Robotics 2

Remote Exam – July 15, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, made by a 3R planar arm mounted on a rail. The robot has the last three links of equal length ℓ. The generalized coordinates $\boldsymbol{q} \in \mathbb{R}^4$ to be used are also shown. Determine the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the dynamic model of this robot (if needed, define symbolically any missing parameters).

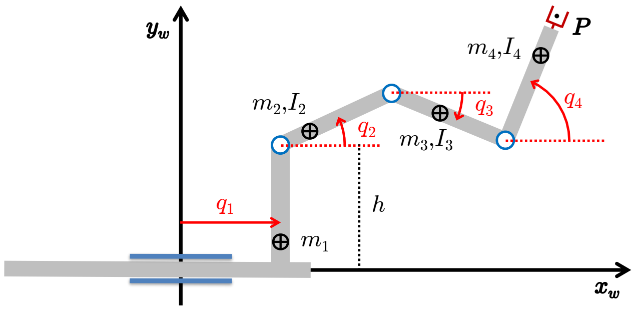

Figure 1: A 4-dof planar robot with generalized coordinates $\boldsymbol{q}$ and relevant parameters.

Exercise #2

For the same robot in Fig. 1, assume $\ell = h = 1$ [m] and consider the following tasks, to be executed using a kinematic control scheme with joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^4$.

Task 1. Trace with the end-effector counterclockwise a circle of radius $R = 3$ [m], centered at $\boldsymbol{C} = (7, 0)$, starting from point $\boldsymbol{P}_0 = (10, 0)$ and with constant speed $v = 1$ [m/s].

Task 2. Keep the second link always horizontal ($q_2(t) = 0$).

Define the augmented Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ for both tasks 1 and 2. Choose a suitable initial robot configuration so as to stay at time $t = 0$ in $\boldsymbol{P}_0$ and compute there the minimum joint velocity norm solution that realizes both tasks simultaneously. Determine the first point $\boldsymbol{P}_s$ on the circular path where an algorithmic singularity of $\boldsymbol{J}_A(\boldsymbol{q})$ necessarily occurs. In that situation, compute the minimum joint velocity norm solution that realizes the first task only. Will the execution of the second task be relaxed or not?

Exercise #3

A 6-dof robot should hold firmly with its three-fingered gripper a cylindric payload, and move it along a desired path on a frictionless plane with one of its bases in full contact with the plane, as shown in Fig. 2. Define an associated task frame where the natural (geometric) constraints and the artificial (control) constraints of this hybrid task can be defined and realized in a decoupled way. Where reasonable, provide also values for the control references.

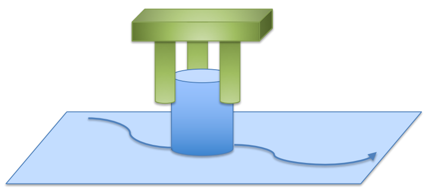

Figure 2: The hybrid task of moving along a path a cylinder in contact with a planar surface.

Exercise #4

During the accurate execution of a smooth joint trajectory $\boldsymbol{q}_d(t)$ lasting $T = 3$ [s] with the 2R planar robot shown in Fig. 3 moving under gravity, the maximum torques of the two joints exceed at some instants their bounds, as given by $|\tau_i| \le \tau_{max,i}$, $i = 1, 2$. We have in particular

$$\tau_1(t_1) = \max_{t\in[0,T]} \tau_1(t) = 140 > 100 = \tau_{max,1}, \qquad \tau_2(t_2) = \max_{t\in[0,T]} \tau_2(t) = 25 > 20 = \tau_{max,2} \qquad \text{[Nm]}.$$

The robot links have equal length $\ell = 0.5$ [m] and equal, uniformly distributed mass $m = 5$ [kg]. The robot configurations at the time instants $t = t_1$ and $t = t_2$ are

$$\boldsymbol{q}(t_1) = \begin{pmatrix} \pi/4 & 0 \end{pmatrix}^T, \qquad \boldsymbol{q}(t_2) = \begin{pmatrix} -\pi/4 & 3\pi/4 \end{pmatrix}^T \qquad \text{[rad]}.$$

In order to recover motion feasibility, a uniform trajectory scaling is used. What will be the minimum feasible motion time $T' = kT$ thus obtained? What are the values of the new joint torques $\tau_1(kt_1)$ and $\tau_2(kt_2)$?

