*Robotics 2*

# Robots with kinematic redundancy

## Part 2: Extensions


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# A general task priority formulation

- consider a large number $p$ of tasks to be executed at best and with strict priorities by a robotic system having many dofs
- everything should run efficiently in real time, with possible addition, deletion, swap, or reordering of tasks
- a recursive formulation that reduces computations is convenient

$$\dot{q} \in \mathbb{R}^n \qquad \dot{r}_k \in \mathbb{R}^{m_k} \qquad \dot{r}_k = J_k(q)\dot{q} \qquad P_k(q) = I - J_k^{\#}(q) J_k(q)$$

$$k = 1, \dots, p$$

$k$-th task

projector in the null-space of $k$-th task

$$i < j \Rightarrow \text{task } i \text{ has higher priority than task } j \qquad \sum_{k=1}^{p} m_k = m (\leq n)$$

even larger!

$$\dot{r}_{A,k} = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \vdots \\ \dot{r}_k \end{pmatrix} \qquad J_{A,k} = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_k \end{pmatrix}$$

**stack** of first $k$ tasks

augmented Jacobian of first $k$ tasks

$$P_{A,k} = I - J_{A,k}^{\#} J_{A,k}$$

projector in the null-space of the augmented Jacobian of the first $k$ tasks

$$J_i P_{A,k} = O \qquad \forall i \leq k$$

$$\Longleftrightarrow \qquad J_{A,k} P_{A,k} = O$$

# Recursive solution with priorities - 1

- start with the first task and reformulate the problem so as to provide always a "solution", at least in terms of minimum error norm

$\boxed{\text{for } k=1}$

$$\begin{cases} \dot{\boldsymbol{q}}_1 = \arg \min\limits_{\dot{\boldsymbol{q}} \in \mathbb{R}^n} \dfrac{1}{2}\|\dot{\boldsymbol{q}}\|^2 \\ \text{s.t.} \quad \boldsymbol{J}_1 \dot{\boldsymbol{q}} = \dot{\boldsymbol{r}}_1 \end{cases} \Rightarrow \begin{cases} \dot{\boldsymbol{q}}_1 = \arg \min\limits_{\dot{\boldsymbol{q}} \in \mathcal{S}_1} \dfrac{1}{2}\|\dot{\boldsymbol{q}}\|^2 \\ \mathcal{S}_1 = \left\{ \arg \min\limits_{\dot{\boldsymbol{q}} \in \mathbb{R}^n} \dfrac{1}{2}\|\boldsymbol{J}_1 \dot{\boldsymbol{q}} - \dot{\boldsymbol{r}}_1\|^2 \right\} \end{cases}$$

$$\Rightarrow \quad \dot{\boldsymbol{q}}_1 = \boldsymbol{J}_1^{\#} \dot{\boldsymbol{r}}_1 \quad \Rightarrow \quad \mathcal{S}_1 = \{\dot{\boldsymbol{q}}_1 + \boldsymbol{P}_1 \boldsymbol{v}_1, \ \boldsymbol{v}_1 \in \mathbb{R}^n\}$$

$\boxed{\text{for } k=2}$

$$\begin{cases} \dot{\boldsymbol{q}}_2 = \arg \min\limits_{\dot{\boldsymbol{q}} \in \mathcal{S}_2} \dfrac{1}{2}\|\dot{\boldsymbol{q}}\|^2 \\ \mathcal{S}_2 = \left\{ \arg \min\limits_{\dot{\boldsymbol{q}} \in \mathcal{S}_1} \dfrac{1}{2}\|\boldsymbol{J}_2 \dot{\boldsymbol{q}} - \dot{\boldsymbol{r}}_2\|^2 \right\} \end{cases} \Rightarrow \begin{array}{l} \dot{\boldsymbol{q}}_2 = \dot{\boldsymbol{q}}_1 + (\boldsymbol{J}_2 \boldsymbol{P}_1)^{\#} (\dot{\boldsymbol{r}}_2 - \boldsymbol{J}_2 \dot{\boldsymbol{q}}_1) \\ \mathcal{S}_2 = \{\dot{\boldsymbol{q}}_2 + \boldsymbol{P}_{A,2} \boldsymbol{v}_2, \ \boldsymbol{v}_2 \in \mathbb{R}^n\} \end{array}$$

# Recursive solution with priorities - 2

generalizing to step $k$

$\dot{\boldsymbol{q}}_{k-1}$

prioritized solution up to task $k-1$

$$\mathcal{S}_{k-1} = \left\{ \dot{\boldsymbol{q}}_{k-1} + \boldsymbol{P}_{A,k-1}\boldsymbol{v}_{k-1},\ \boldsymbol{v}_{k-1} \in \mathbb{R}^n \right\}$$

set of all solutions up to task $k-1$

LQ problem for $k$-th task

$$\dot{\boldsymbol{q}}_k = \arg \min_{\dot{\boldsymbol{q}} \in \mathcal{S}_k} \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2$$

$$\mathcal{S}_k = \left\{ \arg \min_{\dot{\boldsymbol{q}} \in \mathcal{S}_{k-1}} \frac{1}{2}\|\boldsymbol{J}_k\dot{\boldsymbol{q}} - \dot{\boldsymbol{r}}_k\|^2 \right\}$$

**recursive formula** (Siciliano, Slotine: ICAR 1991)

$$\boxed{\underbrace{\dot{\boldsymbol{q}}_k}_{\text{prioritized solution up to task } k} = \dot{\boldsymbol{q}}_{k-1} + \left(\boldsymbol{J}_k\boldsymbol{P}_{A,k-1}\right)^{\#} \underbrace{\left(\dot{\boldsymbol{r}}_k - \boldsymbol{J}_k\dot{\boldsymbol{q}}_{k-1}\right)}_{\text{correction needed when the solution up to task } k-1 \text{ is not satisfying also task } k}}$$

initialization

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{0} \qquad \boldsymbol{P}_{A,0} = \boldsymbol{I}$$

prioritized solution up to task $k$

correction needed when the solution up to task $k-1$ is not satisfying also task $k$

over the steps, the search set is progressively reduced $\Longleftrightarrow$ $\mathbb{R}^n = \mathcal{S}_0 \supseteq \mathcal{S}_1 \supseteq \cdots \supseteq \mathcal{S}_{p-1} \supseteq \mathcal{S}_p$

# Recursive solution with priorities

## properties and implementation

- the solution considering the first $k$ tasks with their priority

$$\dot{\boldsymbol{q}}_k = \dot{\boldsymbol{q}}_{k-1} + \left(\boldsymbol{J}_k \boldsymbol{P}_{A,k-1}\right)^{\#} \left(\dot{\boldsymbol{r}}_k - \boldsymbol{J}_k \dot{\boldsymbol{q}}_{k-1}\right)$$

satisfies also ("does not perturb") the previous $k-1$ tasks

$$\boldsymbol{J}_{A,k-1} \dot{\boldsymbol{q}}_k = \boldsymbol{J}_{A,k-1} \dot{\boldsymbol{q}}_{k-1}$$

since

$$\boldsymbol{J}_{A,k-1} \underbrace{\left(\boldsymbol{J}_k \boldsymbol{P}_{A,k-1}\right)^{\#}} = \underset{=}{\boldsymbol{J}_{A,k-1}} \underbrace{\boldsymbol{P}_{A,k-1} \left(\boldsymbol{J}_k \boldsymbol{P}_{A,k-1}\right)^{\#}} = \boldsymbol{O}$$

(Maciejewski, Klein: IJRR 1985): check the four defining properties of a pseudoinverse

- recursive expression also for the null-space projector

$$\boxed{\boldsymbol{P}_{A,k} = \boldsymbol{P}_{A,k-1} - \left(\boldsymbol{J}_k \boldsymbol{P}_{A,k-1}\right)^{\#} \boldsymbol{J}_k \boldsymbol{P}_{A,k-1}} \qquad \boldsymbol{P}_{A,0} = \boldsymbol{I}$$

(Baerlocher, Boulic: IROS 1998): for the proof, see Appendix A

- when the $k$-th task is (close to be) incompatible with the previous ones (algorithmic singularity), use "DLS" instead of "#" in $k$-th solution...

# A list of extensions

- up to now, only "basic" redundancy resolution schemes
  - defined at first-order differential level (velocity)
    - it is possible to work in acceleration
      - useful for obtaining smoother motion
      - allows including the consideration of dynamics
  - seen within a planning, not a control perspective
    - take into account and recover errors in task execution by using kinematic control schemes
  - applied to robot manipulators with fixed base
    - extend to wheeled mobile manipulators
  - tasks specified only by equality constraints
    - add also linear inequalities in a complete QP formulation
      - very common also for humanoid robots in multiple tasks
    - consider hard limits in joint/command space

# Resolution at acceleration level

$$r = f(q) \;\Rightarrow\; \dot{r} = J(q)\dot{q} \;\Rightarrow\; \boxed{\ddot{r} = J(q)\ddot{q} + \dot{J}(q)\dot{q}}$$

- rewritten in the form

$$\underbrace{J(q)\ddot{q}}_{\text{to be chosen}} = \underbrace{\ddot{r}}_{\substack{\text{given}\\ \text{(at time } t)}} - \underbrace{\dot{J}(q)\dot{q}}_{\substack{\text{known } q,\dot{q}\\ \text{(at time } t)}} \triangleq \ddot{x}$$

to be chosen — given (at time $t$) — known $q, \dot{q}$ (at time $t$)

the problem is formally equivalent to the previous one, with acceleration in place of velocity commands

- for instance, in the null-space method

$$\ddot{q} = \underbrace{J^{\#}(q)\ddot{x}}_{\substack{\text{solution with minimum}\\ \text{acceleration norm } \|\ddot{q}\|^2}} + \left(I - J^{\#}(q)J(q)\right)\underbrace{\ddot{q}_0}_{= \alpha\nabla_q H - K_D\dot{q}}$$

solution with minimum acceleration norm $\|\ddot{q}\|^2$

"preferred" acceleration to damp/stabilize joint motion in the null space $(K_D > 0)$

solution with minimum norm $\|\ddot{q} - \ddot{q}_0\|^2$

# Dynamic redundancy resolution

- dynamic model of a robot manipulator (more later!)

$$M(q)\ddot{q} + n(q,\dot{q}) = \tau \qquad J(q)\ddot{q} = \ddot{x}\ (= \ddot{r} - \dot{J}(q)\dot{q})$$

$N \times N$ symmetric inertia matrix, positive definite for all $q$

input torque vector (provided by the motors)

$M$-dimensional acceleration task

Coriolis/centrifugal vector $c(q,\dot{q})$ + gravity vector $g(q)$

- we can formulate and solve interesting dynamic problems in the general framework of LQ optimization[°]
- closed-form expressions can be obtained by the solution formula[°] (assuming a full rank Jacobian $J$)

[°] in block *Kinematic redundancy - Part 1,* slide #28

# Dynamic redundancy resolution

as Linear-Quadratic optimization problems

- typical dynamic objectives to be locally minimized at $(q,\dot{q})$

torque norm

$$H_1(\ddot{q}) = \frac{1}{2}\|\tau\|^2 = \frac{1}{2}\ddot{q}^T M^2(q)\ddot{q} + n^T(q,\dot{q})M(q)\ddot{q} + \frac{1}{2}\, n^T(q,\dot{q})n(q,\dot{q})$$

