Robotics 2

February 4, 2021

Exercise #1

Consider the RRPR robot in Fig. 1, where all relevant kinematic and dynamic parameters are also shown. The robot moves in a vertical plane. Compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the Lagrangian dynamic model of this robot. Provide a linear factorization of each term, $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}_M$ and $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q})\boldsymbol{a}_g$, introducing dynamic coefficients $\boldsymbol{a}_m \in \mathbb{R}^{p_m}$ and $\boldsymbol{a}_g \in \mathbb{R}^{p_g}$. Find also all open-loop equilibrium configurations $\boldsymbol{q}_e$, i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$.

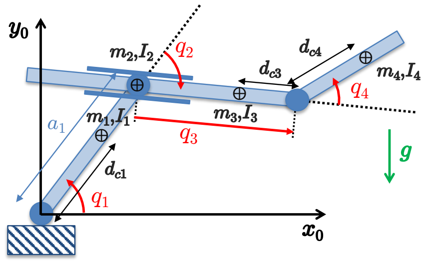

Figure 1: A 4-dof RRPR robot: DH coordinates, kinematic and dynamic parameters.

Exercise #2

Consider the situation depicted in Fig. 2. The two Cartesian robots A and B are commanded by motors and transmissions that generate linear forces $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ along the axes of their prismatic joints. Each control force component is bounded as

$$|\tau_{A,i}| \le \tau_{A,max}, \quad i = 1, 2, \qquad |\tau_{B,i}| \le \tau_{B,max}, \quad i = 1, 2.$$

The two robots hold firmly a payload mass m_p and cooperate in moving the mass in minimum time along a horizontal linear path $\boldsymbol{p} = \boldsymbol{p}(s)$, from point P_{in} to point P_{fin} in a rest-to-rest mode. In addition, it is desired that the sum $H = \|\boldsymbol{\tau}_A\|^2 + \|\boldsymbol{\tau}_B\|^2$ is always minimized instantaneously. Motion occurs in the vertical plane, and we assume that motors are powerful enough to sustain the weight of the respective robot and of the payload. Provide a dynamic model of this cooperating task and determine the optimal time profiles of the four commands $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$.

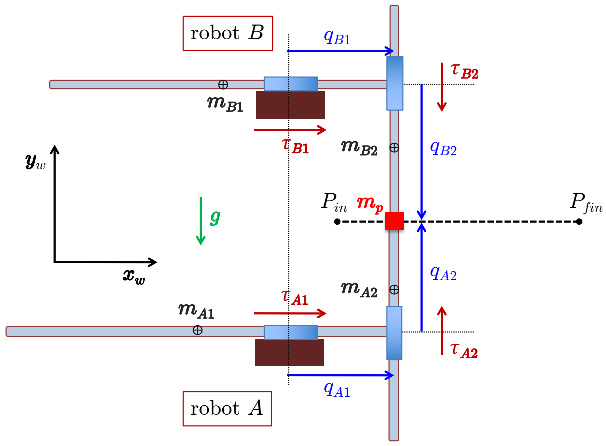

Figure 2: Two Cartesian robots that move a mass m_p under gravity along a linear horizontal path.

Exercise #3

The planar 3R robot shown in Fig. 3 is commanded by joint torques $\boldsymbol{\tau} \in \mathbb{R}^3$ that use feedback from the current state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$. The robot is initially at rest in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\ 11\pi/18,\ -\pi/4)$ and should then perform a self-motion so as to guarantee that the third joint asymptotically reaches the final value $q_{3,fin} = -\pi/2$, while keeping the position of its end-effector always at the same initial point P_{in}. Design a torque control scheme that completes this task in a robust way, i.e., by rejecting transient position and/or velocity errors and without encountering any singular situation for the control law. *Hint: Use an approach based on joint space decomposition.*

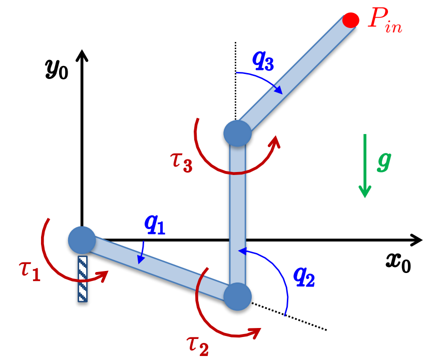

Figure 3: A 3R robot that should perform a self-motion task with constant end-effector position.

