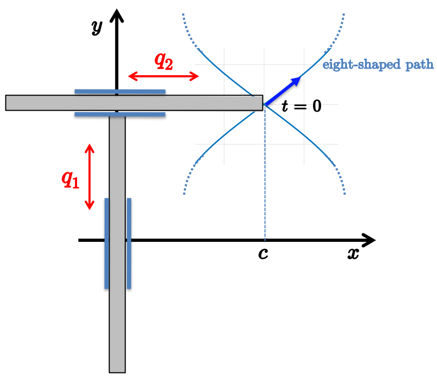

Figure 2: A 2P robot with the end-effector in the initial point of the desired trajectory at $t = 0$.

Give the symbolic expressions of the needed robot joint commands, and determine the maximum value ω_{max} of the angular frequency ω in (1) so that the robot motion satisfies all the constraints. Provide then the numerical value of ω_{max} using the following data: $a = 1$ [m], $b = 1.5$ [m], $c = 3$ [m], $V_1 = V_2 = 2$ [m/s], $V_{c,max} = 1.8$ [m/s], $A_1 = 2$ [m/s^2], $A_2 = 1.5$ [m/s^2].

Exercise #3

For a minimal representation of the orientation of a rigid body given by the YXY sequence of Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$, define the instantaneous mapping between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity $\boldsymbol{\omega}$ of the body. Determine also all the singularities of this mapping.

Exercise #4

With reference to Fig. 3, a 3R planar robot with equal link lengths $\ell = 2$ [m] executes a linear Cartesian path from point $\boldsymbol{A} = (3, 2.5)$ [m] (at $t = 0$) to point $\boldsymbol{B} = (0.75, 1.8)$ [m] with constant speed $v = 0.5$ [m/s], while keeping its end-effector always orthogonal to the path. Provide the value of the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ realizing the task at $t = 1$ [s]. Sketch graphically the situation.

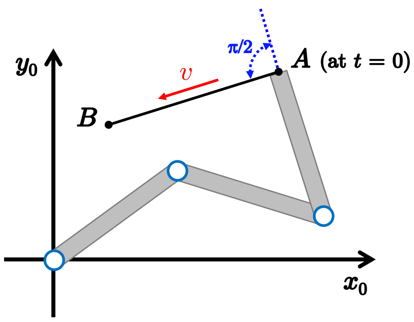

Figure 3: The 3R planar robot and its configuration at the initial point of the desired path.

Exercise #5

This is in the form of a Questionnaire. Please answer with formulas and/or clear and short texts.

A) List all possible Euler sequences of angles around moving axes that can be used to represent the orientation of a rigid body, and associate to each the correct equivalent sequence of Roll-Pitch-Yaw angles around fixed axes.

B) A DC motor has rotor inertia $J_m = 1.2 \cdot 10^{-5}$ [kg$\cdot$m^2] and maximum speed $\dot{\theta}_{max} = 2060$ rpm. It is connected to the driven link through a rigid transmission with reduction ratio $n_r = 100$. Is the link angular velocity $\dot{q} = 3.5$ [rad/s] a feasible one? In the absence of dissipative effects, if the actual value of the reduction ratio is the one that minimizes the required motor torque for a given link angular acceleration $\ddot{q}$, which is then the value of the link inertia J_l? With this numerical value, if the desired link acceleration is $\ddot{q} = 4$ [rad/s^2], compute the torque τ_m that the motor needs to produce on its axis.

[210 minutes (3.5 hours); open books]

Solution

July 15, 2020

Exercise #1

A Denavit-Hartenberg frame assignment is shown in Fig. 4 and the associated parameters are reported in Tab. 1, together with the signs of the constant non-zero parameters (a_i) and the signs of the variables q_i, for $i = 1, \ldots, 4$, when the robot is in the configuration shown in the figure. The transformation between the world frame and the D-H frame 0 is