(squared inverse inertia weighted) torque norm

$$H_2(\ddot{q}) = \frac{1}{2}\|\tau\|^2_{M^{-2}} = \frac{1}{2}\tau^T M^{-2}(q)\tau$$
$$= \frac{1}{2}\ddot{q}^T\ddot{q} + n^T(q,\dot{q})M^{-1}(q)\ddot{q} + \frac{1}{2}\, n^T(q,\dot{q})M^{-2}(q)n(q,\dot{q})$$

(inverse inertia weighted) torque norm

$$H_3(\ddot{q}) = \frac{1}{2}\|\tau\|^2_{M^{-1}} = \frac{1}{2}\tau^T M^{-1}(q)\tau$$
$$= \frac{1}{2}\ddot{q}^T M(q)\ddot{q} + n^T(q,\dot{q})\ddot{q} + \frac{1}{2}\, n^T(q,\dot{q})M^{-1}(q)n(q,\dot{q})$$

# Closed-form solutions

minimum torque norm solution

$$\frac{1}{2}\|\tau\|^2 \quad \Rightarrow \quad \tau_1 = (J(q)M^{-1}(q))^{\#}\big(\ddot{r} - \dot{J}(q)\dot{q} + J(q)M^{-1}(q)n(q,\dot{q})\big)$$

- good for short trajectories (in fact, it is still only a "local" solution!)
- for longer trajectories it leads to torque "oscillation/explosion" (whipping effect)

minimum (squared inverse inertia weighted) torque norm solution

$$\frac{1}{2}\|\tau\|^2_{M^{-2}} \quad \Rightarrow \quad \tau_2 = M(q)J^{\#}(q)\big(\ddot{r} - \dot{J}(q)\dot{q} + J(q)M^{-1}(q)n(q,\dot{q})\big)$$

- good performance in general, to be preferred

minimum (inverse inertia weighted) torque norm solution

$$\frac{1}{2}\|\tau\|^2_{M^{-1}} \quad \Rightarrow \quad \tau_3 = J^T(q)(J(q)M^{-1}(q)J^T(q))^{-1}\big(\ddot{r} - \dot{J}(q)\dot{q} + J(q)M^{-1}(q)n(q,\dot{q})\big)$$

- a solution with a leading $J^T(q)$ term: what is its nice physical interpretation?

May we add also a term $\boldsymbol{\tau}_0$ in a (dynamic) null space? Easy to do in the LQ framework!

# Stabilizing the minimum torque solution

Universal Robots UR-10 (6-dof)

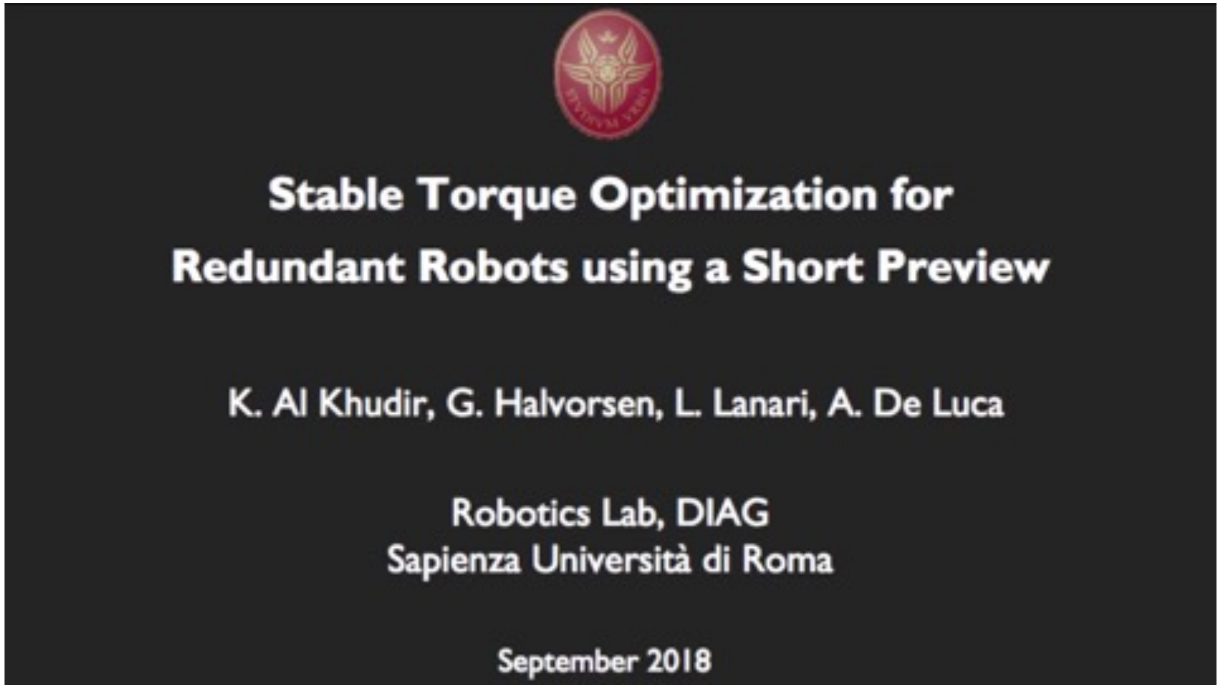

video

$\min \frac{1}{2}\|\tau\|^2$ = MTN

versus

video

KUKA LRW 4 (7-dof, last joint not used)




- MBP = minimizing torque also at a short preview instant
- MTND = damping joint velocity in the null space
- MBPD = … do both

IEEE Robotics and Automation Lett. 2019

# Kinematic control

- given a desired $M$-dimensional task $r_d(t)$, in order to recover a task error $e = r_d - r$ due to initial mismatch or due to
  - disturbances
  - inherent linearization error in using the Jacobian (first-order motion)
  - discrete-time implementation

we need to "close" a feedback loop on task execution, by replacing (with diagonal matrix gains $K > 0$ or $K_P, K_D > 0$)

$$\dot{r} \Rightarrow \dot{r}_d + K(r_d - r)$$ in velocity-based...

$$\ddot{r} \Rightarrow \ddot{r}_d + K_D(\dot{r}_d - \dot{r}) + K_P(r_d - r)$$ ...in acceleration-based methods

where $r = f(q),\ \dot{r} = J(q)\dot{q}$

# Mobile manipulators

- coordinates: $q_b$ of the base and $q_m$ of the manipulator
- differential map: from available commands $u_b$ on the mobile base and $u_m$ on the manipulator to task output velocity

$q_m$

$q_b$

$$r = f(q) \in \mathbb{R}^M$$

(task output, e.g., the E-E pose)

$$q = \begin{pmatrix} q_b \\ q_m \end{pmatrix} \in \mathbb{R}^N$$

kinematic model of the wheeled base (subject to nonholonomic constraints)

$$\dot{q}_b = G(q_b)u_b$$
$$\dot{q}_m = u_m$$

$$u = \begin{pmatrix} u_b \\ u_m \end{pmatrix} \in \mathbb{R}^{N_u} \qquad N_u \le N$$

# Mobile manipulator Jacobian

$$r = f(q) = f(q_b, q_m)$$

$$\dot{r} = \frac{\partial f(q)}{\partial q_b}\dot{q}_b + \frac{\partial f(q)}{\partial q_m}\dot{q}_m = J_b(q)\dot{q}_b + J_m(q)\dot{q}_m$$

$$= J_b(q)G(q_b)u_b + J_m(q)u_m = \begin{pmatrix} J_b(q)G(q_b) & J_m(q) \end{pmatrix}\begin{pmatrix} u_b \\ u_m \end{pmatrix}$$

$$= \boxed{J_{NMM}(q)}\,u$$

Nonholonomic Mobile Manipulator (NMM) Jacobian ($M \times N_u$)

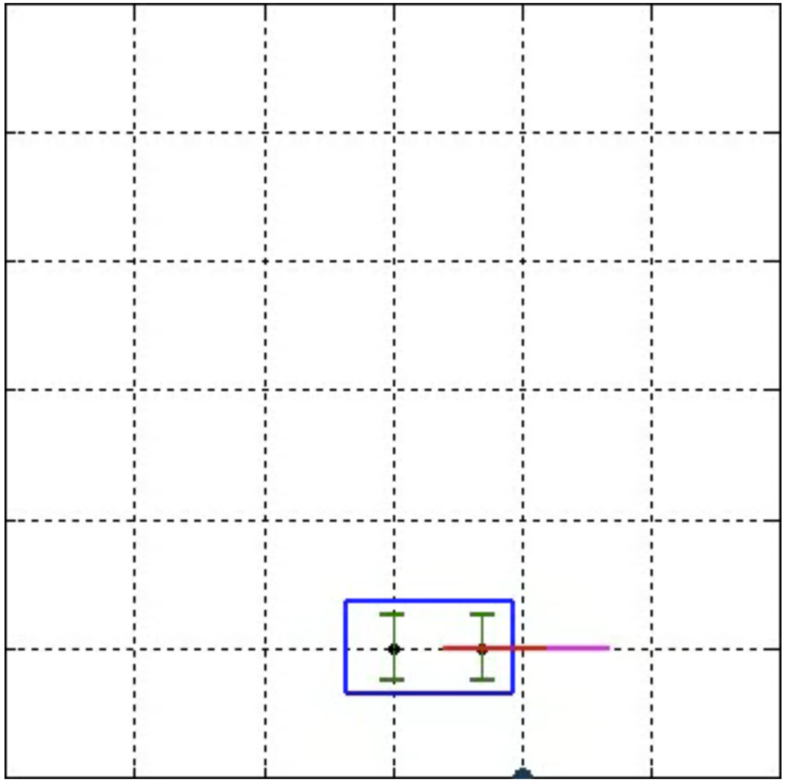
- … most previous results follow by just replacing

$$J \implies J_{NMM} \qquad \dot{q} \implies u$$ (redundancy if $N_u - M > 0$)

↑ namely, the available velocity commands

# Mobile manipulators

video



car-like+2R planar arm
($N = 6, N_u = 4$):
E-E trajectory control on a line ($N_u - M = 2$)
with maximization of $J_{NMM}$ manipulability

Automatica Fair 2008

video

wheeled Justin with centered steering wheels
($N = 3 + 4\times 2, N_u = 8$)
"dancing" in controlled
but otherwise passive mode

# Quadratic Programming (QP)
## with equality and inequality constraints

- minimize a quadratic objective function (typically positive definite, like when using norms of vectors) subject to linear equality and inequality constraints, all expressed in terms of joint velocity commands

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{r}} \qquad \boldsymbol{C}\dot{\boldsymbol{q}} \le \boldsymbol{d} \qquad \dot{\boldsymbol{q}} \in \Omega \subseteq \mathbb{R}^n$$

within a given convex set

solution set, with only equality constraints

$$\mathcal{S}_{eq} = \arg\min_{\dot{\boldsymbol{q}}\in\Omega} \frac{1}{2}\|\boldsymbol{J}\dot{\boldsymbol{q}} - \dot{\boldsymbol{r}}\|^2$$

$$\text{given } \dot{\boldsymbol{q}}^* \in \mathcal{S}_{eq} \quad \Rightarrow \quad \mathcal{S}_{eq} = \{\dot{\boldsymbol{q}} \in \Omega : \boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^*\}$$

solution set, with only inequality constraints

$$\mathcal{S}_{ineq} = \arg\min_{\dot{\boldsymbol{q}}\in\Omega} \frac{1}{2}\|\boldsymbol{w}\|^2$$

$$\text{s.t.} \quad \boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{d} \le \boldsymbol{w} \qquad \boldsymbol{w} \in \mathbb{R}^m_+$$