Exercise #4

Figure 4 shows a simple 1-dof model of the interaction between a robot of mass $m_r > 0$, commanded by a force F, and a rigid environment, with a force sensor of stiffness $k_s > 0$ measuring the contact force F_c. The two coefficients $b_r > 0$ and $b_s > 0$ represent, respectively, the viscous friction affecting robot motion and the viscous damping of the force sensor. The reference position $x_r = 0$ is when the robot mass is in contact with $F_c = 0$. Provide the dynamic model of this system assuming linearity of all effects. Based only on the measured force, design a control law for F that is able to regulate asymptotically the contact force F_c to a desired constant value $F_d > 0$, despite uncertainty in all model parameters. Provide the associated steady-state values of $F = \bar{F}$ and $x_r = \bar{x}_r$.

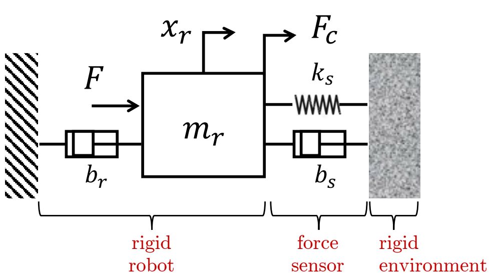

Figure 4: A 1-dof model of interaction between a robot and a rigid environment.

[240 minutes (4 hours); open books]

Solution

February 4, 2021

Exercise #1

The dynamic modeling steps and the requested linear factorizations of terms are quite standard procedures. They are sketched hereafter for the given RRPR planar robot without further comments. We assume that the kinematic parameter a_1 ([m]) and the gravity acceleration $g_0 = 9.81$ [m/s^2] are accurately known. We finally evaluate the open-loop equilibria of the robot under gravity.

Kinetic energy

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2 \qquad T_2 = \frac{1}{2}\,m_2 a_1^2 \dot{q}_1^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

$$T_3 = \frac{1}{2}\,m_3 \|\boldsymbol{v}_{c3}\|^2 + \frac{1}{2}\,I_3 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} a_1 \cos q_1 + \cos\left(q_1 + q_2\right)\left(q_3 - d_{c3}\right) \\ a_1 \sin q_1 + \sin\left(q_1 + q_2\right)\left(q_3 - d_{c3}\right) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -\left(a_1 s_1 + s_{12}(q_3 - d_{c3})\right)\dot{q}_1 - s_{12}(q_3 - d_{c3})\dot{q}_2 + c_{12}\dot{q}_3 \\ \left(a_1 c_1 + c_{12}(q_3 - d_{c3})\right)\dot{q}_1 + c_{12}(q_3 - d_{c3})\dot{q}_2 + s_{12}\dot{q}_3 \end{pmatrix}$$

$$T_4 = \frac{1}{2}\,m_4 \|\boldsymbol{v}_{c4}\|^2 + \frac{1}{2}\,I_4 \left(\dot{q}_1 + \dot{q}_2 + \dot{q}_4\right)^2$$

$$\boldsymbol{p}_{c4} = \begin{pmatrix} a_1 \cos q_1 + q_3 \cos\left(q_1 + q_2\right) + d_{c4}\cos\left(q_1 + q_2 + q_4\right) \\ a_1 \sin q_1 + q_3 \sin\left(q_1 + q_2\right) + d_{c4}\sin\left(q_1 + q_2 + q_4\right) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c4} = \dot{\boldsymbol{p}}_{c4} = \begin{pmatrix} -\left(a_1 s_1 + q_3 s_{12} + d_{c4} s_{124}\right)\dot{q}_1 - \left(q_3 s_{12} + d_{c4} s_{124}\right)\dot{q}_2 + c_{12}\,\dot{q}_3 - d_{c4} s_{124}\,\dot{q}_4 \\ \left(a_1 c_1 + q_3 c_{12} + d_{c4} c_{124}\right)\dot{q}_1 + \left(q_3 c_{12} + d_{c4} c_{124}\right)\dot{q}_2 + s_{12}\,\dot{q}_3 + d_{c4} c_{124}\,\dot{q}_4 \end{pmatrix}$$