$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

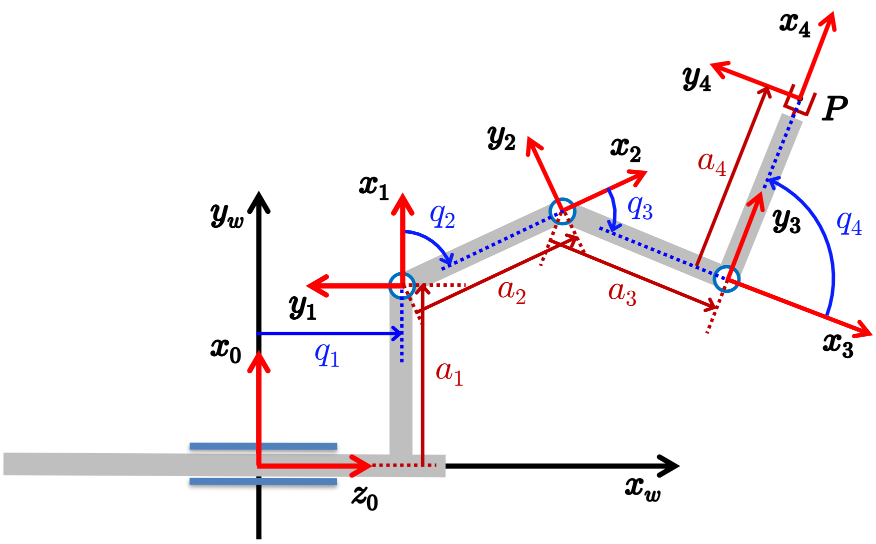

Figure 4: An assignment of D-H frames for the 4-dof robot.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	$a_1 > 0$	$q_1 > 0$	0
2	0	$a_2 > 0$	0	$q_2 < 0$
3	0	$a_3 > 0$	0	$q_3 < 0$
4	0	$a_4 > 0$	0	$q_4 > 0$

Table 1: The D-H table of parameters for the frame assignment in Fig. 4.

Exercise #2

The eight-shaped Cartesian path is plotted in Fig. 5 using the given parameters $a = 1$, $b = 1.5$, and $c = 3$ [m]. This shape is indeed independent from the time/speed at which the path is being traced by the robot end-effector.

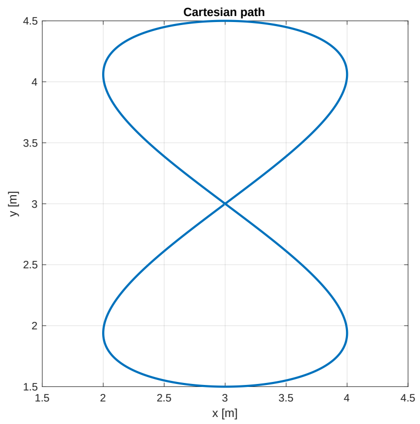

Figure 5: Eight-shaped path traced by the robot end-effector.

The desired Cartesian velocity and acceleration are computed by time derivation of eq. (1). Since the 2P robot has the first joint moving along the y-component and the second joint along the x-component, we have

$$\dot{\boldsymbol{p}}_d(t) = \begin{pmatrix} 2a\omega\cos 2\omega t \\ b\omega\cos\omega t \end{pmatrix} = \begin{pmatrix} \dot{q}_2(t) \\ \dot{q}_1(t) \end{pmatrix}, \tag{2}$$

and

$$\ddot{\boldsymbol{p}}_d(t) = \begin{pmatrix} -4a\omega^2\sin 2\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \begin{pmatrix} \ddot{q}_2(t) \\ \ddot{q}_1(t) \end{pmatrix}, \tag{3}$$

which are also the expressions of the robot joint commands. Moreover, the norm of (2) is

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{4a^2\omega^2\cos^2 2\omega t + b^2\omega^2\cos^2\omega t}. \tag{4}$$

The bounds to be satisfied for all $t \in [0, 2\pi/\omega]$ are then

$$|\dot{q}_1| = |b\omega\cos\omega t| \le V_1 \quad\Rightarrow\quad \omega \le \frac{V_1}{b}, \qquad |\dot{q}_2| = |2a\omega\cos 2\omega t| \le V_2 \quad\Rightarrow\quad \omega \le \frac{V_2}{2a},$$

$$|\ddot{q}_1| = |-b\omega^2\sin\omega t| \le A_1 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_1}{b}}, \qquad |\ddot{q}_2| = |-4a\omega^2\sin 2\omega t| \le A_2 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_2}{4a}},$$

and

$$\|\dot{\boldsymbol{p}}_d(t)\| = \omega\sqrt{4a^2\cos^2 2\omega t + b^2\cos^2\omega t} \le V_{c,max} \quad\Rightarrow\quad \omega \le \frac{V_{c,max}}{\sqrt{4a^2+b^2}}.$$