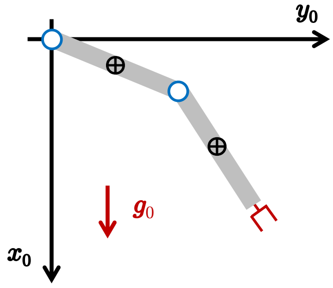

Figure 3: A 2R robot moving under gravity. Joint variables are defined by the D-H convention.

Exercise #5

Consider again the 2R robot in Fig. 3, with the same definition of joint variables and using the same kinematic and dynamic parameters. The robot is initially at rest at $t = 0$ in $\boldsymbol{q}(0) = \boldsymbol{q}_0$. Provide the explicit expressions of all terms in the following three feedback control laws, each achieving its own objective.

a. Global exponential stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, with decoupled transient evolutions of the position errors $e_i(t) = q_{d,i} - q_i(t)$, $i = 1, 2$, of the form $e_1(t) = e_1(0)\,(1 + 5t)\,\exp(-5t)$ and $e_2(t) = e_2(0)\,(2\exp(-5t) - \exp(-10t))$.

b. Global asymptotic stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, without knowledge of the robot inertia matrix.

c. Exponential stabilization of the end-effector position $\boldsymbol{p} = \boldsymbol{p}_d \in \mathbb{R}^2$ with zero velocity $\dot{\boldsymbol{p}} = \boldsymbol{0}$, up to kinematic singularities.

[180 minutes (3 hours); open books]

Solution

July 15, 2020

Exercise #1

Note first that we are using the *absolute* angles w.r.t. to the $\boldsymbol{x}_0$-axis for the orientation of the second to the fourth link (thus, not the Denavit-Hartenberg relative angles). Also, denote by $d_{ci} > 0$ the distance of the center of mass of link i from joint i, for $i = 2, 3, 4$. The individual contributions to the kinetic energy of this 4-dof planar robot are computed as follows.

$$T_1 = \tfrac{1}{2} m_1 \dot{q}_1^2$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_{c2}\cos q_2 \\ h + d_{c1}\sin q_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2}\sin q_2\,\dot{q}_2 \\ d_{c2}\cos q_2\,\dot{q}_2 \end{pmatrix}$$

$$T_2 = \tfrac{1}{2} I_2 \dot{q}_2^2 + \tfrac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2\,\dot{q}_2^2 - 2 d_{c2}\sin q_2\,\dot{q}_1\dot{q}_2\right)$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \ell\cos q_2 + d_{c3}\cos q_3 \\ h + \ell\sin q_2 + d_{c3}\sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - \ell\sin q_2\,\dot{q}_2 - d_{c3}\sin q_3\,\dot{q}_3 \\ \ell\cos q_2\,\dot{q}_2 + d_{c3}\cos q_3\,\dot{q}_3 \end{pmatrix}$$

$$T_3 = \tfrac{1}{2} I_3 \dot{q}_3^2 + \tfrac{1}{2} m_3 \left(\dot{q}_1^2 + \ell^2\,\dot{q}_2^2 + d_{c3}^2\,\dot{q}_3^2 - 2\ell\sin q_2\,\dot{q}_1\dot{q}_2 - 2 d_{c3}\sin q_3\,\dot{q}_1\dot{q}_3 + 2 d_{c3}\ell\cos(q_3 - q_2)\,\dot{q}_2\dot{q}_3\right)$$

$$\boldsymbol{p}_{c4} = \begin{pmatrix} q_1 + \ell(\cos q_2 + \cos q_3) + d_{c4}\cos q_4 \\ h + \ell(\sin q_2 + \sin q_3) + d_{c4}\sin q_4 \end{pmatrix} \Rightarrow \boldsymbol{v}_{c4} = \begin{pmatrix} \dot{q}_1 - \ell(\sin q_2\,\dot{q}_2 + \sin q_3\,\dot{q}_3) - d_{c4}\sin q_4\,\dot{q}_4 \\ \ell(\cos q_2\,\dot{q}_2 + \cos q_3\,\dot{q}_3) + d_{c4}\cos q_4\,\dot{q}_4 \end{pmatrix}$$