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 + T_3 + T_4 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ \vdots & m_{22} & m_{23} & m_{24} \\ \vdots & \ddots & m_{33} & m_{34} \\ \text{symm} & \cdots & \cdots & m_{44} \end{pmatrix}$$

$$\begin{aligned}
m_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_2 + m_3 + m_4\right) a_1^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_3\, q_3 \\
&\quad - 2\, m_3 d_{c3}\, a_1 \cos q_2 + 2\left(m_3 + m_4\right) a_1 q_3 \cos q_2 + 2\, m_4 d_{c4}\left(a_1 \cos(q_2 + q_4) + q_3 \cos q_4\right) \\
m_{12} &= I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_{c3}\, q_3 \\
&\quad - m_3 d_{c3}\, a_1 \cos q_2 + \left(m_3 + m_4\right) a_1 q_3 \cos q_2 + m_4 d_{c4}\left(a_1 \cos(q_2 + q_4) + 2\, q_3 \cos q_4\right) \\
m_{13} &= \left(m_3 + m_4\right) a_1 \sin q_2 - m_4 d_{c4} \sin q_4 \\
m_{14} &= I_4 + m_4 d_{c4}^2 + m_4 d_{d4}\left(a_1 \cos(q_2 + q_4) + q_3 \cos q_4\right) \\
m_{22} &= I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_{c3}\, q_3 + 2\, m_4 d_{c4}\, q_3 \cos q_4 \\
m_{23} &= -m_3 d_{c3} \sin q_4 \\
m_{24} &= I_4 + m_4 d_{c4}^2 + m_4 d_{c4}\, q_3 \cos q_4 \\
m_{33} &= m_3 + m_4 \\
m_{34} &= -m_4 d_{c4} \sin q_4 \\
m_{44} &= I_4 + m_4 d_{c4}^2
\end{aligned}$$

Linear parametrization of the inertia term

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_M, \qquad \boldsymbol{a}_M \in \mathbb{R}^6$$

$$\begin{aligned}
m_{11} &= a_{M1} + a_{M4}\,q_3^2 - 2\,a_{M3}\,q_3 + 2\,(a_{M4}\,q_3 - a_{M3})\,a_1\,c_2 + 2\,a_{M5}\,(a_1c_{24} + q_3c_4)\\
m_{12} &= a_{M2} + a_{M4}\,q_3^2 - 2\,a_{M3}\,q_3 + (a_{M4}\,q_3 - a_{M3})\,a_1\,c_2 + a_{M5}\,(a_1c_{24} + 2\,q_3\,c_4)\\
m_{13} &= a_{M4}\,a_1\,s_2 - a_{M5}\,s_4\\
m_{14} &= a_{M6} + a_{M5}\,(a_1\,c_{24} + q_3\,c_4)\\
m_{22} &= a_{M2} + a_{M4}\,q_3^2 - 2\,a_{M3}\,q_3 + 2\,a_{M5}\,q_3\,c_4\\
m_{23} &= -a_{M3}\,s_4\\
m_{24} &= a_{M6} + a_{M5}\,q_3\,c_4\\
m_{33} &= a_{M4}\\
m_{34} &= -a_{M5}\,s_4\\
m_{44} &= a_{M6}
\end{aligned}$$

$$\boldsymbol{a}_M = \begin{pmatrix} a_{M1}\\ a_{M2}\\ a_{M3}\\ a_{M4}\\ a_{M5}\\ a_{M6} \end{pmatrix} = \begin{pmatrix} I_1 + m_1d_{c1}^2 + I_2 + I_3 + m_3d_{c3}^2 + I_4 + m_4d_{c4}^2 + (m_2+m_3+m_4)\,a_1^2\\ I_2 + I_3 + m_3d_{c3}^2 + I_4 + m_4d_{c4}^2\\ m_3d_{c3}\\ m_3+m_4\\ m_4d_{c4}\\ I_4 + m_4d_{c4}^2 \end{pmatrix}$$