Therefore, the maximum feasible value of ω is

$$\omega_{max} = \min\left(\frac{V_1}{b}, \frac{V_2}{2a}, \sqrt{\frac{A_1}{b}}, \sqrt{\frac{A_2}{4a}}, \frac{V_{c,max}}{\sqrt{4a^2+b^2}}\right). \tag{5}$$

Substituting in (5) the numerical data, we obtain $\omega_{max} = \sqrt{A_2/(4a)} = 0.6124$, corresponding to the saturation of the acceleration bound at joint 2. Figure 6 shows the resulting joint velocities and

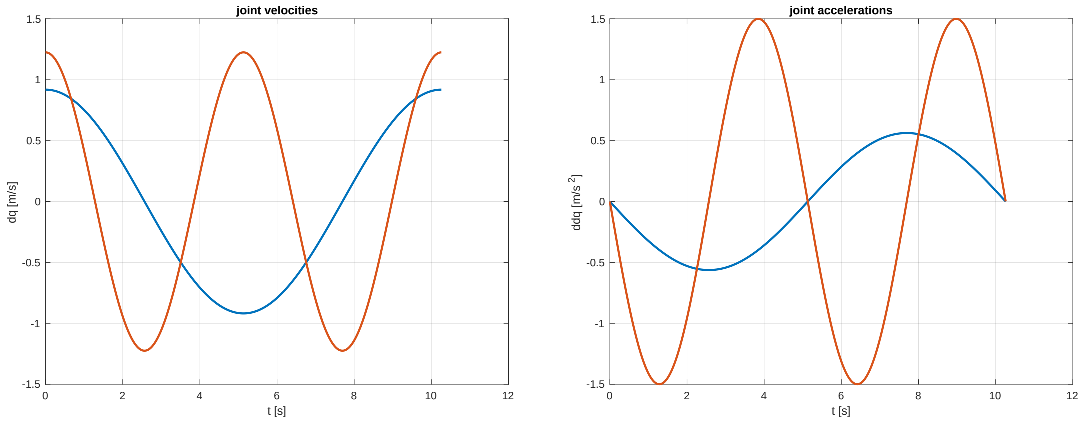

Figure 6: Joint velocities [left] and accelerations [right]. First joint in blue, second joint in red.

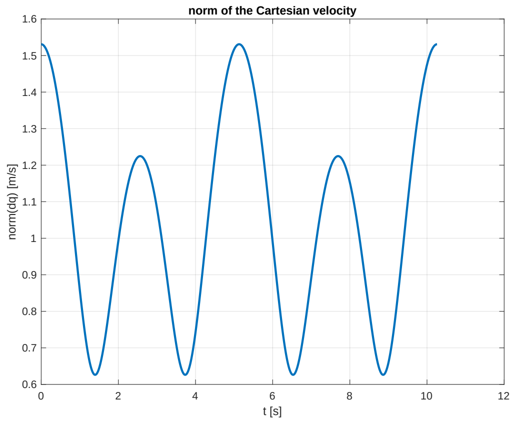

Figure 7: Norm of the Cartesian velocity of the robot end-effector.

accelerations, while the norm of the Cartesian velocity is reported in Fig. 7. Note the (multiple) periodicity of all plots.

Exercise #3

The orientation of a rigid body using the YXY sequence of Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ is given by the rotation matrix

$$\begin{aligned}
\boldsymbol{R}_{YXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma) \\
&= \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma \\ \sin\beta\sin\gamma & \cos\beta & -\sin\beta\cos\gamma \\ -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma \end{pmatrix}.
\end{aligned}$$

The angular velocity $\boldsymbol{\omega}$ of the body can be obtained from the formula $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{YXY}(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}})\boldsymbol{R}^T_{YXY}(\boldsymbol{\phi})$, where $\boldsymbol{S}$ is a skew-symmetric matrix. Using the shorthand notation for trigonometric functions, taking the time derivative of $\boldsymbol{R}_{YXY}$ and post-multiplying by the transpose of the rotation matrix yields