(non-negative) slack variables

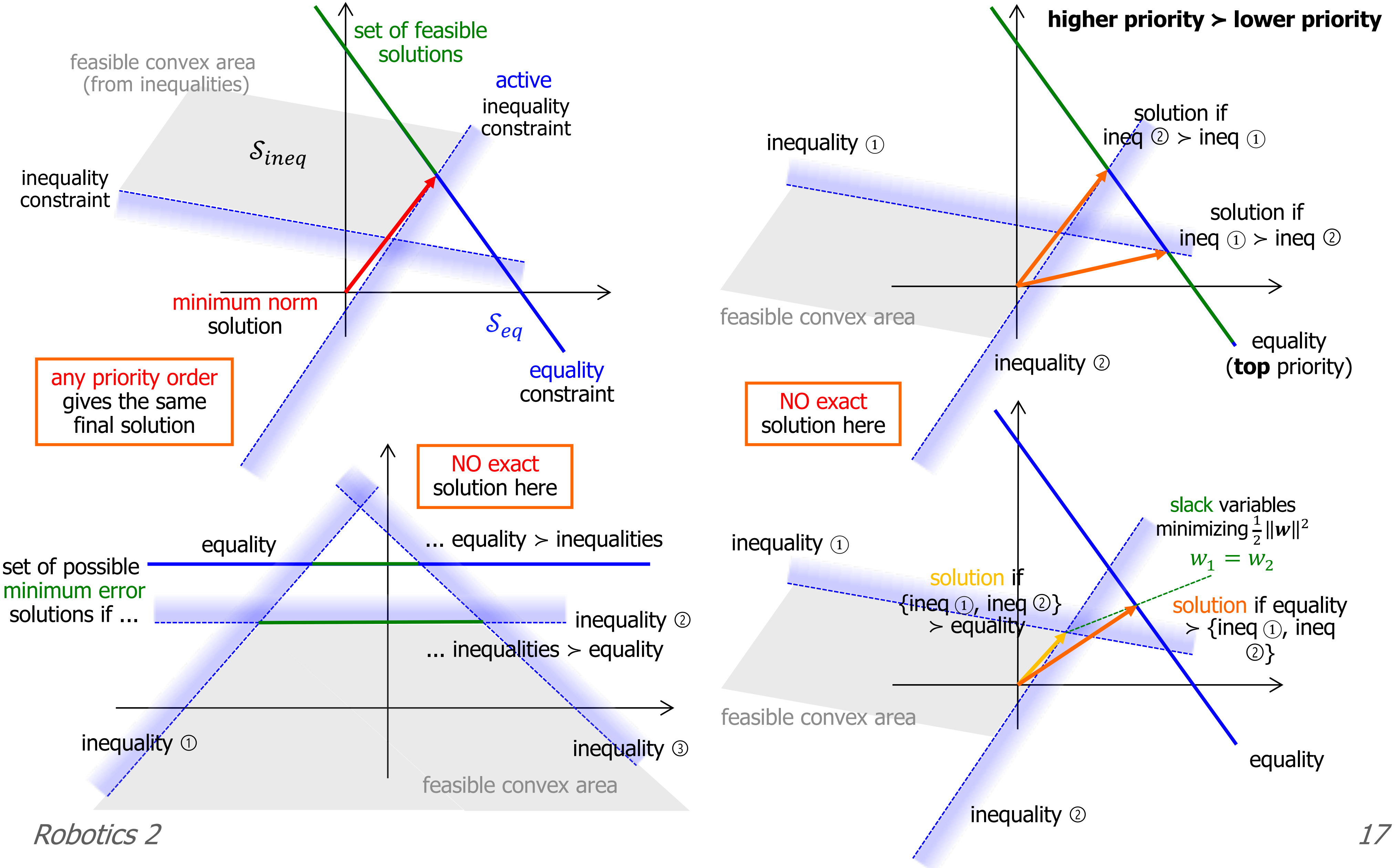
$$\text{given } \dot{\boldsymbol{q}}^* \in \mathcal{S}_{ineq} \quad \Rightarrow \quad \mathcal{S}_{ineq} = \Omega \cap \begin{cases} \boldsymbol{c}_j^T\dot{\boldsymbol{q}} \le d_j, & \text{if } \boldsymbol{c}_j^T\dot{\boldsymbol{q}}^* \le d_j \\ \boldsymbol{c}_j^T\dot{\boldsymbol{q}} = \boldsymbol{c}_j^T\dot{\boldsymbol{q}}^*, & \text{if } \boldsymbol{c}_j^T\dot{\boldsymbol{q}}^* > d_j \end{cases}$$

QP complete formulation

$$\boxed{\begin{aligned} &\min_{\dot{\boldsymbol{q}}\in\Omega} \frac{1}{2}\|\boldsymbol{J}\dot{\boldsymbol{q}} - \dot{\boldsymbol{r}}\|^2 + \frac{1}{2}\|\boldsymbol{w}\|^2 \\ &\text{s.t.} \quad \boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{w} \le \boldsymbol{d} \qquad \boldsymbol{w} \in \mathbb{R}^m_+ \end{aligned}}$$

(possibly with prioritization of constraints)

# Equality and inequality linear constraints

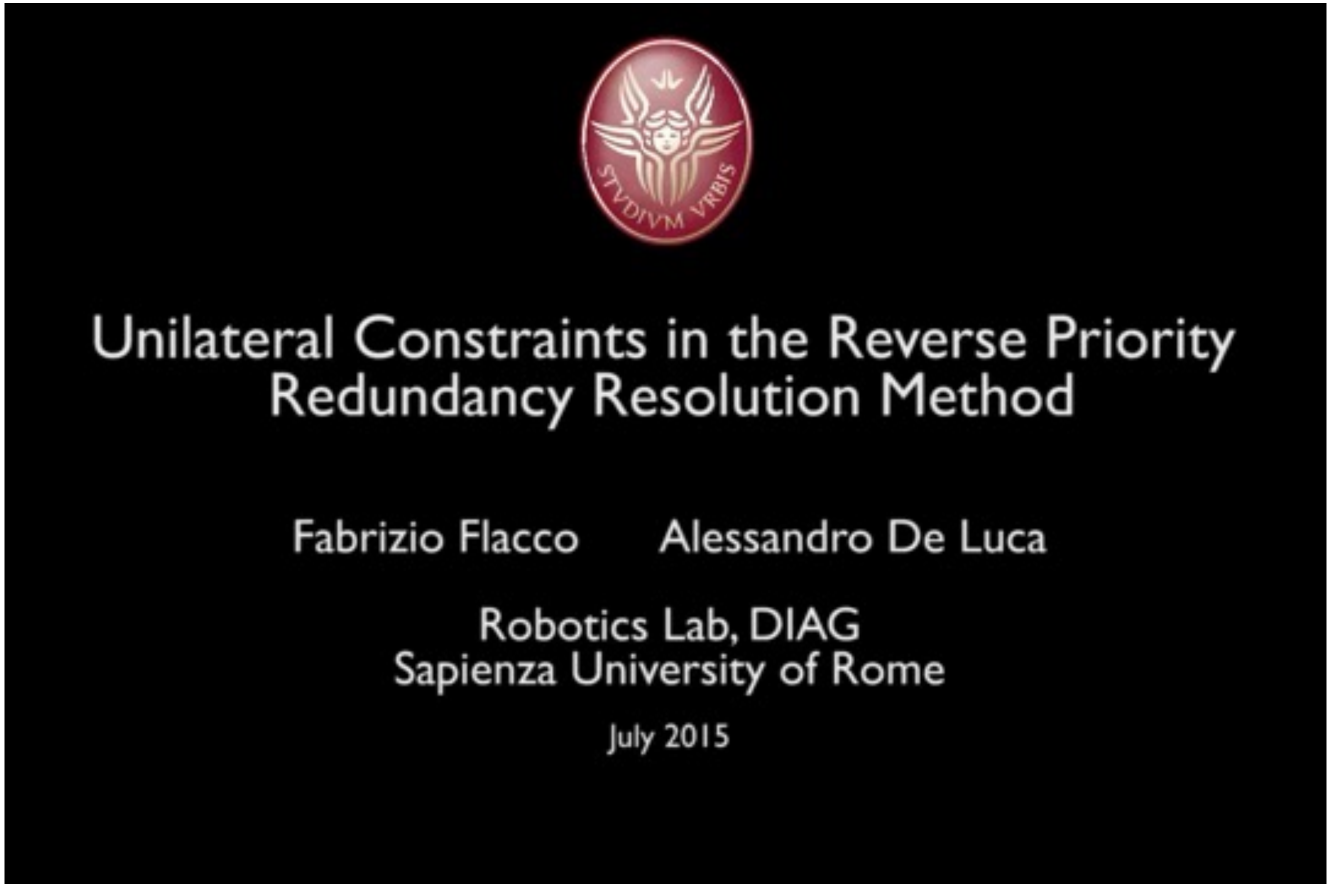


# Equality and Inequality Tasks

## 6R planar robot (simulations) and 7R KUKA LWR (experiment)

- an efficient task priority approach, with simultaneous inequality tasks handled as hard (cannot be violated) or soft (can be relaxed) constraints

