Robotics 2

January 12, 2021

Exercise #1

The 3R robot in Fig. 1 moves in a vertical plane. Its control architecture consists of an ideal low-level controller that is able to execute any (reasonable) reference joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^3$ (including $\dot{\boldsymbol{q}}_r = \mathbf{0}$), as received by a high-level control law. The assigned task requires that the end effector is kept at a desired position P_d in the plane, while the robot minimizes the potential energy $U_g(\boldsymbol{q})$ due to gravity. Design a suitable high-level control law that realizes this task in a robust way (i.e., rejecting also positioning errors for the end effector). Provide the detailed symbolic expression of all terms needed in this law. Compute then the numerical value of $\dot{\boldsymbol{q}}_r$ at the starting instant $t = 0$, assuming that the links have a uniform distribution of mass and using the data:

$$l_1 = 0.5, \quad l_2 = 0.4, \quad l_3 = 0.3 \quad [\text{m}], \qquad m_1 = 5, \quad m_2 = 3, \quad m_3 = 2 \quad [\text{kg}],$$
$$q_1(0) = \pi, \quad q_2(0) = 0, \quad q_3(0) = -\pi/2 \quad [\text{rad}] \qquad \Rightarrow \qquad P_d = (-0.9, 0.3).$$

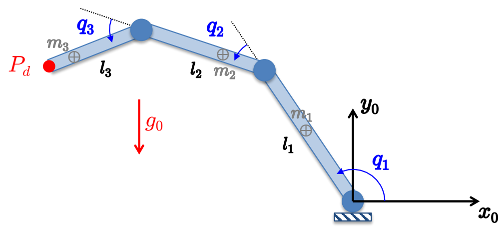

Figure 1: A planar 3R robot with a constant desired position P_d defined for its end effector.

Exercise #2

Consider again the situation of Exercise #1. Assume now that the robot is torque-controlled, namely that the control architecture is able to impose any (reasonable) reference joint torque command $\boldsymbol{\tau}_r \in \mathbb{R}^3$. Design a suitable torque-level control law that realizes the same previous task in a robust way (i.e., rejecting also position and/or velocity errors for the end effector). [*Note: You don't have to detail the expression of the terms in the control law, just define the structure.*]

Exercise #3

Derive the dynamic model of the 2R polar robot in Fig. 2, assuming that $I_{2,xx} \neq I_{2,yy} = I_{2,zz}$. Consider also the presence of viscous friction at the two joints.

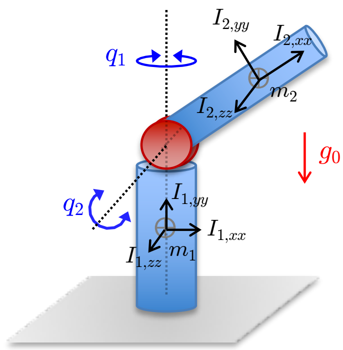

Figure 2: A 2R polar robot. Mass and diagonal barycentric inertia of each link is indicated.

Exercise #4

Consider again the robot of Exercise #3. Answer the following questions on its dynamics.

a) Suppose that the first robot joint is kept at a constant speed $\dot{q}_1 = \Omega > 0$ by a constant torque $\bar{\boldsymbol{\tau}}$ of *minimum* possible norm $\|\bar{\boldsymbol{\tau}}\|$ applied at the joints. Find an associated constant steady-state position $q_2 = \bar{q}_2$ and the expressions of $\bar{\tau}_1$ and $\bar{\tau}_2$. Is the joint angle $\bar{q}_2$ unique for a given $\Omega > 0$? And is the associated minimum norm torque $\bar{\boldsymbol{\tau}}$ unique?

b) Assuming that only the gravity acceleration g_0 and the kinematic parameters are known for this robot, what is the *minimum* number p of uncertain dynamic coefficients $\boldsymbol{a}$ that fully describe the robot dynamics in the linearly parametrized way $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau}$? Give the symbolic expressions for the vector $\boldsymbol{a} \in \mathbb{R}^p$ and the $2 \times p$ matrix $\boldsymbol{Y}$.