$$
\begin{aligned}
&\dot{\boldsymbol{R}}_{YXY}(\boldsymbol{\phi},\dot{\boldsymbol{\phi}})\cdot\boldsymbol{R}_{YXY}^T(\boldsymbol{\phi})\\
&=\begin{pmatrix}
\begin{array}{r}-(s_\alpha c_\gamma+c_\alpha c_\beta s_\gamma)\dot\alpha+s_\alpha s_\beta s_\gamma\dot\beta\\ -(c_\alpha s_\gamma+s_\alpha c_\beta c_\gamma)\dot\gamma\end{array} & c_\alpha s_\beta\dot\alpha+s_\alpha c_\beta\dot\beta & \begin{array}{r}-(s_\alpha s_\gamma-c_\alpha c_\beta c_\gamma)\dot\alpha-s_\alpha s_\beta c_\gamma\dot\beta\\ +(c_\alpha c_\gamma-s_\alpha c_\beta s_\gamma)\dot\gamma\end{array}\\
c_\beta s_\gamma\dot\beta+s_\beta c_\gamma\dot\gamma & -s_\beta\dot\beta & -c_\beta c_\gamma\dot\beta+s_\beta s_\gamma\dot\gamma\\
\begin{array}{r}-(c_\alpha c_\gamma-s_\alpha c_\beta s_\gamma)\dot\alpha+c_\alpha s_\beta s_\gamma\dot\beta\\ +(s_\alpha s_\gamma-c_\alpha c_\beta c_\gamma)\dot\gamma\end{array} & -s_\alpha s_\beta\dot\alpha+c_\alpha c_\beta\dot\beta & \begin{array}{r}-(s_\alpha c_\beta c_\gamma+c_\alpha s_\gamma)\dot\alpha-c_\alpha s_\beta c_\gamma\dot\beta\\ -(c_\alpha c_\beta s_\gamma+s_\alpha c_\gamma)\dot\gamma\end{array}
\end{pmatrix}\\
&\quad\cdot\begin{pmatrix}
c_\alpha c_\gamma-s_\alpha c_\beta s_\gamma & s_\beta s_\gamma & -s_\alpha c_\gamma-c_\alpha c_\beta s_\gamma\\
s_\alpha s_\beta & c_\beta & c_\alpha s_\beta\\
c_\alpha s_\gamma+s_\alpha c_\beta c_\gamma & -s_\beta c_\gamma & c_\alpha c_\beta c_\gamma-s_\alpha s_\gamma
\end{pmatrix}\\
&=\begin{pmatrix}
0 & s_\alpha\dot\beta-c_\alpha s_\beta\dot\gamma & \dot\alpha+c_\beta\dot\gamma\\
-s_\alpha\dot\beta+c_\alpha s_\beta\dot\gamma & 0 & -c_\alpha\dot\beta-s_\alpha s_\beta\dot\gamma\\
-\dot\alpha-c_\beta\dot\gamma & c_\alpha\dot\beta+s_\alpha s_\beta\dot\gamma & 0
\end{pmatrix}=\begin{pmatrix}
0 & -\omega_z & \omega_y\\
\omega_z & 0 & -\omega_x\\
-\omega_y & \omega_x & 0
\end{pmatrix}=\boldsymbol{S}(\boldsymbol{\omega})
\end{aligned}
\tag{6}
$$

The above derivation is greatly simplified by using the symbolic calculation in Matlab. Having defined the rotation matrix $\boldsymbol{R}_{YXY}$ and all the other needed quantities as symbolic variables, the $\boldsymbol{S}$ matrix and the angular velocity $\boldsymbol{\omega}$ are obtained by the following three instructions:

```
Rdot=diff(R_YXY,alfa)*dalfa+diff(R_YXY,beta)*dbeta+diff(R_YXY,gamma)*dgamma
S_omega=simplify(Rdot*R_YXY')}
omega=[S_omega(3,2);S_omega(1,3);S_omega(2,1)]
```

The linear mapping $\boldsymbol{\omega}=\boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ is then extracted from the elements of the $\boldsymbol{S}$ matrix in (6) as

$$
\boldsymbol{\omega}=\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}=\begin{pmatrix}\cos\alpha\,\dot\beta+\sin\alpha\sin\beta\,\dot\gamma\\ \dot\alpha+\cos\beta\,\dot\gamma\\ -\sin\alpha\,\dot\beta+\cos\alpha\sin\beta\,\dot\gamma\end{pmatrix}=\begin{pmatrix}0 & \cos\alpha & \sin\alpha\sin\beta\\ 1 & 0 & \cos\beta\\ 0 & -\sin\alpha & \cos\alpha\sin\beta\end{pmatrix}\begin{pmatrix}\dot\alpha\\ \dot\beta\\ \dot\gamma\end{pmatrix}=\boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.
$$

The singularities of this mapping occur when $\det\boldsymbol{T}(\boldsymbol{\phi})=-\sin\beta=0$, i.e., for $\beta=0$ and $\beta=\pi$.