$$\begin{aligned} T_4 = {}& \tfrac{1}{2} I_4 \dot{q}_4^2 + \tfrac{1}{2} m_4 \left(\dot{q}_1^2 + \ell^2(\dot{q}_2^2 + \dot{q}_3^2 + 2\cos(q_3 - q_2)\,\dot{q}_2\dot{q}_3) + d_{c4}^2\,\dot{q}_4^2\right. \\ & \left. -2\ell(\sin q_2\,\dot{q}_2 + \sin q_3\,\dot{q}_3)\dot{q}_1 - 2 d_{c4}\sin q_4\,\dot{q}_1\dot{q}_4 + 2 d_{c4}\ell(\cos(q_4 - q_2)\,\dot{q}_2 + \cos(q_4 - q_3)\,\dot{q}_3)\,\dot{q}_4\right). \end{aligned}$$

From

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \boldsymbol{M}_{ij}(\boldsymbol{q})\,\dot{q}_i\dot{q}_j,$$

the elements $\boldsymbol{M}_{ij} = \boldsymbol{M}_{ji}$ of the 4×4 symmetric inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of this robot are

$$\begin{aligned} \boldsymbol{M}_{11} &= m_1 + m_2 + m_3 + m_4 \\ \boldsymbol{M}_{12} &= -(m_2 d_{c2} + (m_3 + m_4)\ell)\sin q_2 \\ \boldsymbol{M}_{13} &= -(m_3 d_{c3} + m_4\ell)\sin q_3 \\ \boldsymbol{M}_{14} &= -m_4 d_{c4}\sin q_4 \\ \boldsymbol{M}_{22} &= I_2 + m_2 d_{c2}^2 + (m_3 + m_4)\ell^2 \\ \boldsymbol{M}_{23} &= (m_3 d_{c3} + m_4\ell)\ell\cos(q_3 - q_2) \\ \boldsymbol{M}_{24} &= m_4 d_{c4}\ell\cos(q_4 - q_2) \\ \boldsymbol{M}_{33} &= I_3 + m_3 d_{c3}^2 + m_4\ell^2 \\ \boldsymbol{M}_{34} &= m_4 d_{c4}\ell\cos(q_4 - q_3) \\ \boldsymbol{M}_{44} &= I_4 + m_4 d_{c4}^2. \end{aligned}$$

Exercise #2

The kinematics of the first task of dimension $m_1 = 2$ is given by

$$\boldsymbol{r}_1 = \boldsymbol{p} = \begin{pmatrix} q_1 + \ell(\cos q_2 + \cos q_3 + \cos q_4) \\ h + \ell(\sin q_2 + \sin q_3 + \sin q_4) \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q}),$$

with Jacobian

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -\ell \sin q_2 & -\ell \sin q_3 & -\ell \sin q_4 \\ 0 & \ell \cos q_2 & \ell \cos q_3 & \ell \cos q_4 \end{pmatrix},$$

while the kinematics of the second task of dimension $m_2 = 1$ is given just by

$$r_2 = q_2 = f_2(\boldsymbol{q}),$$

with Jacobian

$$\boldsymbol{J}_2 = \frac{\partial f_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix},$$

The augmented Jacobian is then

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} 1 & -\ell \sin q_2 & -\ell \sin q_3 & -\ell \sin q_4 \\ 0 & \ell \cos q_2 & \ell \cos q_3 & \ell \cos q_4 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $\boldsymbol{J}_A(\boldsymbol{q})$ is singular, i.e., $\operatorname{rank}(\boldsymbol{J}_A(\boldsymbol{q})) < 3 = m_A (= m_1 + m_2)$, if and only if $\cos q_3 = \cos q_4 = 0$. This happens when the third and fourth link are aligned (or folded) along the $\boldsymbol{y}_w$ direction.
With reference to Fig. 4, the augmented task requires

$$\boldsymbol{r}_{1d}(t) = \boldsymbol{C} + R \begin{pmatrix} \cos \dfrac{vt}{R} \\ \sin \dfrac{vt}{R} \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{r}}_{1d}(t) = v \begin{pmatrix} -\sin \dfrac{vt}{R} \\ \cos \dfrac{vt}{R} \end{pmatrix}$$

with $\boldsymbol{r}_{1d}(0) = \boldsymbol{P}_0$, $\dot{\boldsymbol{r}}_{1d}(0) = \begin{pmatrix} 0 & v \end{pmatrix}^T$ and $\|\dot{\boldsymbol{r}}_{1d}(t)\| = v$, as well as

$$r_{2d}(t) = 0 \quad \Rightarrow \quad \dot{r}_{2d} = 0.$$

Setting now $\ell = h = 1$, in order to be consistent with the augmented task at the initial time $t = 0$, there will be multiple robot configurations (q_1, q_3, q_4) that satisfy the desired end-effector positioning with $q_2 = 0$ (second link horizontal and pointing to the right):

$$\begin{pmatrix} q_1 + 1 + \cos q_3 + \cos q_4 \\ \sin q_3 + \sin q_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \boldsymbol{P}_0$$