video



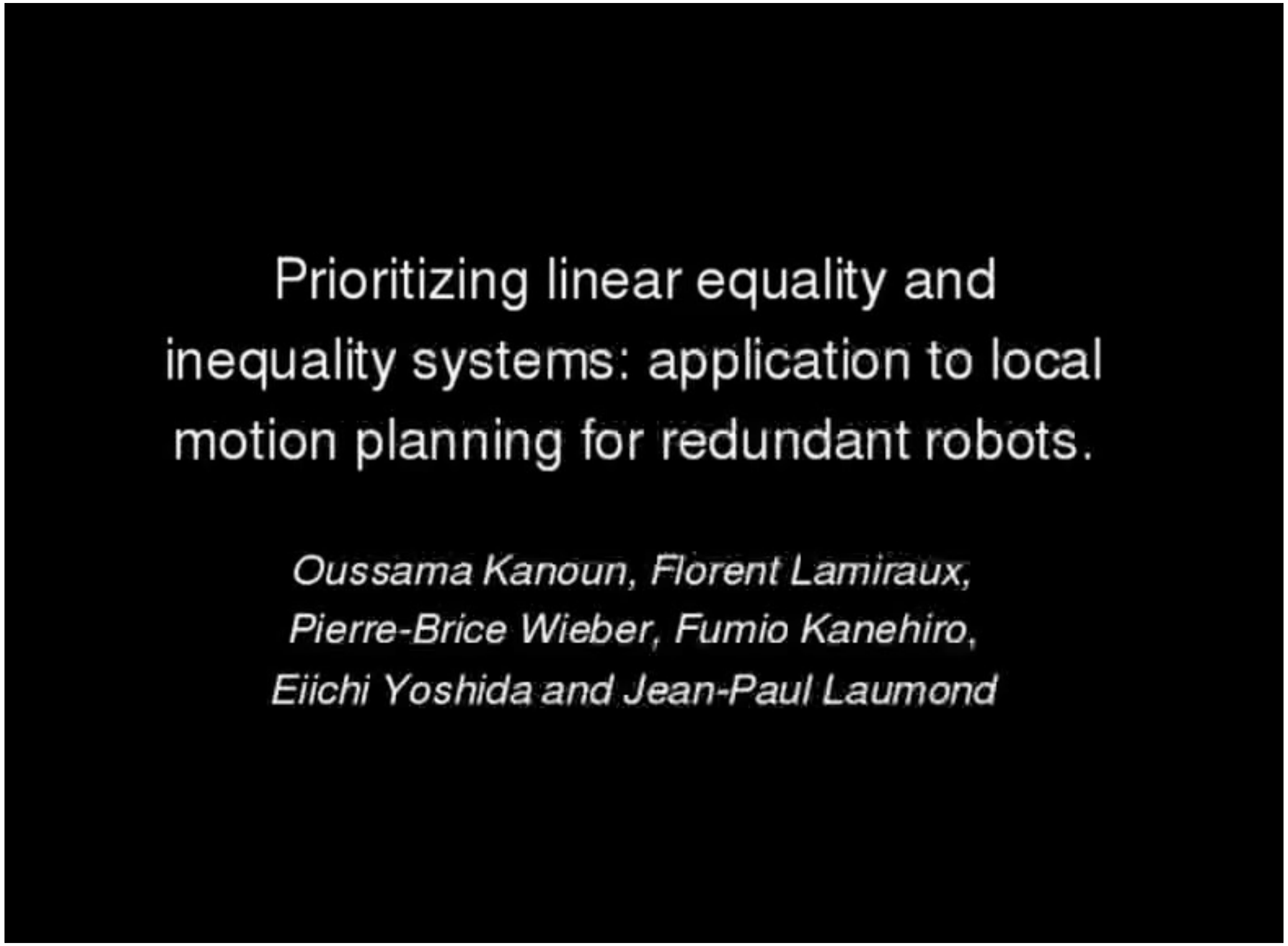

IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS) 2015

# Equality and Inequality Tasks

## for the high-dof humanoid robot HRP2

- a systematic task priority approach, with several simultaneous tasks

video




IEEE Int. Conf. on Robotics and Automation (ICRA) 2009

in any order of priority
- avoid the obstacle
- gaze at the object
- reach the object
- ...

while keeping balance!

all subtasks are locally expressed by linear equalities or inequalities (possibly relaxed when needed) on joint velocities

# Inclusion of hard limits in joint space

## Saturation in the Null Space (SNS) method

- robot has "limited" capabilities: hard limits on joint ranges and/or on joint motion or commands (max velocity, acceleration, torque)
- represented as box inequalities that can never be violated (at most, active constraints or saturated commands) – kept separated from "stack" of tasks
- (equality) tasks are usually executed in full (with priorities, if desired), but can be relaxed (scaled) in case of need (i.e., when robot capabilities are used at their limits)

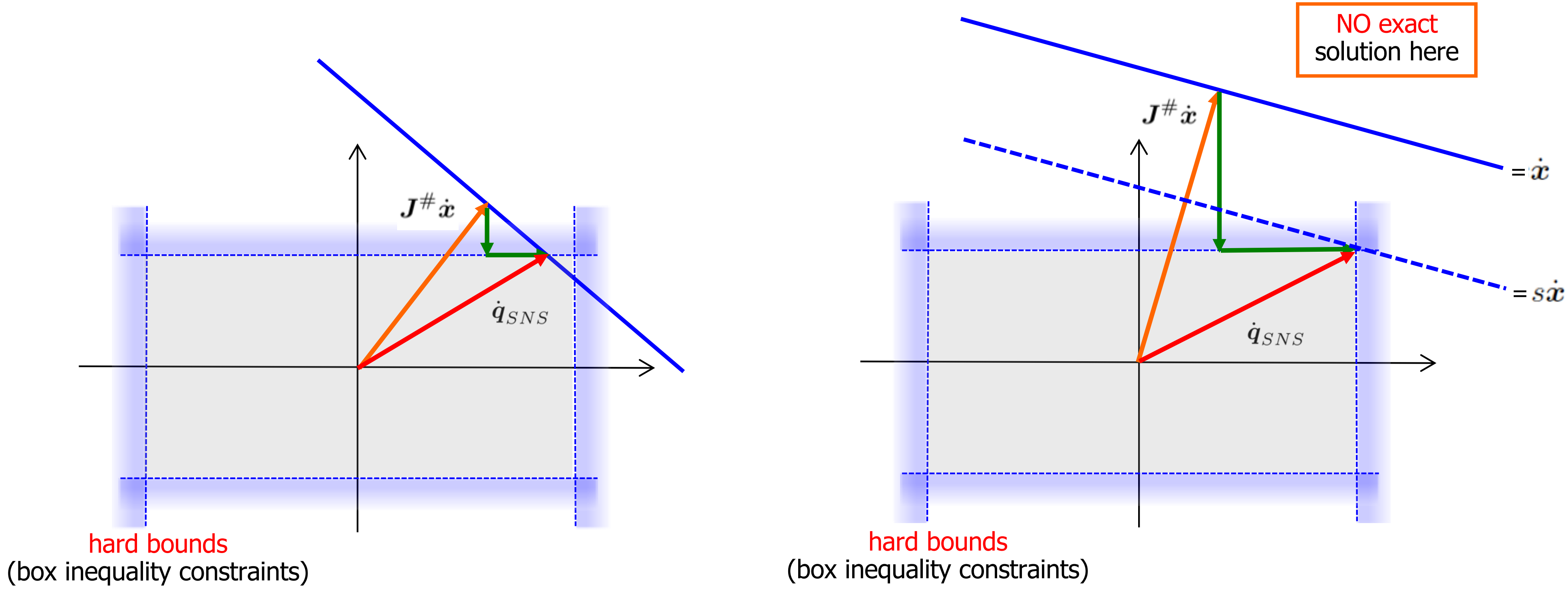
⇓

- saturate one overdriven joint command at a time, until a feasible and better performing solution is found ⇒ Saturation in the Null Space = SNS
- on-line decision: which joint commands to saturate and how, so that this does not affect task execution
- for tasks that are (certainly) not feasible, SNS embeds the selection of a task scaling factor preserving execution of the task direction with minimal scaling

$$\dot{q}_{SNS} = (JW)^{\#}\, \underset{\substack{\uparrow \\ \text{scaling factor}}}{s}\, \dot{x} + \left(I - (\underset{\substack{\uparrow \\ \text{diagonal 0/1 matrix}}}{JW})^{\#} J\right) \dot{q}_N \longleftarrow \text{contains saturated joint velocities}$$

# Geometric view on SNS operation

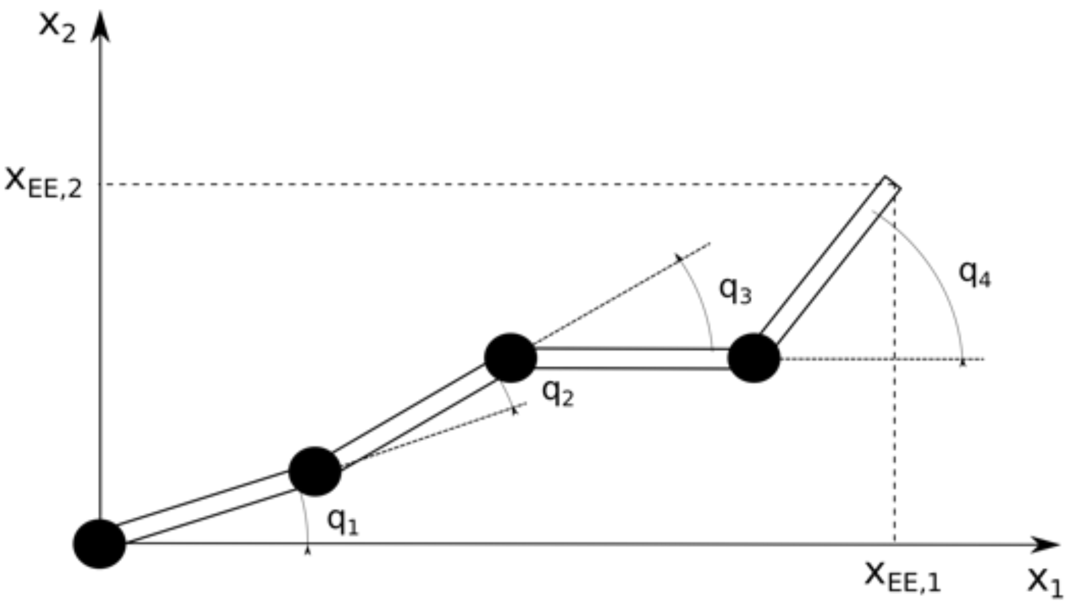
in the space of joint velocity commands




the total correction to the original pseudoinverse solution
is always in the null space of the Jacobian (up to task scaling, if present)

# Illustrative example - 1

consider a 4R robot with equal links of unitary length




task: end-effector Cartesian position

$$\boldsymbol{x} = (x_{EE,1} \; x_{EE,2})$$

manipulator configuration

$$\boldsymbol{q} = (q_1 \; q_2 \; q_3 \; q_4)$$

differential map

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

desired Cartesian velocity $\dot{\boldsymbol{x}} \in \mathcal{R}^2$

commanded joint velocity $\dot{\boldsymbol{q}} \in \mathcal{R}^4$

task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -lS_1 - lS_{12} - lS_{123} - lS_{1234} & -lS_{12} - lS_{123} - lS_{1234} & -lS_{123} - lS_{1234} & -lS_{1234} \\ lC_1 + lC_{12} + lC_{123} + lC_{1234} & lC_{12} + lC_{123} + lC_{1234} & lC_{123} + lC_{1234} & lC_{1234} \end{pmatrix}$$