Exercise #5

A PPR robot moving on a horizontal plane may collide with some unknown obstacle in the environment at an a priori unknown point P_c along its structure, as illustrated in Fig. 3. Suppose that a single collision occurs and that the interaction can be assumed as pointwise, modeled by an unknown and unmeasurable pure force $\boldsymbol{F}_{c,i} \in \mathbb{R}^2$ applied to the robot, respectively with $i = 1$, 2, or 3 according to which link is involved. Define the complete expression of the dynamic terms in a model-based residual vector $\boldsymbol{r} \in \mathbb{R}^3$ that can be used for collision detection and isolation. For every possible situation, analyze if the collision can be detected or not, if the colliding link can be isolated or not, if the colliding force $\boldsymbol{F}_{c,i}$ can be identified (completely or in part) or not, and if the location of the collision point $P_{c,i}$ along the ith link can be determined or not.

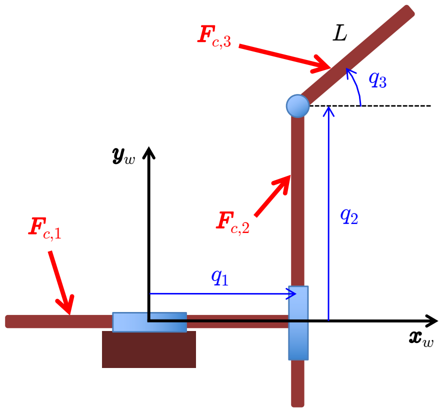

Figure 3: A planar PPR robot undergoing a possible collision during motion.

[240 minutes (4 hours); open books]

Solution

January 12, 2021

Exercise #1

In the absence of position error at the robot end effector level, a suitable high-level control law satisfying the requested self-motion task is obtained by projecting the negative gradient of the gravitational potential energy $U_g(\boldsymbol{q})$ in the null space of the task Jacobian $\boldsymbol{J}(\boldsymbol{q})$, or

$$\dot{\boldsymbol{q}}_r = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right) \nabla_{\boldsymbol{q}} U_g(\boldsymbol{q}) = \alpha \left(\boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) - \boldsymbol{I}\right) \boldsymbol{g}(\boldsymbol{q}). \tag{1}$$

The scalar $\alpha > 0$ is a step size in the anti-gradient direction $-\nabla_{\boldsymbol{q}} U_g = -\left(\partial U_g/\partial \boldsymbol{q}\right)^T$ and $\boldsymbol{g}(\boldsymbol{q})$ is the gravity term in the robot dynamic model[1]. To make the control law robust w.r.t. transient Cartesian position errors $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$, where $\boldsymbol{f}(\boldsymbol{q})$ is the (positional) task kinematics of the robot, we modify (1) as

$$\dot{\boldsymbol{q}}_r = \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{K}_P\boldsymbol{e}_p + \alpha \left(\boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) - \boldsymbol{I}\right) \boldsymbol{g}(\boldsymbol{q}) = -\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\boldsymbol{K}_P\boldsymbol{e}_p + \alpha\, \boldsymbol{J}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q})\right), \tag{2}$$

where $\boldsymbol{K}_P > 0$ is a (typically, diagonal) control gain matrix. Out of singularities, this produces an exponentially converging error dynamics since $\dot{\boldsymbol{e}}_p = -\dot{\boldsymbol{p}} = -\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_r = -\boldsymbol{K}_P\boldsymbol{e}_p$. In order to evaluate (2), we need the following terms (using the usual compact trigonometric notation):