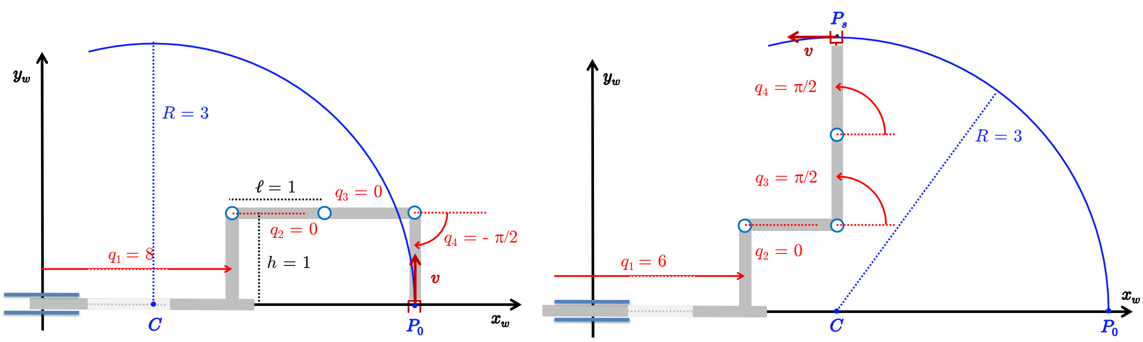

Figure 4: The 4-dof planar robot in the initial configuration $\boldsymbol{q}_0$ [left] and in the singular configuration $\boldsymbol{q}_s$ [right] for the two tasks of tracing the circle with its end effector (task 1) while keeping the second link horizontal (task 2).

A simple choice is to pick $q_1 = 8$ [m], $q_3 = 0$, $q_4 = -\pi/2$ [rad], as in Fig. 4 [left]. The configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0) = \begin{pmatrix} 8 & 0 & 0 & -\pi/2 \end{pmatrix}^T$ is not singular. Accordingly, any augmented task velocity $\dot{\boldsymbol{r}}_d \in \mathbb{R}^3$ can be instantaneously realized (actually in a infinite number of ways). The minimum norm joint velocity solution is obtained using pseudoinversion ($\boldsymbol{J}_A^\# = \boldsymbol{J}_A^T(\boldsymbol{J}_A\boldsymbol{J}_A^T)^{-1}$):

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{J}_A^\#(\boldsymbol{q}_0)\dot{\boldsymbol{r}}_d(0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^\# \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0.5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

As a result, only the third joint moves, rotating counterclockwise. As shown in Fig. 4 [right], the first point on the circle where the augmented task necessarily encounters a singularity is at $\boldsymbol{P}_s = (\boldsymbol{C}_x, \boldsymbol{C}_y + R) = (7, 3)$. The robot arrives there at some instant $t = t_s > 0$ and can satisfy the positional/orientation tasks in only one configuration $\boldsymbol{q}_s = \boldsymbol{q}(t_s) = \begin{pmatrix} 6 & 0 & \pi/2 & \pi/2 \end{pmatrix}^T$, which is indeed singular. Note that this is a true *algorithmic singularity*, since both tasks are full rank ($\text{rank}\,(\boldsymbol{J}_1(\boldsymbol{q}_s)) = 2$, $\text{rank}\,(\boldsymbol{J}_2) = 1$) but $\text{rank}\,(\boldsymbol{J}_A(\boldsymbol{q}_s)) = 2 < 3 = m_A$. Indeed, one can still compute the pseudoinverse solution, which provides

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}_A^\#(\boldsymbol{q}_s)\dot{\boldsymbol{r}}_d(t_s) = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^\# \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3333 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ -0.3333 & 0 & 0 \\ -0.3333 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