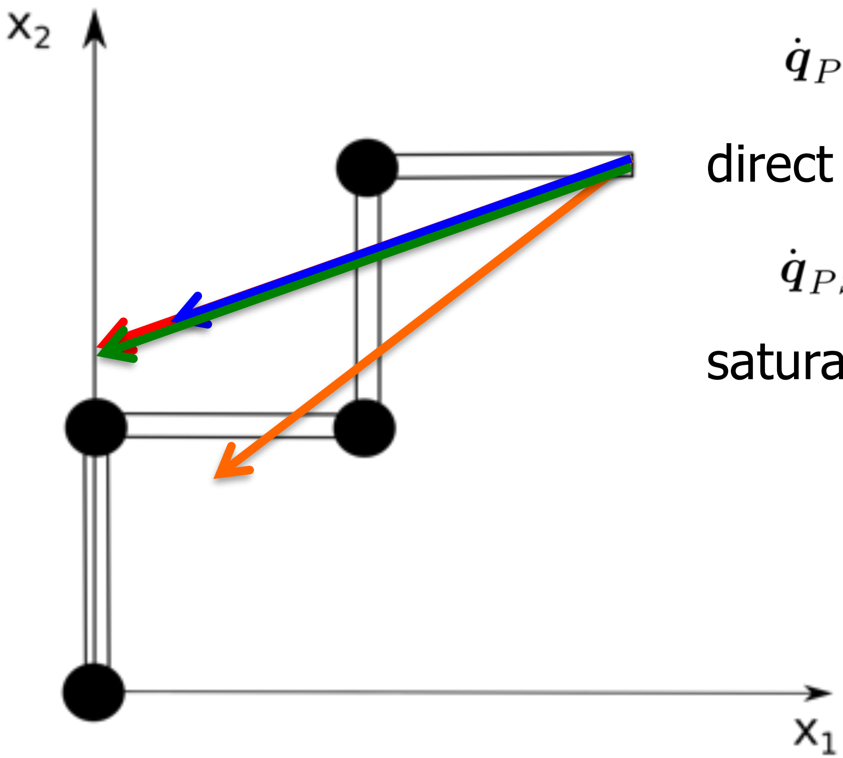
velocity limits $\quad |\dot{q}_i| \leq V_i \, , i = 1 \ldots 4 \qquad \boxed{V_1 = V_2 = 2 \quad V_3 = V_4 = 4 \; [\mathrm{rad/s}]}$

# Illustrative example - 2

current configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/2 & \pi/2 & -\pi/2 \end{pmatrix}^T$

associated Jacobian $\boldsymbol{J} = \begin{pmatrix} J_1 & J_2 & J_3 & J_4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 & 0 \\ 2 & 2 & 1 & 1 \end{pmatrix}$

desired end-effector velocity $\dot{\boldsymbol{x}} = \begin{pmatrix} -4 & -1.5 \end{pmatrix}^T$




$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\dot{\boldsymbol{x}} = \begin{pmatrix} \overset{2.0}{2.4545} & \overset{-2.0}{-2.1364} & 1.2273 & -3.3636 \end{pmatrix}^T$$

direct (velocity =) task scaling? $s = 0.8148$

$$\dot{\boldsymbol{q}}_{PS} = s\boldsymbol{J}^{\#}\dot{\boldsymbol{x}} = \begin{pmatrix} 2.0 & -1.74 & 1.0 & -2.74 \end{pmatrix}^T$$

saturating **only** the most violating velocity? $\dot{q}_1 = V_1 = 2$

$$\dot{\boldsymbol{x}}_{SNS} = \dot{\boldsymbol{x}} - J_1 V_1 = \begin{pmatrix} J_2 & J_3 & J_4 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix}$$

$$\begin{aligned} \dot{\boldsymbol{q}}_{SNS} &= \begin{pmatrix} V_1 & \left[\begin{pmatrix} J_2 & J_3 & J_4 \end{pmatrix}^{\#} \dot{\boldsymbol{x}}_{SNS}\right]^T \end{pmatrix}^T \\ &= \begin{pmatrix} 2 & -1.8333 & 1.8333 & -3.6667 \end{pmatrix}^T \end{aligned}$$

# Joint velocity bounds

joint space **limits**

$$\begin{aligned} Q_{min,i} &\leq q_i \leq Q_{max,i} \\ -V_{max,i} &\leq \dot{q}_i \leq V_{max,i} \qquad i = 1, \dots, n \\ -A_{max,i} &\leq \ddot{q}_i \leq A_{max,i} \end{aligned}$$

joint velocity bounds

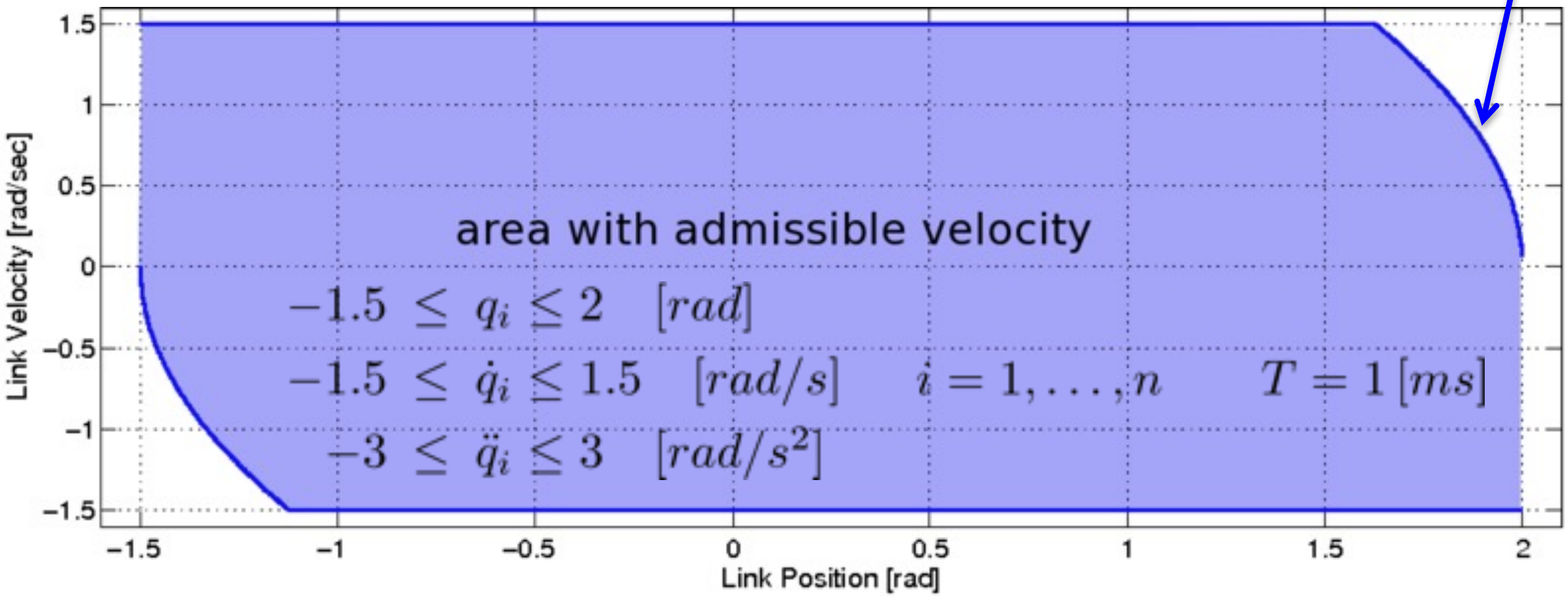
$$\boxed{\dot{\boldsymbol{Q}}_{min}(t_k) \leq \dot{\boldsymbol{q}} \leq \dot{\boldsymbol{Q}}_{max}(t_k)}$$

*conversion:* control sampling (piece-wise constant velocity commands) + max feasible velocities and decelerations to stay/stop within the joint range

$$\dot{Q}_{min,i} = \max\left\{\frac{Q_{min,i} - q_{k,i}}{T}, -V_{max,i}, -\sqrt{2A_{max,i}\left(q_{k,i} - Q_{min,i}\right)}\right\}$$

$$\dot{Q}_{max,i} = \min\left\{\frac{Q_{max,i} - q_{k,i}}{T}, V_{max,i}, \sqrt{2A_{max,i}\left(Q_{max,i} - q_{k,i}\right)}\right\}$$

smooth velocity bound "anticipates" the reaching of a hard limit



# SNS at velocity level

## Algorithm 1

$W = I,\ \dot{q}_N = 0,\ s = 1,\ s^* = 0$

**repeat**

  limit_exceeded = FALSE

  $\dot{\bar{q}} = \dot{q}_N + (JW)^{\#}(\dot{x} - J\dot{q}_N)$

  **if** $\left\{ \begin{array}{l} \exists\, i \in [1:n]: \\ \dot{\bar{q}}_i < \dot{Q}_{min,i}\ .\text{OR}.\ \dot{\bar{q}}_i > \dot{Q}_{max,i} \end{array} \right\}$ **then**

    limit_exceeded = TRUE

    $a = (JW)^{\#}\dot{x}$

    $b = \dot{\bar{q}} - a$

    getTaskScalingFactor($a$, $b$) (***call Algorithm 2***)

    **if** {task scaling factor} $> s^*$ **then**

      $s^* =$ {task scaling factor}

      $W^* = W,\ \dot{q}_N^* = \dot{q}_N$

    **end if**

    $j =$ {the most critical joint}

    $W_{jj} = 0$

    $$\dot{q}_{N,j} = \begin{cases} \dot{Q}_{max,j} & \text{if } \dot{\bar{q}}_j > \dot{Q}_{max,j} \\ \dot{Q}_{min,j} & \text{if } \dot{\bar{q}}_j < \dot{Q}_{min,j} \end{cases}$$

    **if** rank$(JW) < m$ **then**

      $s = s^*,\ W = W^*,\ \dot{q}_N = \dot{q}_N^*$

      $\dot{\bar{q}} = \dot{q}_N + (JW)^{\#}(s\dot{x} - J\dot{q}_N)$

      limit_exceeded = FALSE    (**outputs solution**)

    **end if**

  **end if**

**until** limit_exceeded = TRUE

$\dot{q}_{SNS} = \dot{\bar{q}}$

initialization

$W$ : diagonal matrix with $(j,j)$ element

= 1 if joint $j$ is enabled

= 0 if joint $j$ is disabled

$\dot{q}_N$ : vector with saturated velocities in correspondence of disabled joints

$s$ : current task scale factor

$s^*$: largest task scale factor so far

# SNS at velocity level

## Algorithm 1

$\boldsymbol{W} = \boldsymbol{I}$, $\dot{\boldsymbol{q}}_N = \boldsymbol{0}$, $s = 1$, $s^* = 0$

**repeat**

  limit_exceeded = FALSE

  $\dot{\bar{\boldsymbol{q}}} = \dot{\boldsymbol{q}}_N + (\boldsymbol{J}\boldsymbol{W})^{\#}\,(\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}_N)$

  **if** $\left\{ \begin{array}{l} \exists\, i \in [1:n] : \\ \dot{\bar{q}}_i < \dot{Q}_{min,i} \text{ .OR. } \dot{\bar{q}}_i > \dot{Q}_{max,i} \end{array} \right\}$ **then**

    limit_exceeded = TRUE

    $\boldsymbol{a} = (\boldsymbol{J}\boldsymbol{W})^{\#}\,\dot{\boldsymbol{x}}$

    $\boldsymbol{b} = \dot{\bar{\boldsymbol{q}}} - \boldsymbol{a}$

    getTaskScalingFactor($\boldsymbol{a}$, $\boldsymbol{b}$) (***call Algorithm 2***)

    **if** {task scaling factor} $> s^*$ **then**

      $s^*$ = {task scaling factor}

      $\boldsymbol{W}^* = \boldsymbol{W}$, $\dot{\boldsymbol{q}}_N^* = \dot{\boldsymbol{q}}_N$

    **end if**

    $j$ = {the most critical joint}

    $W_{jj} = 0$

    $\dot{q}_{N,j} = \left\{ \begin{array}{ll} \dot{Q}_{max,j} & \text{if } \dot{\bar{q}}_j > \dot{Q}_{max,j} \\ \dot{Q}_{min,j} & \text{if } \dot{\bar{q}}_j < \dot{Q}_{min,j} \end{array} \right.$