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{q}) &= \begin{pmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \end{pmatrix}, \\
\boldsymbol{J}(\boldsymbol{q}) &= \left(\frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} -(l_1s_1 + l_2s_{12} + l_3s_{123}) & -(l_2s_{12} + l_3s_{123}) & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \end{pmatrix}, \\
U_g(\boldsymbol{q}) &= g_0\left(m_1d_{c1}s_1 + m_2(l_1s_1 + d_{c2}s_{12}) + m_3(l_1s_1 + l_2s_{12} + d_{c3}s_{123})\right), \\
\boldsymbol{g}(\boldsymbol{q}) &= \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} \left(m_1d_{c1} + (m_2+m_3)l_1\right)c_1 + \left(m_2d_{c2} + m_3l_2\right)c_{12} + m_3d_{c3}\,c_{123} \\ \left(m_2d_{c2} + m_3l_2\right)c_{12} + m_3d_{c3}\,c_{123} \\ m_3d_{c3}\,c_{123} \end{pmatrix},
\end{aligned}$$

where $d_{ci} > 0$ is the distance of the center of mass of link i from the joint axis i, for $i = 1, 2, 3$. With the given data, it is $\boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} 0.9 & 0.3 \end{pmatrix}^T = \boldsymbol{p}_d$, so that $\boldsymbol{e}_p = \boldsymbol{0}$ and we don't need to know the actual value of $\boldsymbol{K}_p$. For the remaining terms in (2), we compute

$$\boldsymbol{J}(\boldsymbol{q}(0)) = \begin{pmatrix} -0.3 & -0.3 & -0.3 \\ -0.9 & -0.4 & 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}^{\#}(\boldsymbol{q}(0)) = \begin{pmatrix} 0.5464 & -1.1475 \\ -1.2295 & 0.0820 \\ -2.6503 & 1.0656 \end{pmatrix}$$

and

$$\boldsymbol{g}(\boldsymbol{q}(0)) = g_0 \begin{pmatrix} -5.15 \\ -1.4 \\ 0 \end{pmatrix} = \begin{pmatrix} -50.522 \\ -13.734 \\ 0 \end{pmatrix} \text{ [Nm]},$$

[1]Note that α converts here a joint torque into a joint velocity. Thus, it has dimensional units [rad$\cdot$(Nm$\cdot$s)$^{-1}$].

where $g_0 = 9.81$ [m/s^{-2}]. Setting for instance $\alpha = 1$, we finally obtain

$$\dot{\boldsymbol{q}}_r(0) = \begin{pmatrix} 2.5731 \\ -5.7895 \\ 3.2164 \end{pmatrix} \text{ [rad/s]}.$$

The first and third link start moving counterclockwise, while the second link will rotate clockwise. Robot motion will continue until the projection of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the null space of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ will vanish (in general, not implying that $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$).

Exercise #2

When the robot is in a torque-controlled mode, its full dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

has to be taken into account. To address the same task as in Exercise #1 with torque control, we apply first a feedback linearization law in the joint space, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{a}. \tag{3}$$

A joint acceleration command performing a robot self-motion, as driven by the negative gradient of the potential energy due to gravity[2], is then designed as

$$\boldsymbol{a} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}}). \tag{4}$$

In (4), a damping velocity term $-\boldsymbol{K}_v\dot{\boldsymbol{q}}$, with $\boldsymbol{K}_v > 0$ and diagonal, has been added in the null space of the task Jacobian in order to stabilize the joint motion. This is customary (and almost mandatory) when resolving redundancy at the acceleration level.