$$\boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}}) =$$

$$\begin{pmatrix}
\ddot{q}_1 & \ddot{q}_2 & \begin{matrix} -2\,(q_3 + a_1s_2)\,\ddot{q}_1\\ -\,(2\,q_3 + a_1s_2)\,\ddot{q}_2 \end{matrix} & \begin{matrix} \left(q_3^2 + 2\,q_3\,a_1c_2\right)\ddot{q}_1\\ +\left(q_3^2 + q_3\,a_1c_2\right)\ddot{q}_2\\ +a_1s_2\,\ddot{q}_3 \end{matrix} & \begin{matrix} 2\,(a_1c_{24} + q_3c_4)\,\ddot{q}_1\\ +\,(a_1c_{24} + 2\,q_3\,c_4)\,\ddot{q}_2\\ -s_4\,\ddot{q}_3\,(a_1\,c_{24} + q_3\,c_4)\,\ddot{q}_4 \end{matrix} & \ddot{q}_4\\
0 & \ddot{q}_1 + \ddot{q}_2 & \begin{matrix} \left(q_3^2 + q_3\,a_1c_2\right)\ddot{q}_1\\ -2\,q_3\,\ddot{q}_2 \end{matrix} & \begin{matrix} \left(q_3^2 + q_3\,a_1c_2\right)\ddot{q}_1\\ +q_3^2\,\ddot{q}_2 \end{matrix} & \begin{matrix} (a_1c_{24} + 2\,q_3\,c_4)\,\ddot{q}_1\\ +q_3\,c_4\,(2\,\ddot{q}_2 + \ddot{q}_4) \end{matrix} & \ddot{q}_4\\
0 & 0 & -s_3\ddot{q}_2 & a_1s_2\,\ddot{q}_1 + \ddot{q}_3 & -s_4\,(\ddot{q}_1 + \ddot{q}_4) & 0\\
0 & 0 & 0 & 0 & \begin{matrix} (a_1\,c_{24} + q_3\,c_4)\,\ddot{q}_1\\ +q_3\,c_4\,\ddot{q}_2 - s_4\,\ddot{q}_3 \end{matrix} & \ddot{q}_1 + \ddot{q}_2 + \ddot{q}_4
\end{pmatrix}$$

Potential energy

$$U_1 = m_1d_{c1}g_0\sin q_1 \qquad U_2 = m_2a_1g_0\sin q_1$$

$$U_3 = m_3g_0\,(a_1\sin q_1 + (q_3 - d_{c3})\sin(q_1+q_2))$$

$$U_4 = m_4g_0\,(a_1\sin q_1 + q_3\sin(q_1+q_2) + d_{c4}\sin(q_1+q_2+q_4))$$

$$U(\boldsymbol{q}) = U_1 + U_2 + U_3 + U_4$$

Gravity vector

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4 \end{pmatrix}$$

$$
\begin{aligned}
g_1 &= (m_1 d_{c1} + (m_2 + m_3 + m_4)\, a_1)\, g_0 \cos q_1 - m_3 d_{c3}\, g_0\, c_{12} \\
&\quad + (m_3 + m_4)\, g_0\, q_3\, c_{12} + m_4 d_{c4}\, g_0\, c_{124} \\
g_2 &= -m_3 d_{c3}\, g_0\, c_{12} + (m_3 + m_4)\, g_0\, q_3\, c_{12} + m_4 d_{c4}\, g_0\, c_{124} \\
g_3 &= (m_3 + m_4)\, g_0\, s_{12} \\
g_4 &= m_4 d_{c4}\, g_0\, c_{124}
\end{aligned}
$$

Linear parametrization of the gravity vector

$$
\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q})\, \boldsymbol{a}_g, \qquad \boldsymbol{a}_g \in \mathbb{R}^4
$$

$$
\boldsymbol{a}_g = \begin{pmatrix} a_{g1} \\ a_{g2} \\ a_{g3} \\ a_{g4} \end{pmatrix} = \begin{pmatrix} m_1 d_{c1} + (m_2 + m_3 + m_4)\, a_1 \\ m_3 d_{c3} \\ m_3 + m_4 \\ m_4 d_{c4} \end{pmatrix}
$$

$$
\boldsymbol{Y}_g(\boldsymbol{q}) = \begin{pmatrix} g_0\, c_1 & -g_0\, c_{12} & g_0\, q_3\, c_{12} & g_0\, c_{124} \\ 0 & -g_0\, c_{12} & g_0\, q_3\, c_{12} & g_0\, c_{124} \\ 0 & 0 & g_0\, s_{12} & 0 \\ 0 & 0 & 0 & g_0\, c_{124} \end{pmatrix}
$$

Note. We kept separated the two linear parametrizations, as requested. Indeed, $a_{g2} = a_{M3}$, $a_{g3} = a_{M4}$ and $a_{g4} = a_{M5}$, so that only $6 + 1 = 7$ different dynamic coefficients would be needed in total.

Open-loop equilibria

$$
\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0} \quad \Longleftrightarrow \quad \begin{cases} q_{e1} = \pm\dfrac{\pi}{2} \\ (m_3 + m_4)\, q_{e3} = m_3 d_{c3} \\ q_{e1} + q_{e2} = \{0, \pi\} \\ q_{e1} + q_{e2} + q_{e4} = \pm\dfrac{\pi}{2} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} q_{e1} = \pm\dfrac{\pi}{2} \\ q_{e2} = \pm\dfrac{\pi}{2} \\ q_{e3} = \dfrac{m_3 d_{c3}}{m_3 + m_4} < d_{c3} \\ q_{e4} = \pm\dfrac{\pi}{2} \end{cases}
$$

One of the equilibria is shown in Fig. 5.