    **if** rank($\boldsymbol{J}\boldsymbol{W}$) $< m$ **then**

      $s = s^*$, $\boldsymbol{W} = \boldsymbol{W}^*$, $\dot{\boldsymbol{q}}_N = \dot{\boldsymbol{q}}_N^*$

      $\dot{\bar{\boldsymbol{q}}} = \dot{\boldsymbol{q}}_N + (\boldsymbol{J}\boldsymbol{W})^{\#}\,(s\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}_N)$

      limit_exceeded = FALSE   (*outputs solution*)

    **end if**

  **end if**

**until** limit_exceeded = TRUE

$\dot{\boldsymbol{q}}_{SNS} = \dot{\bar{\boldsymbol{q}}}$

compute the joint velocity with initialized values

$$\dot{\bar{\boldsymbol{q}}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{x}}$$

check the joint velocity bounds

compute the task scaling factor and the most critical joint

if a larger task scaling factor is obtained, save the current solution

disable the most critical joint by forcing it at its saturated velocity

# SNS at velocity level

## Algorithm 1

$\boldsymbol{W} = \boldsymbol{I}$, $\dot{\boldsymbol{q}}_N = \boldsymbol{0}$, $s = 1$, $s^* = 0$

**repeat**

limit_exceeded = FALSE

$\dot{\bar{\boldsymbol{q}}} = \dot{\boldsymbol{q}}_N + (\boldsymbol{J}\boldsymbol{W})^{\#}\,(\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}_N)$

**if** $\left\{ \begin{array}{l} \exists\, i \in [1\!:\!n] : \\ \dot{\bar{q}}_i < \dot{Q}_{min,i} \text{ .OR. } \dot{\bar{q}}_i > \dot{Q}_{max,i} \end{array} \right\}$ **then**

limit_exceeded = TRUE

$\boldsymbol{a} = (\boldsymbol{J}\boldsymbol{W})^{\#}\,\dot{\boldsymbol{x}}$

$\boldsymbol{b} = \dot{\bar{\boldsymbol{q}}} - \boldsymbol{a}$

getTaskScalingFactor($\boldsymbol{a}$, $\boldsymbol{b}$) (***call Algorithm 2***)

**if** {task scaling factor} $> s^*$ **then**

$s^* =$ {task scaling factor}

$\boldsymbol{W}^* = \boldsymbol{W}$, $\dot{\boldsymbol{q}}_N^* = \dot{\boldsymbol{q}}_N$

**end if**

$j =$ {the most critical joint}

$W_{jj} = 0$

$\dot{q}_{N,j} = \begin{cases} \dot{Q}_{max,j} & \text{if } \dot{\bar{q}}_j > \dot{Q}_{max,j} \\ \dot{Q}_{min,j} & \text{if } \dot{\bar{q}}_j < \dot{Q}_{min,j} \end{cases}$

**if** rank($\boldsymbol{J}\boldsymbol{W}$) $< m$ **then**

$s = s^*$, $\boldsymbol{W} = \boldsymbol{W}^*$, $\dot{\boldsymbol{q}}_N = \dot{\boldsymbol{q}}_N^*$

$\dot{\bar{\boldsymbol{q}}} = \dot{\boldsymbol{q}}_N + (\boldsymbol{J}\boldsymbol{W})^{\#}\,(s\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}_N)$

limit_exceeded = FALSE (**outputs solution**)

**end if**

**end if**

**until** limit_exceeded = TRUE

$\dot{\boldsymbol{q}}_{SNS} = \dot{\bar{\boldsymbol{q}}}$

check if task can be accomplished with the remaining enabled joints

if NOT, use the parameters that allow the largest task scaling factor and exit

repeat until at least one joint limit is exceeded (exit if there is none!)

# Task scaling factor

## Algorithm 2

**function** getTaskScalingFactor($\boldsymbol{a}, \boldsymbol{b}$)

**for** $i = 1 \rightarrow n$ **do**

$S_{min,i} = \left(\dot{Q}_{min,i} - b_i\right)/a_i$

$S_{max,i} = \left(\dot{Q}_{max,i} - b_i\right)/a_i$

**if** $S_{min,i} > S_{max,i}$ **then**

{switch $S_{min,i}$ and $S_{max,i}$}

**end if**

**end for**

$s_{max} = \min_i \{S_{max,i}\}$

$s_{min} = \max_i \{S_{min,i}\}$

the most critical joint $= \operatorname{argmin}_i \{S_{max,i}\}$

**if** $s_{min} > s_{max}$ .OR. $s_{max} < 0$ .OR. $s_{min} > 1$ **then**

task scaling factor $= 0$

**else**

task scaling factor $= s_{max}$

**end if**

called with current $\boldsymbol{a} = (\boldsymbol{JW})^{\#}\dot{\boldsymbol{x}}$ and $\boldsymbol{b} = \left(\boldsymbol{I} - (\boldsymbol{JW})^{\#}\boldsymbol{J}\right)\dot{\boldsymbol{q}}_N \Rightarrow \dot{\boldsymbol{q}}_{SNS} = \boldsymbol{a}s + \boldsymbol{b}$

$\dot{Q}_{min,i} \leq a_i s + b_i \leq \dot{Q}_{max,i}$ with $s \in [0,1]$

restore the correct order of inequalities (possibly modified by the sign of $a_i$)

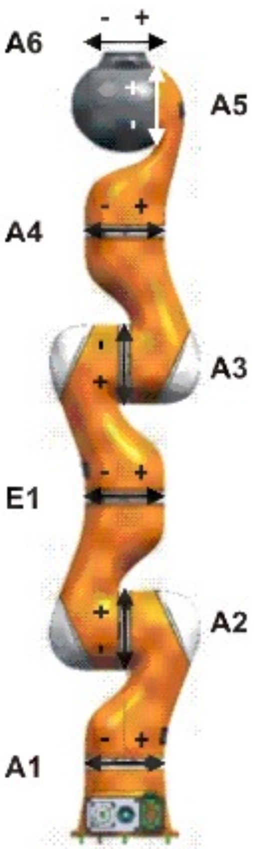
yields the best task scaling factor (i.e., closest to the ideal value = 1) due to the most critical among the currently enabled joint velocity components

no variation of the scaling factor currently used in Algorithm 1 is needed (it will keep the previous $s^*$)

always take the largest value for task scaling …

# Simulation results

| Axis | Range of motion, software-limited | Velocity without payload |
|---|---|---|
| A1 (J1) | +/-170° | 100°/s |
| A2 (J2) | +/-120° | 110°/s |
| E1 (J3) | +/-170° | 100°/s |
| A3 (J4) | +/-120° | 130°/s |
| A4 (J5) | +/-170° | 130°/s |
| A5 (J6) | +/-120° | 180°/s |
| A6 (J7) | +/-170° | 180°/s |




**7-dof KUKA LWR IV**

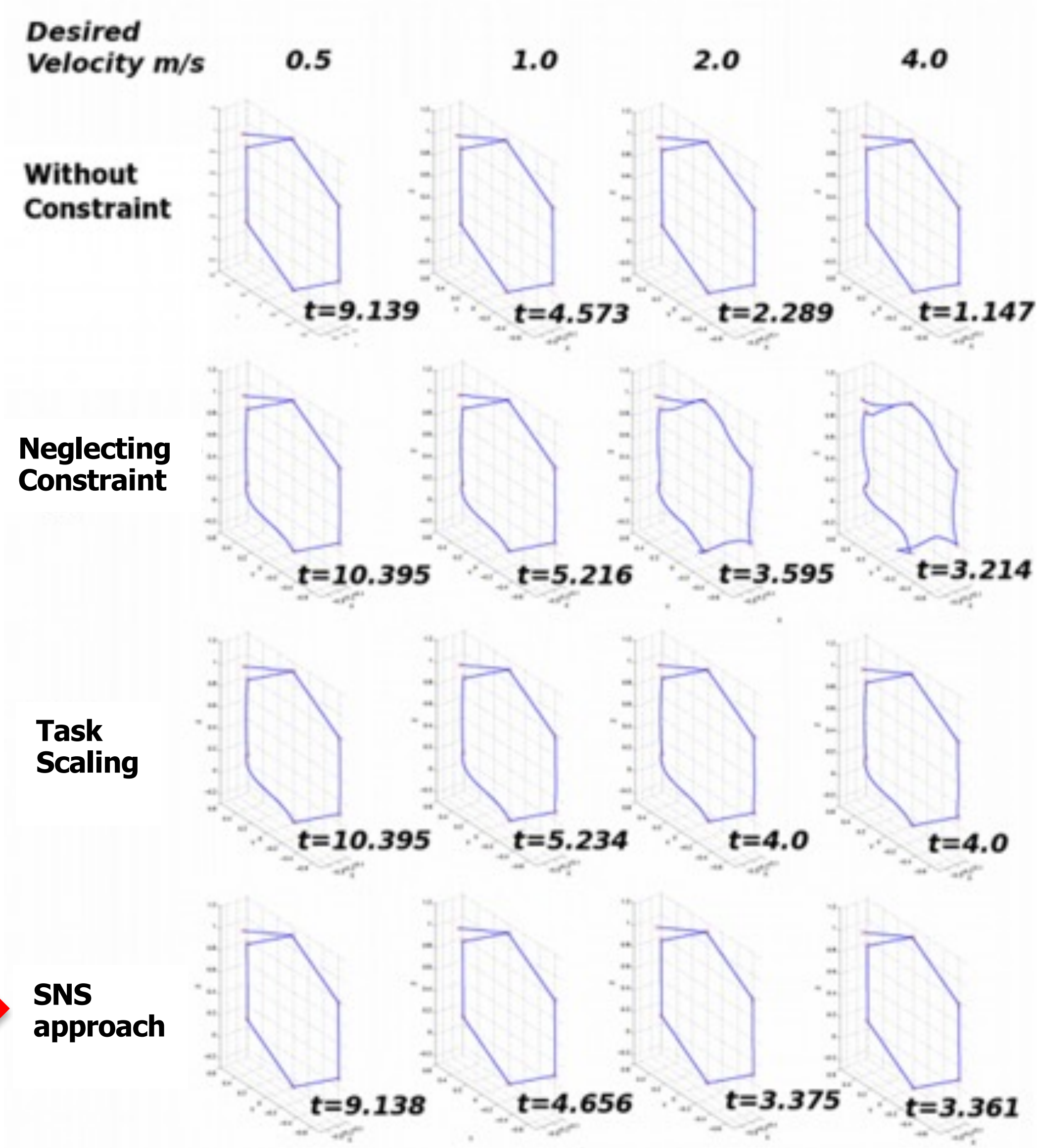
$$\boldsymbol{Q}_{max} = (170, 120, 170, 120, 170, 120, 170) \text{ [deg]}$$

$$\boldsymbol{V}_{max} = (100, 110, 100, 130, 130, 180, 180) \text{ [deg/s]}$$

$$A_{max,i} = 300 \text{ [deg/s}^2\text{]} \quad \forall i = 1 \ldots n$$

$$T = 1 \text{ [ms]}$$

# Simulation results




for increasing $V$

requested task
move the end-effector through six desired Cartesian positions along linear paths with constant speed $V$

$$\dot{\boldsymbol{x}} = V \frac{\boldsymbol{x}_r - \boldsymbol{x}}{\|\boldsymbol{x}_r - \boldsymbol{x}\|}$$

task redundancy degree = 7 – 3 = 4

robot starts at the configuration

$\boldsymbol{q}(0) = (0, 45, 45, 45, 0, 0, 0)$ [deg]