For rejecting position and/or velocity errors that may occur around the desired constant end-effector position, the command (4) is modified as

$$\boldsymbol{a} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}}), \tag{5}$$

including thus a PD action, with (typically diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, on the Cartesian position error, and taking into account that $\dot{\boldsymbol{p}}_d = \boldsymbol{0}$. Plugging (5) into (3) yields finally the desired torque control law

$$\begin{aligned}\boldsymbol{\tau}_r = \boldsymbol{M}(\boldsymbol{q})\Big[&\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ &- \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}})\Big] + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}).\end{aligned} \tag{6}$$

Exercise #3

Kinetic energy

$$T_1 = \frac{1}{2}\, I_{1,yy}\, \dot{q}_1^2 \qquad T_2 = \frac{1}{2}\, m_2\|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\,\boldsymbol{\omega}_2^T\boldsymbol{I}_2\boldsymbol{\omega}_2 = \frac{1}{2}\, m_2\|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\,{}^2\boldsymbol{\omega}_2^T\,{}^2\boldsymbol{I}_2\,{}^2\boldsymbol{\omega}_2$$

[2] In this case, α converts a joint torque into a joint acceleration. Thus, it has dimensional units [rad·Nm^{-1}·s^{-2}]. Similarly, the units of $\boldsymbol{K}_v$ are [s^{-1}].

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_{c2}\cos q_2 \cos q_1 \\ d_{c2}\cos q_2 \sin q_1 \\ d_{c2}\sin q_2 \end{pmatrix} = \begin{pmatrix} d_{c2}\, c_1 c_2 \\ d_{c2}\, s_1 c_2 \\ d_{c2}\, s_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_{c2}(s_1 c_2\, \dot{q}_1 + c_1 s_2\, \dot{q}_2) \\ d_{c2}(c_1 c_2\, \dot{q}_1 - s_1 s_2\, \dot{q}_2) \\ d_{c2}\, c_2\, \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = d_{c2}^2 \left(\dot{q}_2^2 + c_2^2 \dot{q}_1^2\right)$$

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \;\Rightarrow\; {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \;\Rightarrow\; {}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\, {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow\; T_2 = \frac{1}{2} m_2\, d_{c2}^2 \left(\dot{q}_2^2 + c_2^2\dot{q}_1^2\right) + \frac{1}{2} \begin{pmatrix} s_2\dot{q}_1 & c_2\dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} I_{2,xx} & 0 & 0 \\ 0 & I_{2,yy} & 0 \\ 0 & 0 & I_{2,zz} \end{pmatrix} \begin{pmatrix} s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$= \frac{1}{2} m_2\, d_{c2}^2 \left(\dot{q}_2^2 + c_2^2\dot{q}_1^2\right) + \frac{1}{2} I_{2,zz}\, \dot{q}_2^2 + \frac{1}{2} \left(I_{2,xx} s_2^2 + I_{2,yy} c_2^2\right) \dot{q}_1^2$$

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

Inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{1,yy} + I_{2,xx}s_2^2 + \left(I_{2,yy} + m_2 d_{c2}^2\right) c_2^2 & 0 \\ 0 & I_{2,zz} + m_2 d_{c2}^2 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} I_{1,yy} + I_{2,xx} + \left(I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx}\right) c_2^2 & 0 \\ 0 & I_{2,zz} + m_2 d_{c2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2\, c_2^2 & 0 \\ 0 & a_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1(q_2) & \boldsymbol{m}_2 \end{pmatrix}$$

Potential energy and gravity vector

$$U_1 = 0 \qquad U_2 = g_0 m_2 d_{c2} s_2 \qquad U(\boldsymbol{q}) = U_1 + U_2$$

$$\Rightarrow\; \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ g_0 m_2 d_{c2} c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_4\, c_2 \end{pmatrix}$$

Coriolis and centrifugal vector

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_1}\right)\right) = \begin{pmatrix} 0 & -a_2 s_2 c_2 \\ -a_2 s_2 c_2 & 0 \end{pmatrix}$$

$$c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} = -2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 = -a_2 \sin(2q_2) \dot{q}_1 \dot{q}_2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_2}\right)\right) = \begin{pmatrix} a_2 s_2 c_2 & 0 \\ -0 & 0 \end{pmatrix}$$

$$c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \dot{\boldsymbol{q}} = a_2 s_2 c_2\, \dot{q}_1^2$$

$$\Rightarrow\; \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 \\ a_2 s_2 c_2\, \dot{q}_1^2 \end{pmatrix}$$