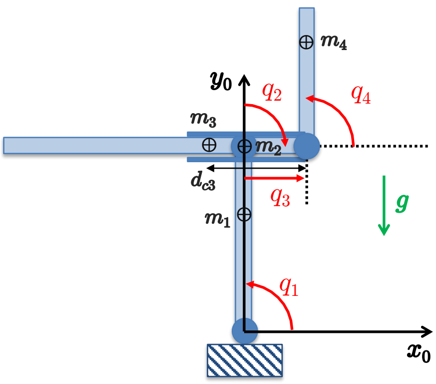

Figure 5: An equilibrium configuration of the RRPR robot: $\boldsymbol{q}_e = \left(\dfrac{\pi}{2}, -\dfrac{\pi}{2}, \dfrac{m_3 d_{c3}}{m_3 + m_4}, \dfrac{\pi}{2}\right)$.

Exercise #2

We define first a path parametrization (for the payload motion) and see how the direct kinematics of each robot is related to the common cooperative task. The problem is then addressed by considering the dynamics of each of the two robots and of the payload separately. However, the three subsystems will interact among each other via exchanged forces. For these, the principle of action and reaction holds: a force applied from a robot to the payload is equal to the same force applied by the payload to that robot.

Once we put everything together, the original optimal control problem is naturally decomposed in *i)* a minimum-time motion problem for the total mass of the system moving along the path, and *ii)* a minimum internal force problem along the normal to the path (while compensating for gravity). The first problem is solved by a common bang-bang profile for the command forces acting on the first (horizontal) prismatic joints of the two robots. The second problem is solved by equally distributing the total gravity load between the two command forces on the second (vertical) prismatic joints of the two robots. The resulting commands provide the minimum value of the objective function H (sum of the squared norms of the robot inputs) among all force commands that produce the same motion in minimum time along the given path.

Path parametrization

The simplest parametrization of the desired path is a linear one. Expressing it in the world reference frame RF_w, we have

$$\boldsymbol{p}(s) = \boldsymbol{p}_{in} + \frac{\boldsymbol{p}_{fin} - \boldsymbol{p}_{in}}{L}\, s = \begin{pmatrix} p_{in,x} + s \\ p_{in,y} \end{pmatrix}, \qquad s \in [0, L], \qquad L = \left\| \boldsymbol{p}_{fin} - \boldsymbol{p}_{in} \right\|.$$

The acceleration along the path is then given by

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \begin{pmatrix} \ddot{s} \\ 0 \end{pmatrix}. \tag{1}$$

Task kinematics

Indeed, the payload position $\boldsymbol{p}$ coincides with the end effector position of both robots. As seen from each robot side, the direct kinematics of each robot is related to the task by $\boldsymbol{p} = \boldsymbol{f}_A(\boldsymbol{q}_A) = \boldsymbol{f}_B(\boldsymbol{q}_B)$, with

$$\boldsymbol{f}_A(\boldsymbol{q}_A) = \begin{pmatrix} q_{A1} + q_{A1,0} \\ q_{A2,0} \end{pmatrix} = \begin{pmatrix} q_{B1} + q_{B1,0} \\ q_{B2,0} \end{pmatrix} = \boldsymbol{f}_B(\boldsymbol{q}_B),$$

for some constant (but irrelevant hereafter) values $q_{A1,0}$, $q_{A2,0}$, $q_{B1,0}$, and $q_{B2,0}$. Differentiating twice w.r.t. time and using (1), yields

$$\ddot{q}_{A1} = \ddot{q}_{B1} = \ddot{p}_x = \ddot{s}, \qquad \ddot{q}_{A2} = \ddot{q}_{B2} = \ddot{p}_y = 0. \tag{2}$$

Payload dynamics

With reference to the free-body diagram in Fig. 6, we shall consider the forces applied by the two robots A and B to the payload as decomposed in those that contribute to motion (along the path tangent) and those that may generate internal forces on the rigid payload (along the normal to the path) —and will produce in any event no motion. In this simple cooperative task, such decomposition occurs along two fixed directions, i.e., those of the world frame RF_w. Thus, we have

$$\begin{aligned} m_p\, \ddot{p}_x &= F_{A,motion} + F_{B,motion} \\ m_p\, \ddot{p}_y + m_p\, g_0 &= F_{A,internal} - F_{B,internal}. \end{aligned}$$

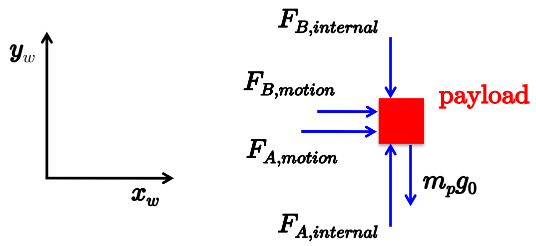

Figure 6: Free-body diagram of the forces applied to the payload.