(with a small initial approaching phase)

# Experimental results

KUKA LWR IV with hard joint-space limits

video


IEEE Transactions on Robotics 2015

# Variations of the SNS method

SNS at the **acceleration** command level + consideration of **multiple tasks** with priority

video


IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS) 2012

# Inclusion of hard Cartesian bounds

- SNS at the velocity command level, with hard bounds on joint position, velocity, and acceleration and task scaling factor (just one task is considered here …)
- additional (possibly, time-varying) Cartesian box inequalities on position, velocity, and acceleration of $\boldsymbol{r}$ control points along the structure (including end effector)
- generalized treatment of all bounds in a unified way (conversions like in slide #24)

$$Q_j^{min} \le q_j \le Q_j^{max} \qquad \boldsymbol{P}_{cp,i}^{min} \le \boldsymbol{p}_{cp,i} \le \boldsymbol{P}_{cp,i}^{max}$$

$$V_j^{min} \le \dot{q}_j \le V_j^{max} \qquad \boldsymbol{V}_{cp,i}^{min} \le \dot{\boldsymbol{p}}_{cp,i} \le \boldsymbol{V}_{cp,i}^{max}$$

$$\Lambda_j^{min} \le \ddot{q}_j \le \Lambda_j^{max} \qquad \boldsymbol{\Lambda}_{cp,i}^{min} \le \ddot{\boldsymbol{p}}_{cp,i} \le \boldsymbol{\Lambda}_{cp,i}^{max}$$

$$j = 1, \ldots, n \qquad i = 1, \ldots, r$$

$$\boxed{\boldsymbol{p}_{cp,i} \in \mathbb{R}^{d_i}} \qquad d_i \in \{1, 2, 3\}$$

generalized vector $\boxed{\boldsymbol{a} = \begin{pmatrix} \boldsymbol{q}^T & \boldsymbol{p}_{cp,1}^T & \boldsymbol{p}_{cp,2}^T & \ldots & \boldsymbol{p}_{cp,r}^T \end{pmatrix}^T}$

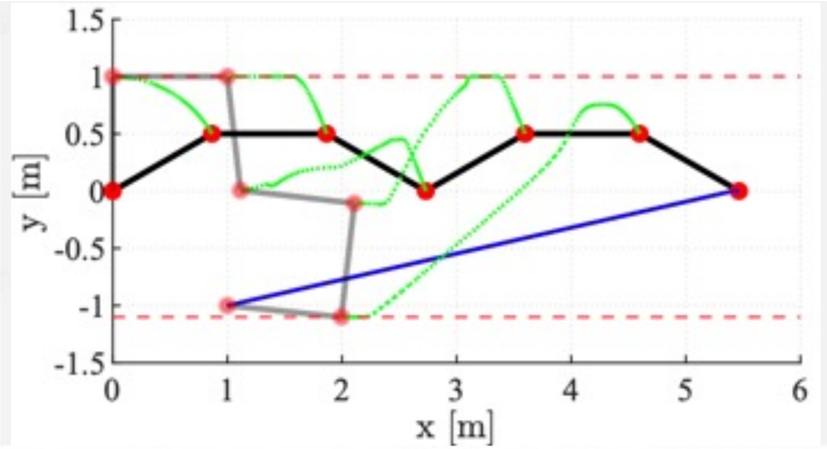
additional processing of $\dot{\boldsymbol{q}}$ in Algorithm 1 (rather than by $\boldsymbol{I}$ only) $\longrightarrow$ $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{J}_{cp,1}^T & \boldsymbol{J}_{cp,2}^T & \ldots & \boldsymbol{J}_{cp,r}^T \end{pmatrix}^T$

$$\Longrightarrow \quad \boldsymbol{B}_{min}(t_k) \le \boxed{\dot{\boldsymbol{a}}(\boldsymbol{q}, \dot{\boldsymbol{q}})} \le \boldsymbol{B}_{max}(t_k)$$ unified joint/Cartesian bounds

# Inclusion of hard Cartesian bounds

simulation on a 6R planar manipulator with $r = 5$ control points (at joints from 2 to 6)

video




$$Q_j^{max} = -Q_j^{min} = 90 \quad [\mathrm{deg}]$$

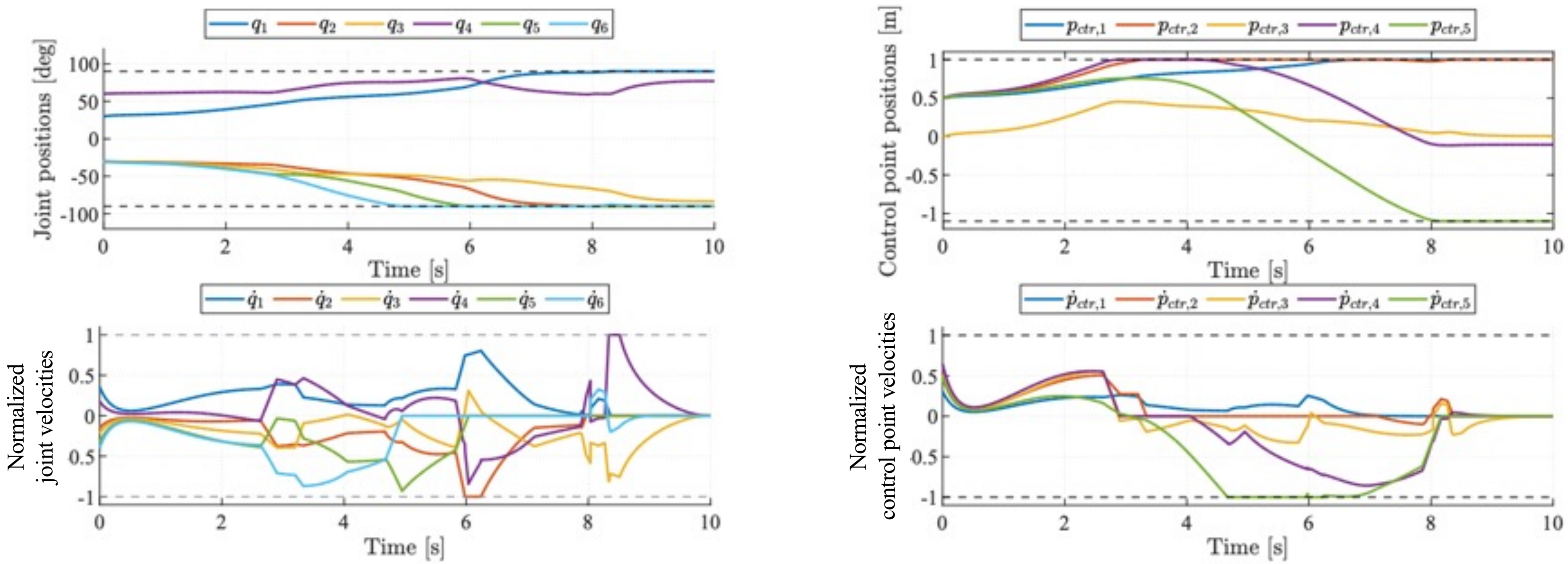
$$V_j^{max} = -V_j^{min} = \frac{90}{\pi} \quad [\mathrm{deg/s}]$$

joint limits on position and velocity ($j = 1,\ldots,6$)

$$P_{cp,i}^{max,y} = 1, \quad P_{cp,i}^{min,y} = -1.1 \ [\mathrm{m}]$$

$$V_{cp,i}^{max,y} = 0.5, \quad V_{cp,i}^{min,y} = -0.5 \ [\mathrm{m/s}]$$

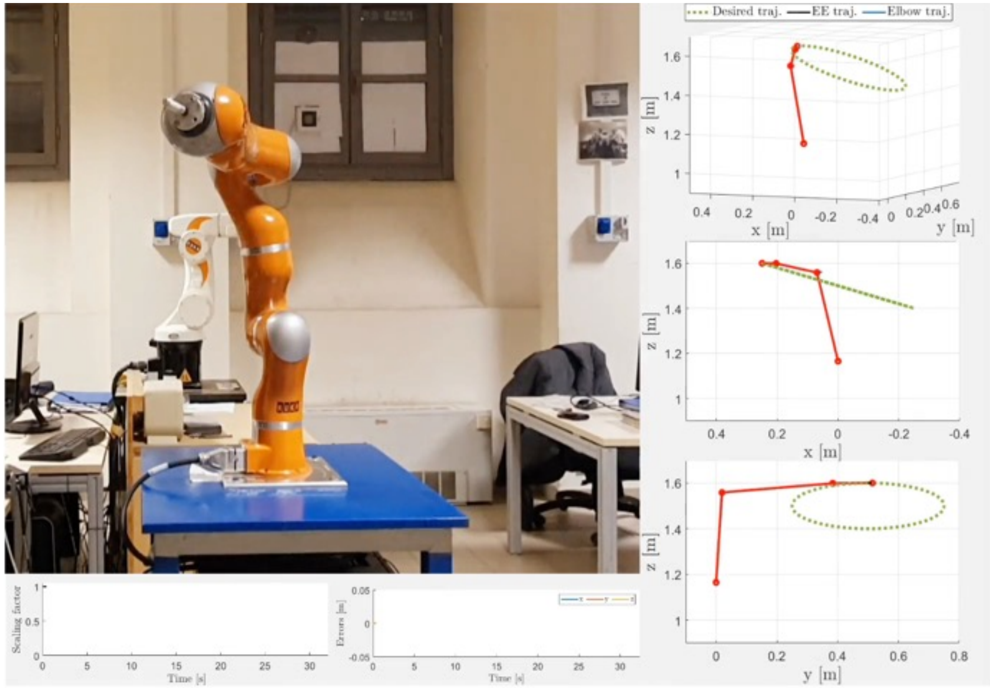
control point limits on position and velocity along $y$ axis ($i = 1,\ldots,5$)




Proc. 3rd Italian Conference on Robotics and Intelligent Machines (I-RIM 2021)