In alternative to the above procedure, and perhaps more quickly, we can build the matrix $\boldsymbol{T}(\boldsymbol{\phi})$ by noting the individual contributions to the angular velocity $\boldsymbol{\omega}$ of $\dot\alpha$ (a rotation around the initial, fixed Y-axis), $\dot\beta$ (a rotation around the X'-axis, i.e., the X-axis after the rotation $\boldsymbol{R}_Y(\alpha)$), and $\dot\gamma$ (a rotation around the Y''-axis, i.e., the Y-axis after the first two rotations $\boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)$). We have

$$
\begin{aligned}
\boldsymbol{\omega}=\boldsymbol{\omega}_{\dot\alpha,Y}+\boldsymbol{\omega}_{\dot\beta,X'}+\boldsymbol{\omega}_{\dot\gamma,Y''}&=\begin{pmatrix}0\\1\\0\end{pmatrix}\dot\alpha+\boldsymbol{R}_Y(\alpha)\begin{pmatrix}1\\0\\0\end{pmatrix}\dot\beta+\boldsymbol{R}_Y(\alpha)\boldsymbol{R}_X(\beta)\begin{pmatrix}0\\1\\0\end{pmatrix}\dot\gamma\\
&=\begin{pmatrix}0\\1\\0\end{pmatrix}\dot\alpha+\begin{pmatrix}\cos\alpha\\0\\-\sin\alpha\end{pmatrix}\dot\beta+\begin{pmatrix}\sin\alpha\sin\beta\\ \cos\beta\\ \cos\alpha\sin\beta\end{pmatrix}\dot\gamma=\boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.
\end{aligned}
$$

Note also that, being each contribution to $\boldsymbol{\omega}$ a vector itself, the order in the sum is irrelevant.

Exercise #4

The solution requires to compute the position $\boldsymbol{p}_d(t) \in \mathbb{R}^2$ and orientation $\phi_d(t) \in \mathbb{R}$ of the end-effector at $t = 1$ [s] during the execution of the assigned task, together with the task velocity vector $\dot{\boldsymbol{r}}_d(t) = \begin{pmatrix} \dot{\boldsymbol{p}}_d^T(t) & \dot{\phi}_d(t) \end{pmatrix}^T$. An inverse kinematics problem is solved then analytically to obtain at that time instant a unique value of $\boldsymbol{q}$, which is used to evaluate the 3×3 task Jacobian $\boldsymbol{J}(\boldsymbol{q})$. Finally, inversion of the differential kinematics map provides the commanded joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$.

Note first that the motion trajectory from $\boldsymbol{A}$ to $\boldsymbol{B}$ lasts $T = \|\boldsymbol{B} - \boldsymbol{A}\|/v = 2.3564/0.5 = 4.7127$ [s], so that $t = 1$ [s] corresponds to an instant when the end-effector is actually on the linear path. The Cartesian trajectory and the desired position $\bar{\boldsymbol{p}}_d$ are thus

$$\boldsymbol{p}_d(t) = \boldsymbol{A} + \frac{vt}{\|\boldsymbol{B} - \boldsymbol{A}\|}(\boldsymbol{B} - \boldsymbol{A}) = \begin{pmatrix} 3 \\ 2.5 \end{pmatrix} - \frac{0.5t}{2.3564}\begin{pmatrix} 2.25 \\ 0.7 \end{pmatrix}, \quad \text{at } t = 1 \ \Rightarrow\ \bar{\boldsymbol{p}}_d = \boldsymbol{p}_d(1) = \begin{pmatrix} 2.5226 \\ 2.3515 \end{pmatrix}.$$

The orientation remains instead constant at all times and, according to Fig. 3, is given by[1]

$$\phi_d = \text{ATAN2}\left\{\boldsymbol{A}_y - \boldsymbol{B}_y, \boldsymbol{A}_x - \boldsymbol{B}_x\right\} + \frac{\pi}{2} = \text{ATAN2}\left\{0.7, 2.25\right\} + \frac{\pi}{2} = 1.8724 \text{ [rad]} = 107.28^\circ.$$

Accordingly, the desired task velocity (at $t = 1$, as well as at any other instant) is constant and is specified by