Using (1), these simplify to

$$m_p\,\ddot{s} = F_{A,motion} + F_{B,motion} \tag{3}$$

$$m_p\,g_0 = F_{A,internal} - F_{B,internal} \tag{4}$$

Dynamics of the cooperating robots

For the two robots A and B, one has

$$\begin{aligned}(m_{A1}+m_{A2})\,\ddot{q}_{A1} &= \tau_{A1} - F_{A,motion}\\ m_{A2}\,\ddot{q}_{A2} + m_{A2}\,g_0 &= \tau_{A2} - F_{A,internal},\end{aligned} \tag{5}$$

and, respectively,

$$\begin{aligned}(m_{B1}+m_{B2})\,\ddot{q}_{B1} &= \tau_{B1} - F_{B,motion}\\ m_{B2}\,\ddot{q}_{B2} - m_{B2}\,g_0 &= \tau_{B2} - F_{B,internal}.\end{aligned} \tag{6}$$

As already mentioned, the force components F_i appearing on the right-hand sides of these equations are those applied to the robots by the payload dynamics.

Minimum time motion

Putting together (3) with the first components of (5) and (6), and using (2), we obtain

$$m_p\,\ddot{s} = \tau_{A1} - (m_{A1}+m_{A2})\,\ddot{s} + \tau_{B1} - (m_{B1}+m_{B2})\,\ddot{s}$$

or

$$m_{tot}\,\ddot{s} = \tau_{A1} + \tau_{B1} = \tau_{motion}, \qquad \text{with} \quad m_{tot} = m_p + m_{A1} + m_{A2} + m_{B1} + m_{B2}. \tag{7}$$

As a result, the rest-to-rest minimum time solution for the total mass m_{tot} moving under an equivalent force command τ_{motion}, bounded as

$$|\tau_{motion}| \le \tau_{A,max} + \tau_{B,max} = \tau_{max},$$

will be given by the bang-bang profiles τ^*_{A1}, τ^*_{B1}, and τ^*_{motion}, as illustrated in Fig. 7. Accordingly, the minimum motion time is

$$T^* = \sqrt{\frac{4L}{\ddot{s}_{max}}} = 2\sqrt{\frac{L\,m_{tot}}{\tau_{max}}}.$$

Minimum internal force

Consider now (4) together with the second components of (5) and (6). Using again (2), we obtain

$$m_p\,g_0 = \tau_{A2} - m_{A2}g_0 - \tau_{B2} - m_{B2}g_0,$$

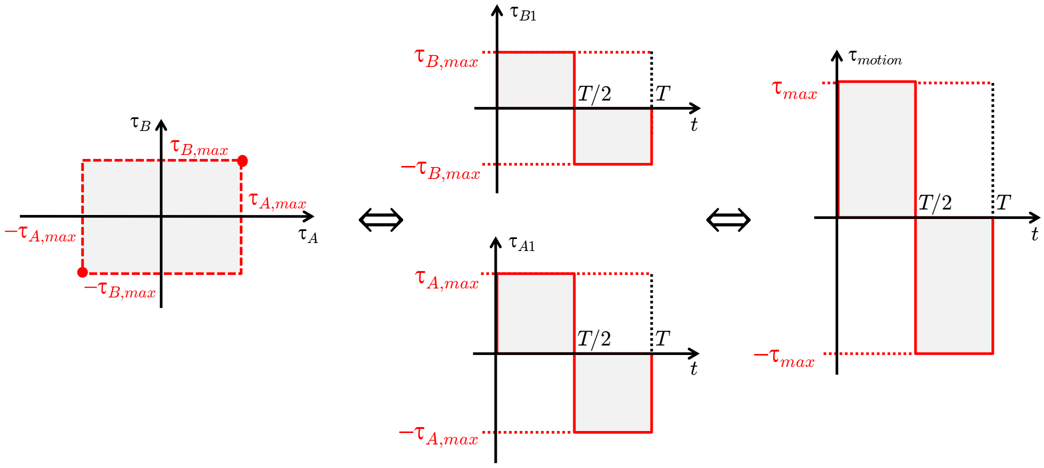

Figure 7: Bang-bang profiles of the individual robot force commands τ_{A1}^* and τ_{B1^*} and of the equivalent total force τ_{motion}^* in the minimum motion time solution.