Dynamic model (including viscous friction)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}\dot{\boldsymbol{q}} = \boldsymbol{\tau} \quad \Longleftrightarrow \quad \begin{cases} \left(a_1 + a_2\, c_2^2\right)\ddot{q}_1 - 2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 + f_1 \dot{q}_1 = \tau_1 \\ a_3 \ddot{q}_2 + a_2 s_2 c_2\, \dot{q}_1^2 + a_4\, c_2 + f_2 \dot{q}_2 = \tau_2 \end{cases} \tag{8}$$

Exercise #4

a) With reference to eqs. (8), set $\dot{q}_1 = \Omega > 0$, $\ddot{q}_1 = 0$, $q_2 = \bar{q}_2$, $\dot{q}_2 = \ddot{q}_2 = 0$ for the desired steady state. We get

$$f_1 \Omega = \tau_1 \tag{9}$$

$$a_2 \sin \bar{q}_2 \cos \bar{q}_2\, \Omega^2 + a_4 \cos \bar{q}_2 = \tau_2 \tag{10}$$

From (9), the torque on joint 1 should necessarily compensate the loss of energy due to viscous friction when the joint is rotating with a speed Ω, so $\bar{\tau}_1 = f_1 \Omega$. Because of the minimum norm requirement for the steady-state torque $\bar{\boldsymbol{\tau}}$, we seek then a solution $\bar{q}_2$ in (10) for $\tau_2 = \bar{\tau}_2 = 0$. This is possible in two cases.

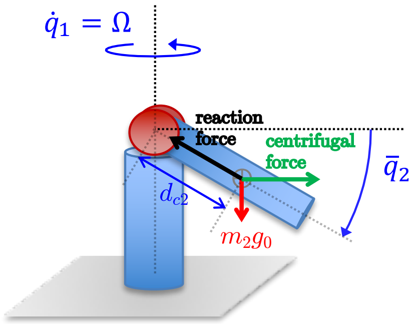

Figure 4: Equilibrium of forces for a polar robot spinning its first joint at a constant $\dot{q}_1 = \Omega > 0$.

- For $\bar{q}_2 = \pm\pi/2$ (second link vertical, up or down along the first joint axis). Indeed, $\cos \bar{q}_2 = 0$ and (10) will be an identity with zero applied torque. However, these equilibria are both unstable: small perturbations to this steady-state condition will let the second robot joint deviate from any of these two configurations.

- When $a_2 \sin \bar{q}_2\, \Omega^2 + a_4 = 0$. This corresponds to a special balancing between the vector sum of the gravity force (pointing vertically and downward) and centrifugal force (pointing horizontally and radially from the first joint axis), both applied to the center of mass of link 2, and the internal reaction force by the rigid robot structure (see Fig. 4). This balance is obtained for

$$\bar{q}_2 = \arcsin \frac{-a_4}{a_2\, \Omega^2} = -\arcsin \frac{g_0 m_2 d_{c2}}{\left(I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx}\right)\Omega^2} \in \left(-\frac{\pi}{2}, 0\right). \tag{11}$$

  The domain of definition for $\bar{q}_2$ follows from $d_{c2} > 0$ and from the fact that $I_{2,yy} + m_2 d_{c2}^2 > I_{2,xx}$ always holds, namely that the inertia of link 2 around an axis belonging to its base is less than the baricentral inertia around an axis stretching along the link length. Such inequality can be proven for any rigid body, no matter what mass it has and how the mass is distributed in the body volume[3]. For low values of Ω, the angle in (11) would be close to $-\pi/2$; for large values of Ω, we have instead $\bar{q}_2 \to 0^-$. It can be shown that this equilibrium is dynamically stable.