The prismatic joint retracts, while joints 3 and 4 will rotate counterclockwise. When evaluating the execution of the augmented task with this joint velocity, we find

$$\boldsymbol{J}_A(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_d(t_s)!$$

Thus, the entire velocity task is still satisfied. In fact, despite the loss of rank of the augmented Jacobian, it is easy to see that $\dot{\boldsymbol{r}}_d(t_s) \in \mathcal{R}\{\boldsymbol{J}_A(\boldsymbol{q}_s)\}$. The pseudoinverse joint velocity returns then the exact solution also in this case. We note finally that the robot will not be able to trace the entire circle, being the lower part outside its workspace.

Exercise #3

With reference to Fig. 5, we define the task frame with axis $\boldsymbol{z}_t$ normal to the plane of motion and passing through the center of the cylinder base, and axis $\boldsymbol{x}_t$ tangential to the path on the plane. The natural constraints are then

$$f_x = 0, \qquad f_y = 0, \qquad v_z = 0, \qquad \omega_x = 0, \qquad \omega_y = 0, \qquad \mu_z = 0,$$

in which we neglected any friction effect at the contact. The complementary artificial constraints are

$$v_x = v_{x,d}(t) > 0, \qquad v_y = 0, \qquad f_z = f_{z,d} \neq 0, \qquad \mu_x = 0, \qquad \mu_y = 0, \qquad \omega_z = \omega_{z,d}.$$

The value of the velocity v_y (normal to the path) is chosen to be zero, signifying that the robot end-effector/payload should strictly follow the path on the plane. A non-zero $f_{z,d}$ can be chosen

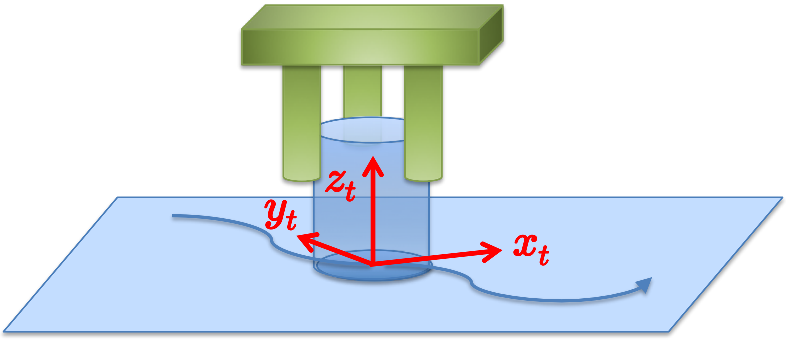

Figure 5: The task frame assignment for the contact motion of a cylinder following a path on a frictionless plane.

so as to enforce full surface contact with the base, despite of the presence of friction (and other disturbances) in the real world. The two reaction torques μ_x and μ_y are set to zero, in order not to stress the object while in contact. Finally, $\omega_{z,d}$ can be set to zero or not, depending on whether the cylinder should keep its orientation or rotate around its major axis while the center of its base is following the path traced on the plane.

Exercise #4

In this exercise, we just need to derive the gravity term in the dynamic model of the 2R planar robot. No information is required in fact on the inertial terms. Using the Denavit-Hartenberg coordinates, the $\boldsymbol{q} = \mathbf{0}$ configuration will correspond to the robot arm being stretched downward along the $\boldsymbol{x}_0$-axis, the configuration of minimum potential energy. Therefore, being $m_1 = m_2 = m$ and $d_{c1} = d_{c2} = \ell/2$, the potential energy due to gravity is

$$\begin{aligned} U = U_1 + U_2 &= -m_1 g_0\, d_{c1} \cos q_1 - m_2 g_0 \left(\ell \cos q_1 + d_{c2} \cos(q_1 + q_2)\right) \\ &= -m g_0 \ell \left(\frac{3}{2} \cos q_1 + \frac{1}{2} \cos(q_1 + q_2) \right), \end{aligned}$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} m g_0 \ell \left(\frac{3}{2} \sin q_1 + \frac{1}{2} \sin(q_1 + q_2) \right) \\ m g_0 \frac{\ell}{2} \sin(q_1 + q_2) \end{pmatrix} = \begin{pmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{pmatrix}. \tag{1}$$