or

$$(m_p + m_{A2} + m_{B2})\, g_0 = \tau_{A2} - \tau_{B2} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \tau_{A2} \\ \tau_{B2} \end{pmatrix} = \boldsymbol{J}_{internal} \begin{pmatrix} \tau_{A2} \\ \tau_{B2} \end{pmatrix}. \tag{8}$$

The requirement of minimizing the objective function

$$H = \|\boldsymbol{\tau}_A\|^2 + \|\boldsymbol{\tau}_B\|^2 = \tau_{A1}^2 + \tau_{A2}^2 + \tau_{B1}^2 + \tau_{B2}^2,$$

in view of the (unique) minimum time solution already found for $\tau_{A1} = \tau_{A1}^*$ and $\tau_{B1} = \tau_{B1}^*$, reduces to the minimization of the quadratic sub-function

$$H' = \tau_{A2}^2 + \tau_{B2}^2,$$

subject to the linear constraint (8). It is easy to see that the (unique) solution to this simple LQ problem for the remaining robot commands is

$$\tau_{A2}^* = \frac{1}{2}\,(m_p + m_{A2} + m_{B2})\, g_0, \qquad \tau_{B2}^* = -\frac{1}{2}\,(m_p + m_{A2} + m_{B2})\, g_0 = -\tau_{A2}^*. \tag{9}$$

Indeed, any other force pair of the perturbed form

$$\tau_{A2} = \tau_{A2}^* + \Delta, \qquad \tau_{B2} = \tau_{B2}^* + \Delta, \qquad \forall \Delta \in \mathbb{R},$$

will still satisfy the linear constraint (8) of the reduced problem. These force perturbations are in fact in the null space of the Jacobian of the constraint, or

$$\Delta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}_{internal}\right\},$$

and therefore produce solutions that are larger in norm (thus, with a higher H'). Such perturbations have the meaning of internal forces that may arise when the payload is rigidly held by the two robots. Thus, the optimal solution (9) has the nice physical interpretation of minimizing the internal forces ($\Delta = 0$). We finally note that, if gravity were not present (e.g., for motions occurring on a horizontal plane), the solution that minimizes the internal forces would be $\tau_{A2}^* = \tau_{B2}^* = 0$.

Exercise #3

With the dynamics of this planar 3R robot given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

we apply first a feedback linearization law in the joint space, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{a}, \tag{10}$$

to convert the self-motion task into a purely kinematic one. The robot should always keep the position of its end effector

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1+q_2) + l_3 \cos(q_1+q_2+q_3) \\ l_1 \sin q_1 + l_2 \sin(q_1+q_2) + l_3 \sin(q_1+q_2+q_3) \end{pmatrix}$$

at

$$\boldsymbol{p}_{in} = \boldsymbol{f}(\boldsymbol{q}_{in}) = \begin{pmatrix} l_1 \cos\dfrac{\pi}{9} + \dfrac{\sqrt{2}}{2} l_3 \\ -l_1 \cos\dfrac{\pi}{9} + l_2 + \dfrac{\sqrt{2}}{2} l_3 \end{pmatrix}.$$

Indeed, the 3R robot has $n - m = 3 - 2 = 1$ degree of redundancy for the positioning task in the plane. A joint acceleration command performing a robot self-motion, as driven by the target position $q_{3,fin}$ for the third joint, can then be designed as

$$\boldsymbol{a} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\left(\begin{pmatrix} 0 \\ 0 \\ \alpha\,(q_{3,fin} - q_3) \end{pmatrix} - \boldsymbol{K}_v\dot{\boldsymbol{q}}\right), \tag{11}$$

for some $\alpha > 0$. In (11), a damping velocity term $-\boldsymbol{K}_v\dot{\boldsymbol{q}}$, with diagonal gain matrix $\boldsymbol{K}_v > 0$, has been added in the null space of the task Jacobian so as to stabilize the joint motion[1]. In order to reject also position and/or velocity errors that may occur around the desired constant end-effector position $\boldsymbol{p}_{in}$, a more robust version of the command (11) is given by

$$\boldsymbol{a} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\begin{pmatrix} -k_{v,1}\,\dot{q}_1 \\ -k_{v,2}\,\dot{q}_2 \\ \alpha\,(q_{3,fin} - q_3) - k_{v,3}\,\dot{q}_3 \end{pmatrix}, \tag{12}$$