[3]Triangular inequalities hold among the elements of a diagonal barycentric inertia matrix of a rigid body (or for the elements on its principal axes), such as $I_{yy} + I_{zz} > I_{xx}$. In addition, the inertia I around any barycentric axis of a body of mass m is smaller than md^2, where d is the distance from the CoM to a parallel axis at the body end. These two physical properties together prove the inequality in the text.

In any case, we note that the (minimum norm) torque for this dynamic equilibrium will be the same:

$$\bar{\boldsymbol{\tau}} = \begin{pmatrix} f_1\Omega \\ 0 \end{pmatrix}, \qquad \|\bar{\boldsymbol{\tau}}\| = f_1\,|\Omega|\,.$$

Indeed, when considering also a constant torque $\bar{\tau}_2 \neq 0$ applied at the second joint (and thus, a total torque with $\|\bar{\boldsymbol{\tau}}\| > f_1\,|\Omega|$), we may find other steady-state solutions for $\bar{q}_2$.

b) The minimum number of dynamic coefficients required is $p = 6$. From eqs. (8), one has

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & c_2^2\ddot{q}_1 - 2s_2c_2\,\dot{q}_1\dot{q}_2 & 0 & 0 & \dot{q}_1 & 0 \\ 0 & s_2c_2\,\dot{q}_1^2 & \ddot{q}_2 & c_2 & 0 & \dot{q}_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} I_{1,yy} + I_{2,xx} \\ I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx} \\ I_{2,zz} + m_2 d_{c2}^2 \\ g_0 m_2 d_{c2} \\ f_1 \\ f_2 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}.\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau}.$$

A comment is in place on the alternative definition of coefficients in (7), before introducing the trigonometric substitution $s_2^2 = 1 - c_2^2$. When $I_{2,yy} = I_{2,zz}$, as in the present case, only three independent dynamic coefficients would appear anyway in the inertia matrix, although with the different definitions

$$a_1' = I_{1,yy}, \quad a_2' = I_{2,xx}, \quad a_3' = I_{2,yy} + m_2 d_{c2}^2 = I_{2,zz} + m_2 d_{c2}^2 = a_3,$$

whereas $a_i' = a_i$ for the remaining $i = 4, 5, 6$. Although the associated regressor matrix $\boldsymbol{Y}'$ would look slightly different, both parametrizations are minimal ($p = 6$). On the other hand, this would no longer be true for the parametrization suggested by (7) in case $I_{2,yy} \neq I_{2,zz}$: 4 dynamic coefficients would then be used in $\boldsymbol{M}$, leading to a total of 7 coefficients in the dynamic model.

Exercise #5

We shall derive first the terms in the dynamic model of the planar PPR robot in Fig. 3, following a Lagrangian approach. In the absence of gravity, we have

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\tau} + \boldsymbol{\tau}_c, \qquad \boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c,$$

where $\boldsymbol{S}\dot{\boldsymbol{q}}$ is a factorization of the quadratic velocity terms such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is a skew-symmetric matrix (or, equivalently, $\dot{\boldsymbol{M}} = \boldsymbol{S} + \boldsymbol{S}^T$), $\boldsymbol{\tau}_c$ is the joint torque resulting from a collision with a force $\boldsymbol{F}_c$, and $\boldsymbol{J}_c$ is the Jacobian of the collision point along the structure.

The kinetic energy of the first two links, moved by two prismatic joints with orthogonal axes, is

$$T_1 + T_2 = \frac{1}{2}\,m_1\dot{q}_1^2 + \frac{1}{2}\,m_2\left(\dot{q}_1^2 + \dot{q}_2^2\right).$$

From

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + d_{c3}c_3 \\ q_2 + d_{c3}s_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_1 - d_{c3}s_3\dot{q}_3 \\ \dot{q}_2 + d_{c3}c_3\dot{q}_3 \end{pmatrix}, \qquad \omega_3 = \dot{q}_3,$$

the kinetic energy of the third (rotational) link is computed as

$$T_3 = \frac{1}{2}\,m_3\|\boldsymbol{v}_{c3}\|^2 + \frac{1}{2}\,I_3\,\omega_3^2 = \frac{1}{2}\,m_3\left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\dot{q}_3^2 + 2\,d_{c3}(-s_3\dot{q}_1 + c_3\dot{q}_2)\dot{q}_3\right) + \frac{1}{2}\,I_3\dot{q}_3^2.$$