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} v\,\dfrac{\boldsymbol{B} - \boldsymbol{A}}{\|\boldsymbol{B} - \boldsymbol{A}\|} \\ \dot{\phi}_d \end{pmatrix} = \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix}.$$

Using the standard D-H joint variables, the task kinematics of the 3R robot at hand is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} l\left(\cos q_1 + \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3)\right) \\ l\left(\sin q_1 + \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3)\right) \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

with its 3×3 Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l\left(s_1 + s_{12} + s_{123}\right) & -l\left(s_{12} + s_{123}\right) & -l s_{123} \\ l\left(c_1 + c_{12} + c_{123}\right) & l\left(c_{12} + c_{123}\right) & l c_{123} \\ 1 & 1 & 1 \end{pmatrix}, \tag{7}$$

where the shorthand notation for trigonometric functions has been used, e.g., $s_{123} = \sin(q_1+q_2+q_3)$.

With reference to Fig. 8, the inverse kinematics problem can be decomposed in two parts. First, we solve for the two joint variables q_1 and q_2 in order to place the tip of the second link (or, the base of the third link) in the necessary position

$$\bar{\boldsymbol{p}}_{t2} = \bar{\boldsymbol{p}}_d - \ell \begin{pmatrix} \cos \phi_d \\ \sin \phi_d \end{pmatrix} = \begin{pmatrix} 3.1167 \\ 0.4418 \end{pmatrix} \text{ [m]}.$$

Since the robot arm has not crossed a singularity while moving the end-effector from $\boldsymbol{A}$ to $\bar{\boldsymbol{p}}_d$ (this can be easily verified), the configuration of the first two joints should remain the initial one, or *elbow up*. Thus, we find a unique solution for the pair (q_1, q_2) given by

$$c_2 = \frac{\bar{\boldsymbol{p}}_{t2,x}^2 + \bar{\boldsymbol{p}}_{t2,y}^2 - 2\ell^2}{2\ell^2} = 0.2386, \quad s_2 = -\sqrt{1 - c_2^2} = -0.9711$$

$$\Rightarrow\ q_2 = \text{ATAN2}\left\{s_2, c_2\right\} = -1.3298 \text{ [rad]},$$

[1] In the two arguments of the ATAN2 function, we have eliminated the common denominator $\|\boldsymbol{B} - \boldsymbol{A}\| > 0$.

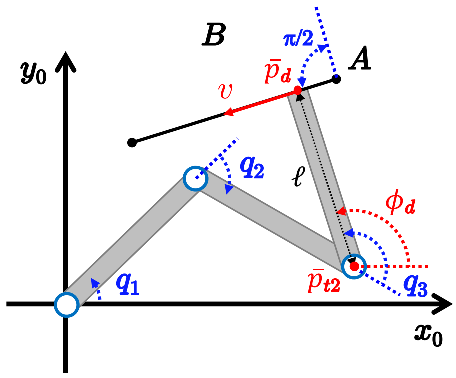

Figure 8: Solution approach to the inverse kinematics for the 3R planar robot.

and[2]

$$s_1 = \frac{\bar{\boldsymbol{p}}_{t2,y}\,(\ell + \ell c_2) - \bar{\boldsymbol{p}}_{t2,x}\ell s_2}{2\ell^2(1+c_2)} = 0.7213, \quad c_1 = \frac{\bar{\boldsymbol{p}}_{t2,x}\,(\ell + \ell c_2) + \bar{\boldsymbol{p}}_{t2,y}\ell s_2}{2\ell^2(1+c_2)} = 0.6926$$

$$\Rightarrow \; q_1 = \text{ATAN2}\,\{s_1, c_1\} = 0.8057\,[\text{rad}].$$

At this point, with $(q_1, q_2) = (0.8057, -1.3298)$ [rad] $= (46.16°, -76.19°)$, the third joint variable q_3 is recovered from the specification $\phi_d = 1.8724$ [rad] on the end-effector orientation:

$$q_3 = \phi_d - (q_1 + q_2) = 2.3965 \text{ [rad]} = 137.31°.$$

The above solution of the inverse kinematics problem is coded in Matlab by the instructions:

```
p_t2=p_d-l*[cos(phi_d); sin(phi_d)]
px=p_t2(1);
py=p_t2(2);
c2=(px^2+py^2-2*l^2)/(2*l^2)
s2=-sqrt(1-c2^2)  % elbow up solution (as the initial configuration)
q2=atan2(s2,c2)
s1=py*(l+l*c2)-px*l*s2  % denominator (> 0) discarded in s1 and c1
c1=px*(l+l*c2)+py*l*s2
q1=atan2(s1,c1)
q3=phi_d-(q1+q2)
```