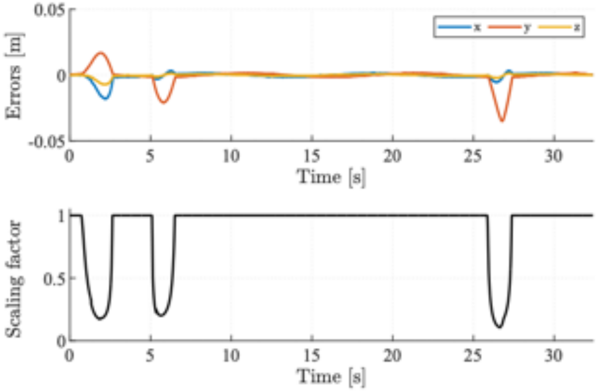
# Inclusion of hard Cartesian bounds

experiment #1 on KUKA LWR IV robot with $r = 1$ control point at the robot elbow (with $d_1 = 2$)




video

task error and scaling




joint limits on position, velocity and acceleration

$$\boldsymbol{Q}^{max} = -\boldsymbol{Q}^{min} = (170\ 105\ 170\ 120\ 170\ 85\ 170)^T \ \ [\text{deg}]$$
$$\boldsymbol{V}^{max} = -\boldsymbol{V}^{min} = (20\ 22\ 20\ 26\ 26\ 36\ 36)^T \ \ [\text{deg/s}]$$
$$\boldsymbol{\Lambda}^{max} = -\boldsymbol{\Lambda}^{min} = (30\ 30\ 30\ 30\ 30\ 30\ 30)^T \ \ [\text{deg/s}^2]$$

control point limits on position, velocity and acceleration

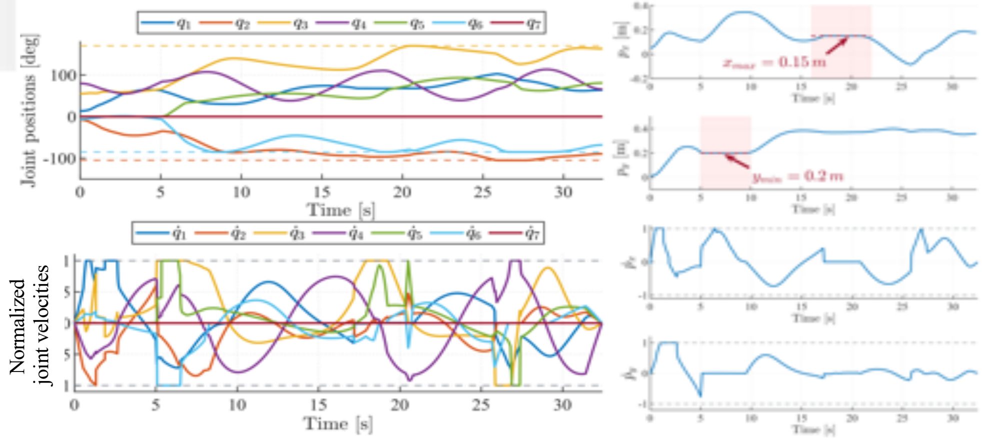
$$-0.1 \le \dot{\boldsymbol{p}}_{cp_x,1} \le 0.1, \quad -0.1 \le \dot{\boldsymbol{p}}_{cp_y,1} \le 0.1 \ [\text{m/s}]$$
$$-0.5 \le \ddot{\boldsymbol{p}}_{cp_x,1} \le 0.5, \quad -0.5 \le \ddot{\boldsymbol{p}}_{cp_y,1} \le 0.5 \ [\text{m/s}^2]$$

permanent

$$\boldsymbol{p}_{cp_x,1} \le 0.15 \ [\text{m}], \quad 16 \le t \le 22 \ [\text{s}]$$
$$\boldsymbol{p}_{cp_y,1} \le 0.2 \ [\text{m}], \quad 5 \le t \le 10 \ [\text{s}]$$

(online) time-varying

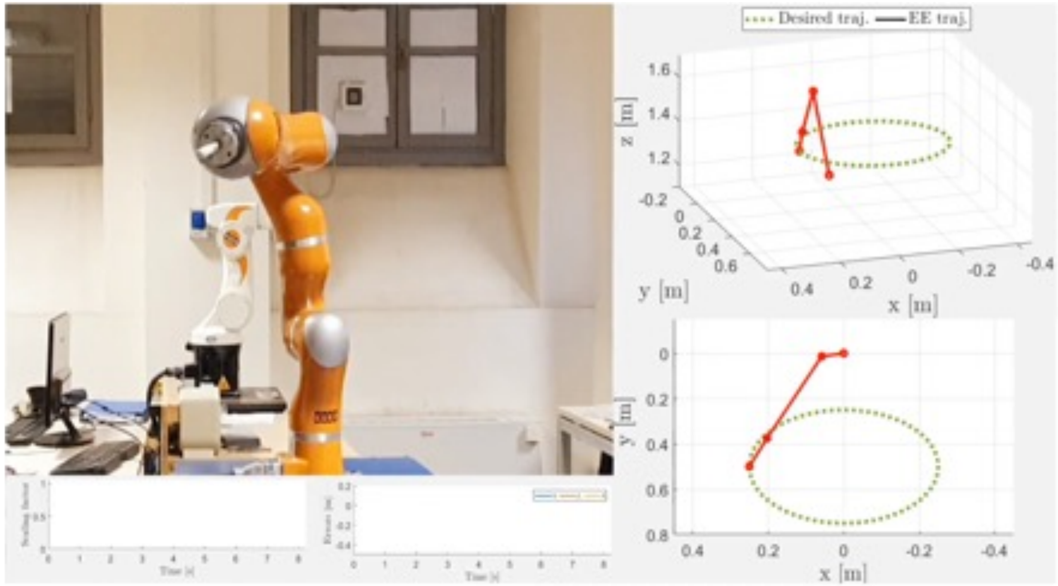
joint-space and Cartesian control point behaviors




IEEE Robotics and Automation Letters, 2022

# Inclusion of hard Cartesian bounds

experiment #2 on KUKA LWR IV robot with $r = 1$ control point at the robot elbow (with $d_1 = 2$)




video

joint limits on position, velocity and acceleration

$$\boldsymbol{Q}^{max} = -\boldsymbol{Q}^{min} = (170\ 120\ 170\ 120\ 170\ 120\ 170)^T \ [\text{deg}]$$

$$\boldsymbol{V}^{max} = -\boldsymbol{V}^{min} = (100\ 110\ 100\ 130\ 130\ 180\ 180)^T \ [\text{deg/s}]$$

$$\boldsymbol{\Lambda}^{max} = -\boldsymbol{\Lambda}^{min} = (300\ 300\ 300\ 300\ 300\ 300\ 300)^T \ [\text{deg/s}^2]$$

much higher than before ⇒ faster motion!

control point limits on position, velocity and acceleration

$$-0.7 \le \dot{\boldsymbol{p}}_{cp_x,1} \le 0.7, \quad -0.7 \le \dot{\boldsymbol{p}}_{cp_y,1} \le 0.7 \ [\text{m/s}]$$

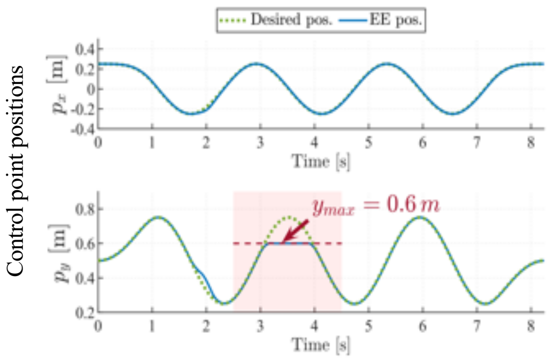
$$-1.5 \le \ddot{\boldsymbol{p}}_{cp_x,1} \le 1.5, \quad -1.5 \le \ddot{\boldsymbol{p}}_{cp_y,1} \le 1.5 \ [\text{m/s}^2]$$

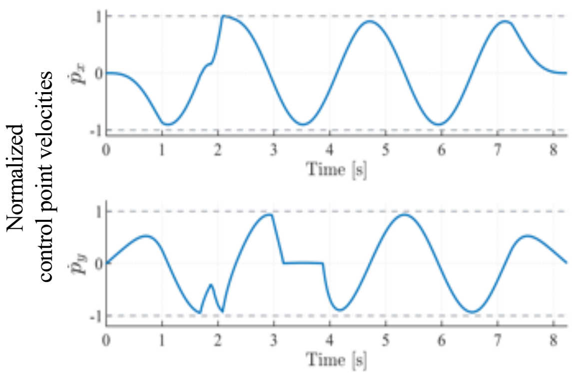
permanent

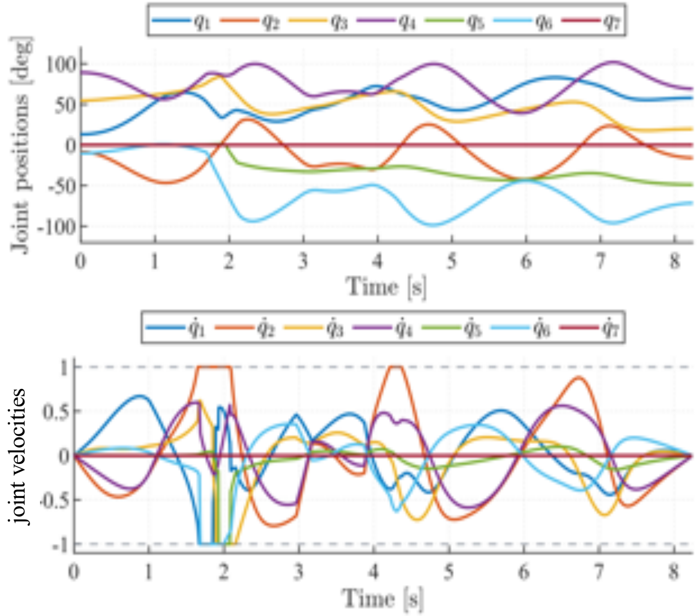
$$\boldsymbol{p}_{cp_y,1} \le 0.6 \ [\text{m}], \quad 2.5 \le t \le 4.5 \ [\text{s}]$$

(online) time-varying

circle will be "cut" during second turn!



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# Appendix A - Recursive Task Priority

## proof of recursive expression for null-space projector

$$\boxed{P_{A,k} = P_{A,k-1} - (J_k P_{A,k-1})^{\#} J_k P_{A,k-1}}$$

- proof based on a result on pseudoinversion of partitioned matrices (Cline: J. SIAM 1964)

$$\begin{pmatrix} A \\ B \end{pmatrix}^{\#} = \begin{pmatrix} A^{\#} - TBA^{\#} & T \end{pmatrix} \qquad T = E^{\#} + X\left(I - EE^{\#}\right) \quad X \text{ is irrelevant here}$$

$$E = B(I - A^{\#}A)$$

- (i) $P_{A,k} = I - J^{\#}_{A,k} J_{A,k} = I - \begin{pmatrix} J_{A,k-1} \\ J_k \end{pmatrix}^{\#} \begin{pmatrix} J_{A,k-1} \\ J_k \end{pmatrix}$

$$= I - \begin{pmatrix} J^{\#}_{A,k-1} - TJ_k J^{\#}_{A,k-1} & T \end{pmatrix} \begin{pmatrix} J_{A,k-1} \\ J_k \end{pmatrix}$$

$$= I - J^{\#}_{A,k-1} J_{A,k-1} + TJ_k J^{\#}_{A,k-1} J_{A,k-1} - TJ_k$$

$$= P_{A,k-1} - TJ_k P_{A,k-1}$$

- (ii) $T = (J_k P_{A,k-1})^{\#} + X\left(I - (J_k P_{A,k-1})(J_k P_{A,k-1})^{\#}\right)$

$$\Rightarrow \quad TJ_k P_{A,k-1} = (J_k P_{A,k-1})^{\#} J_k P_{A,k-1}$$

- (i) + (ii) ⇒ Q.E.D.

- if $k$-th task is scalar

$$J_k = \text{single row } j_k^T$$

$$P_{A,k} = P_{A,k-1} - \frac{P_{A,k-1} j_k j_k^T P_{A,k-1}}{\|P_{A,k-1} j_k\|^2}$$

(Greville formula)