From the kinetic energy of the system,

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

the inertia matrix is extracted as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & -m_3 d_{c3} s_3 \\ 0 & m_2 + m_3 & m_3 d_{c3} c_3 \\ -m_3 d_{c3} s_3 & m_3 d_{c3} c_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

The (purely) centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ are derived using the Christoffel symbols, i.e., for each component

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i}\right), \qquad i = 1, 2, 3$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} c_3 \end{pmatrix} && \Rightarrow && c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} c_3 \dot{q}_3^2 \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} s_3 \end{pmatrix} && \Rightarrow && c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} s_3 \dot{q}_3^2 \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \boldsymbol{0} && \Rightarrow && c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0,
\end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_3 d_{c3} c_3 \dot{q}_3^2 \\ -m_3 d_{c3} s_3 \dot{q}_3^2 \\ 0 \end{pmatrix}.$$

Using again the Christoffel symbols, a suitable factorization matrix for $\boldsymbol{c}$ is computed as

$$\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -m_3 d_{c3} c_3 \dot{q}_3 \\ 0 & 0 & -m_3 d_{c3} s_3 \dot{q}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

The model-based residual for collision detection and isolation can then be evaluated in all its terms, namely

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left(\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{r}\right) ds\right), \qquad \boldsymbol{K}_I > 0, \tag{12}$$

where we have assumed that $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (the robot starts at rest). In particular, we have for the second term in the integral

$$\boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 0 \\ -m_3 d_{c3} \left(s_3 \dot{q}_2 + c_3 \dot{q}_1\right) \dot{q}_3 \end{pmatrix}.$$

The residual $\boldsymbol{r}$ in (12) is affected, through the joint torque $\boldsymbol{\tau}_{ci}$, by a collision force $\boldsymbol{F}_{ci} = \begin{pmatrix} F_x & F_y \end{pmatrix}^T$ acting at the collision point P_{ci} on the robot link i as

$$\dot{\boldsymbol{r}} = \boldsymbol{K}_I\left(\boldsymbol{\tau}_{ci} - \boldsymbol{r}\right) = \boldsymbol{K}_I\left(\boldsymbol{J}_{ci}^T(\boldsymbol{q})\boldsymbol{F}_{ci} - \boldsymbol{r}\right), \qquad i = 1, 2, 3,$$

except for some singular cases. Essentially, there are directions along which the point P_{ci} cannot be given a linear instantaneous velocity in the motion plane by means of a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Next, we shall distinguish between collisions on the first, second, or third link and analyze the various possible situations in terms of collision detection and isolation, as well as collision force identification and localization. Assuming that sufficiently large gains can be chosen in the diagonal matrix $\boldsymbol{K}_I$, we will have

$$\boldsymbol{r} \approx \boldsymbol{\tau}_{ci},$$

and the residual $\boldsymbol{r}$ (in particular, its components r_i) can be used as a proxy for $\boldsymbol{\tau}_{ci}$ when reasoning about the nature of collisions. In the following, all quantities will be expressed in the world frame RF_w of Fig. 3.