Setting $m = 5$ [kg], $\ell = 0.5$ [m] and $g_0 = 9.81$ [m/s^2], by evaluating (1) at $\boldsymbol{q}(t_1) = (\pi/4, 0)$ and $\boldsymbol{q}(t_2) = (-\pi/4, 3\pi/4)$ we obtain the gravity torques at the two joints

$$g_1(\boldsymbol{q}(t_1)) = 34.6836 \;\; [\text{Nm}] \qquad \text{and} \qquad g_2(\boldsymbol{q}(t_2)) = 12.2625 \;\; [\text{Nm}].$$

The uniform time scaling factor $k > 1$ needed to recover feasibility of the entire motion is computed from

$$k_1 = \sqrt{\frac{\tau_1(t_1) - g_1(\boldsymbol{q}(t_1))}{\tau_{max,1} - g_1(\boldsymbol{q}(t_1))}} = 1.2698, \qquad k_2 = \sqrt{\frac{\tau_2(t_2) - g_2(\boldsymbol{q}(t_2))}{\tau_{max,2} - g_2(\boldsymbol{q}(t_2))}} = 1.2830,$$

as

$$k = \max\{k_1, k_2\} = 1.2830\,(= k_2).$$

Thus, the second joint is the one with higher relative violation of the torque limit (once gravity is removed). The motion time is then increased from $T = 3$ [s] to the new value $T' = kT = 3.8491$ [s], which is the minimum feasible one under uniform time scaling. The values of the new joint torques (expressed in [Nm]) at the scaled instants $t_1' = kt_1$ and $t_2' = kt_2$ are computed as

$$\tau_1(t_1') = \frac{\tau_1(t_1) - g_1(\boldsymbol{q}(t_1))}{k^2} + g_1(\boldsymbol{q}(t_1) = 98.6589 < 100 = \tau_{max,1},$$

$$\tau_2(t_2') = \frac{\tau_2(t_2) - g_2(\boldsymbol{q}(t_2))}{k^2} + g_2(\boldsymbol{q}(t_2) = 20 = \tau_{max,2}.$$

As expected, the second joint torque will be in saturation at the scaled instant t_2'.

Exercise #5

The three requested motion tasks are all regulation problems. The dynamic terms needed for the various feedback control laws are listed first. We make reference to the 2R robot in Fig. 3, with the same definition of joint variables and using the same parameters. The inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix},$$

with $a_1 = I_1 + I_2 + \frac{3}{2}\, m\ell^2$, $a_2 = \frac{1}{2}\, m\ell^2$, $a_3 = I_2 + \frac{1}{4} m\ell^2$. The Coriolis and centrifugal terms are

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 \sin q_2 (\dot{q}_2 + 2\dot{q}_1)\dot{q}_2 \\ a_2 \sin q_2 \dot{q}_1^2 \end{pmatrix}.$$

The gravity vector has been already computed in Exercise #4, and is rewritten here as

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_4 \sin q_1 + a_5 \sin(q_1 + q_2) \\ a_5 \sin(q_1 + q_2) \end{pmatrix}$$

with $a_4 = \frac{1}{2}\, m g_0 \ell$, $a_5 = \frac{1}{2}\, m g_0 \ell$. From the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$, the robot Jacobian that maps the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ to the velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^2$ of the end effector is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell(\sin q_1 + \sin(q_1 + q_2)) & -\ell \sin(q_1 + q_2) \\ \ell(\cos q_1 + \cos(q_1 + q_2)) & \ell \cos(q_1 + q_2) \end{pmatrix}.$$

Finally, we shall need also the time derivative of the Jacobian matrix, namely

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(\cos q_1\, \dot{q}_1 + \cos(q_1 + q_2)\,(\dot{q}_1 + \dot{q}_2)) & -\ell \cos(q_1 + q_2)\,(\dot{q}_1 + \dot{q}_2) \\ -\ell\,(\sin q_1\, \dot{q}_1 + \sin(q_1 + q_2)\,(\dot{q}_1 + \dot{q}_2)) & -\ell \sin(q_1 + q_2)\,(\dot{q}_1 + \dot{q}_2) \end{pmatrix}.$$

a. Global exponential stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, with decoupled transient evolutions of the position errors $e_i(t) = q_{d,i} - q_i(t)$, $i = 1, 2$, of the form $e_1(t) = e_1(0)\,(1 + 5t)\,\exp(-5t)$ and $e_2(t) = e_2(0)\,(2\exp(-5t) - \exp(-10t))$.