Evaluating the Jacobian in (7) for the obtained $\boldsymbol{q} = (q_1, q_2, q_3)$ and inverting the differential mapping yields finally the joint velocity

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{r}}_d = \begin{pmatrix} -2.3515 & -0.9088 & -1.9097 \\ 2.5226 & 1.1374 & -0.5941 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -0.4458 & 0.2577 & -0.6982 \\ 0.8024 & 0.1137 & 1.5998 \\ -0.3566 & -0.3714 & 0.0983 \end{pmatrix} \begin{pmatrix} -0.4774 \\ -0.1485 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1745 \\ -0.4000 \\ 0.2254 \end{pmatrix} \text{ [rad/s]}.$$

[2]The common (positive) denominator $2\ell^2(1+c_2)$ in the expressions of s_1 and c_1 can be discarded without affecting the final result.

Exercise #5

A) The possible sequences of Euler angles are 12. They are listed in Tab. 2, together with their one-to-one correspondence with RPY-type angles (with the reverse order in the products of elementary rotation matrices). The rotation matrix produced in one or in the corresponding sequence of angles (around the moving or fixed axes) will be the same.

i	Euler sequences	Roll-Pitch-Yaw sequences
1	$X(\alpha)Y'(\beta)X''(\gamma)$	$X(\gamma)Y(\beta)X(\alpha)$
2	$X(\alpha)Y'(\beta)Z''(\gamma)$	$Z(\gamma)Y(\beta)X(\alpha)$
3	$X(\alpha)Z'(\beta)X''(\gamma)$	$X(\gamma)Z(\beta)X(\alpha)$
4	$X(\alpha)Z'(\beta)Y''(\gamma)$	$Y(\gamma)Z(\beta)X(\alpha)$
5	$Y(\alpha)X'(\beta)Y''(\gamma)$	$Y(\gamma)X(\beta)Y(\alpha)$
6	$Y(\alpha)X'(\beta)Z''(\gamma)$	$Z(\gamma)X(\beta)Y(\alpha)$
7	$Y(\alpha)Z'(\beta)X''(\gamma)$	$X(\gamma)Z(\beta)Y(\alpha)$
8	$Y(\alpha)Z'(\beta)Y''(\gamma)$	$Y(\gamma)Z(\beta)Y(\alpha)$
9	$Z(\alpha)X'(\beta)Y''(\gamma)$	$Y(\gamma)X(\beta)Z(\alpha)$
10	$Z(\alpha)X'(\beta)Z''(\gamma)$	$Z(\gamma)X(\beta)Z(\alpha)$
11	$Z(\alpha)Y'(\beta)X''(\gamma)$	$X(\gamma)Y(\beta)Z(\alpha)$
12	$Z(\alpha)Y'(\beta)Z''(\gamma)$	$Z(\gamma)Y(\beta)Z(\alpha)$

Table 2: Correspondence between Euler and RPY minimal representations of orientation.

B) The maximum angular velocity that the driven link can reach is equal to

$$\dot{q}_{max} = \frac{\dot{\theta}_{max}(\text{rpm})}{n_r} \cdot \frac{2\pi}{60} = \frac{4120}{6000}\,\pi = 2.1572\,[\text{rad/s}].$$

Thus, the link velocity $\dot{q} = 3.5$ [rad/s] is unfeasible. The optimal reduction ratio that minimizes the required motor torque for a given link angular acceleration $\ddot{q}$ satisfies the relation $n_r = \sqrt{J_l/J_m}$. If $n_r = 100$ is such an optimal value for the given motor inertia $J_m = 1.2 \cdot 10^{-5}$ [kg·m^2], then the value of the link inertia is

$$J_l = n_r^2\, J_m = 1.2 \cdot 10^{-1}\ [\text{kg}\cdot\text{m}^2].$$

Therefore, the motor torque needed to produce a link angular acceleration $\ddot{q} = 4$ [rad/s^2] is

$$\tau_m = \left(J_m n_r + \frac{J_l}{n_r}\right)\ddot{q} = 2\sqrt{J_m J_l}\,\ddot{q} = 9.6 \cdot 10^{-3}\ [\text{Nm}],$$

where the second (equivalent) expression follows from the optimality of the reduction ratio.

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