- **Collision on link 1**. The position of the collision point along the first link[4] and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c1} = \begin{pmatrix} q_1 - \rho_1 \\ 0 \end{pmatrix}, \text{ with } \rho_1 \in [0, 2l_{1,max}] \;\Rightarrow\; \boldsymbol{J}_{c1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\Rightarrow\; \boldsymbol{\tau}_{c1} = \boldsymbol{J}_{c1}^T\boldsymbol{F}_{c1} = \begin{pmatrix} F_x \\ 0 \\ 0 \end{pmatrix}.$$

  Clearly, a collision is not detected at all when

$$\boldsymbol{F}_{c1}^0 = \begin{pmatrix} 0 \\ F_y \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{J}_{c1}^T\boldsymbol{F}_{c1}^0 = \mathbf{0}.$$

  Only the intensity F_x of $\boldsymbol{F}_{c1}$ can be identified. The closer is the alignment of $\boldsymbol{F}_{c1}$ to the axis of joint 1, the poorer will be the detection. Moreover, we will never have an information on the localization of the collision point P_{c1}.

- **Collision on link 2**. The position of the contact point along the second link and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 \\ q_2 - \rho_2 \end{pmatrix}, \text{ with } \rho_2 \in [0, 2l_{2,max}] \;\Rightarrow\; \boldsymbol{J}_{c2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;\Rightarrow\; \boldsymbol{\tau}_{c2} = \boldsymbol{J}_{c2}^T\boldsymbol{F}_{c2} = \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix}.$$

  Thus, the collision will always be detected and the collision force $\boldsymbol{F}_{c2}$ fully identified. However, once again, no information on the localization of the collision point P_{c2} is provided by the residual. This is also critical for the isolation of the actual link in collision. In fact, when $F_y = 0$ in $\boldsymbol{F}_{c2}$, we obtain $\boldsymbol{\tau}_{c2} = \boldsymbol{\tau}_{c1}$ and there will be no way to understand whether the collision occurred on link 1 (with the force intensity being possibly identified only in part) or on link 2 (with direction and intensity of the collision force being fully identified).

- **Collision on link 3**. The position of the contact point along the third link and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \rho_3\, c_3 \\ q_2 + \rho_3\, s_3 \end{pmatrix}, \quad \text{with } \rho_3 \in [0, L] \quad\Rightarrow\quad \boldsymbol{J}_{c3}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & -\rho_3\, s_3 \\ 0 & 1 & \rho_3\, c_3 \end{pmatrix}$$

[4] For simplicity, assume that the prismatic joints have limited excursions, i.e., $q_i \in [-l_{i,max}, l_{i,max}]$, for $i = 1, 2$.

$$\Rightarrow \quad \boldsymbol{\tau}_{c3} = \boldsymbol{J}_{c3}^T(\boldsymbol{q})\boldsymbol{F}_{c3} = \begin{pmatrix} F_x \\ F_y \\ \rho_3 \left(c_3 F_y - s_3 F_x \right) \end{pmatrix}.$$

The first two components of $\boldsymbol{\tau}_{c3}$ (in practice, of its proxy $\boldsymbol{r}$) show again that the collision is always detected, and that the collision force $\boldsymbol{F}_{c3}$ is fully identified as well. In this case, localization of the actual collision point P_{c3} (i.e., the value ρ_3) is also possible, provided that the third component $\tau_{c3,3} \neq 0$. In fact, we can estimate then ρ_3 as

$$\rho_3 = \frac{\tau_{c3,3}}{c_3 F_y - s_3 F_x} = \frac{\tau_{c3,3}}{c_3 \tau_{c3,2} - s_3 \tau_{c3,1}} \approx \frac{r_3}{c_3 r_2 - s_3 r_1} = \hat{\rho}_3.$$

Such localization will fail when

$$\boldsymbol{F}_{c3} = \|\boldsymbol{F}_{c3}\| \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} \qquad \Rightarrow \qquad \tau_{c3,3} = 0,$$

namely $\boldsymbol{F}_{c3}$ is aligned with the third link. Moreover, in this situation we obtain $\boldsymbol{\tau}_{c3} = \boldsymbol{\tau}_{c2}$ and also the isolation of the actual link in collision will fail. In fact, we cannot distinguish between a collision occurred on link 2 or 3. The same happens when a force $\boldsymbol{F}_{c3}$ hits the third link at its base (being then ρ_3).

* * * * *