This is obtained by feedback linearization control in the joint space:

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \quad \text{with } \boldsymbol{a} = -\boldsymbol{K}_D\, \dot{\boldsymbol{q}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}),$$

with $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ and both diagonal. The desired error transients are obtained by choosing suitable gains in the linear and decoupled second-order dynamics

$$\ddot{e}_i + K_{D,i}\dot{e}_i + K_{P,i}e_i = 0, \qquad i = 1, 2, \tag{2}$$

for the two position errors $e_i(t) = q_{d,i} - q_i(t)$. For joint 1, substitute $e_1(t) = e_1(0)\,(1+5t)\,\exp(-5t)$ and its first and second time derivatives in (2):

$$e_1(0)\,(-25+125t)\,\exp(-5t) - K_{D,1}\,e_1(0)\,25t\exp(-5t) + K_{P,1}\,e_1(0)\,(1+5t)\,\exp(-5t) = 0.$$

Since $e_1(0)\exp(-5t) \neq 0$ for any finite $t \geq 0$, this common factor can be eliminated so as to obtain

$$(-25+125t) - 25K_{D,1}t + (1+5t)\,K_{P,1} = 0.$$

By the principle of polynomial identity (w.r.t. the powers of t), this implies

$$125 - 25\,K_{D,1} + 5\,K_{P,1} = 0, \quad -25 + K_{P,1} = 0 \qquad \Rightarrow \qquad K_{P,1} = 25,\ K_{D,1} = 10.$$

Moreover, transforming eq. (2) for $i = 1$ in the Laplace domain and using these values leads to

$$\left(s^2 + 10s + 25\right) e_1(s) = (s+5)^2 e_1(s) = 0,$$

namely, the error dynamics at the first joint is characterized by two real and coincident negative eigenvalues in -5.

We proceed similarly for joint 2. Substitute $e_2(t) = e_2(0)\,(2\exp(-5t) - \exp(-10t))$ and its first and second time derivatives in (2):

$$\begin{aligned} e_2(0)\,(50\exp(-5t) - 100\exp(-10t)) + K_{D,2}\,e_2(0)\,(-10\exp(-5t) + 10\exp(-10t)) \\ + K_{P,2}\,e_2(0)\,(2\exp(-5t) - \exp(-10t)) = 0. \end{aligned}$$

Being $e_2(0) \neq 0$, in order for this expression to vanish identically at all times $t \geq 0$, we should zero the coefficients multiplying the two different exponentials $\exp(-5t)$ and $\exp(-10t)$. This yields

$$50 - 10\,K_{D,2} + 2\,K_{P,2} = 0, \quad -100 + 10\,K_{D,2} - K_{P,2} = 0 \qquad \Rightarrow \qquad K_{P,2} = 50,\ K_{D,2} = 15.$$

Moreover, transforming eq. (2) for $i = 2$ in the Laplace domain and using these values leads to

$$\left(s^2 + 15s + 50\right) e_2(s) = (s+5)(s+10)\,e_2(s) = 0,$$

namely, the error dynamics at the second joint has two real and distinct negative eigenvalues in -5 and -10.

b. Global asymptotic stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, without knowledge of the robot inertia matrix.

This can be obtained by multiple choices, the most common being a PD control with gravity cancellation

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}),$$

with symmetric $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, typically chosen diagonal. In alternative, one can use gravity compensation

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$$

further requiring that the minimum eigenvalue of $\boldsymbol{K}_P$ is larger than a finite upper bound $\alpha > 0$ on the norm of the Hessian $\partial^2 U(\boldsymbol{q})/\partial \boldsymbol{q}^2$ of the gravitational potential energy U.

c. Exponential stabilization of the end-effector position $\boldsymbol{p} = \boldsymbol{p}_d \in \mathbb{R}^2$ with zero velocity $\dot{\boldsymbol{p}} = \boldsymbol{0}$, up to kinematic singularities.

In this case, we require a feedback linearization control in the Cartesian space:

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}),$$

with

$$\boldsymbol{a} = -\boldsymbol{K}_D\, \dot{\boldsymbol{p}} + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) = -\boldsymbol{K}_D\, \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}))$$

and where $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ are both chosen diagonal.

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