including thus a PD action, with (typically diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, on the Cartesian position error, and taking into account that $\dot{\boldsymbol{p}}_{in} = \boldsymbol{0}$. Plugging (12) into (10) yields the torque control law

$$\begin{aligned}\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\Big[&\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ &+ \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\begin{pmatrix} -k_{v,1}\,\dot{q}_1 \\ -k_{v,2}\,\dot{q}_2 \\ \alpha\,(q_{3,fin} - q_3) - k_{v,3}\,\dot{q}_3 \end{pmatrix}\Big] + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}).\end{aligned} \tag{13}$$

It should be noted that the control law (13), or simply the acceleration law (11) and its robust version (12), may or may not guarantee the reaching of the final desired value for the third joint.

[1] This is rather customary (and almost mandatory) when resolving redundancy at the acceleration level.

In principle, at a steady-state with $\dot{\boldsymbol{q}} = \mathbf{0}$, the projection operator $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}^{\#}\boldsymbol{J}$ (with its columns $\boldsymbol{P}_i$, $i = 1, 2, 3$) may still mask the presence of a residual configuration error. Thus, one should also show that there exists no configuration $\boldsymbol{q}^*$ such that $\boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{p}_{in}$ (i.e., belonging to the self-motion manifold in the joint space associated to the initial point P_{in}) for which $\boldsymbol{P}_3(\boldsymbol{q}^*)\, e_3 = \mathbf{0}$ while $e_3 = q_{3,fin} - q_3^* \neq 0$. Such statement about non-existence is in fact true, but its formal proof is not trivial. Therefore, an alternative approach that directly guarantees also the convergence of q_3 to $q_{3,fin}$ may be more attractive.

In the following, we will illustrate the use of a joint space decomposition approach at the acceleration level (i.e., to be applied, after feedback linearization, to $\ddot{\boldsymbol{q}} = \boldsymbol{a}$). In this case, one focuses on the command to be given to the third joint, the one that has a special target assigned, leaving to the other two joints the task of keeping the end effector at the desired position P_{in}. First, for the third joint we choose the control law

$$\ddot{q}_3 = a_3 = k_{p,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3, \qquad k_{p,3} > 0,\ k_{d,3} > 0. \tag{14}$$

As a result, the error e_3 will satisfy the linear differential equation

$$\ddot{e}_3 + k_{d,3}\dot{e}_3 + k_{p,3}e_3 = 0,$$

which guarantees that q_3 will converge exponentially from any initial state to the desired $q_{3,fin}$, with $\dot{q}_3 = \dot{q}_{3,fin} = 0$. Moreover, by a suitable choice of the gains $k_{p,3}$ and $k_{d,3}$, the natural motion of q_3 will always remain in the interval $[q_{3,fin}, q_{3,in}] = [-\pi/2, -\pi/4]$ (i.e., without overshooting or wandering). Next, decompose the second-order differential kinematics as follows:

$$\begin{aligned}\ddot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_{12}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{12} + \boldsymbol{J}_3(\boldsymbol{q})\,\ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &= \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -l_3 s_{123} \\ l_3 c_{123} \end{pmatrix} \ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}.\end{aligned} \tag{15}$$

The square sub-Jacobian $\boldsymbol{J}_{12}$ made by the first two columns of $\boldsymbol{J}$ has

$$\det \boldsymbol{J}_{12}(\boldsymbol{q}) = l_1\,(l_2 \sin q_2 + l_3 \sin(q_2 + q_3))\,.$$

As long as this determinant is different from zero, we can set $\ddot{\boldsymbol{p}} = \mathbf{0}$ in (15) and solve for $\ddot{\boldsymbol{q}}_{12}$ so as to realize our self-motion task by

$$\ddot{\boldsymbol{q}}_{12} = -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\left(\boldsymbol{J}_3(\boldsymbol{q})\,\ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right), \tag{16}$$

for any motion $\ddot{q}_3$, in particular that given by (14). To introduce more robustness in the task of keeping the end-effector position at $\boldsymbol{p}_{in}$, we replace

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_{in} = \mathbf{0} \qquad \Rightarrow \qquad \ddot{\boldsymbol{p}} = \boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{17}$$

with (diagonal) 2×2 gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ weighting, respectively, the position error $\boldsymbol{e}_P = \boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})$ and the velocity error $\boldsymbol{e}_D = \dot{\boldsymbol{p}}_{in} - \dot{\boldsymbol{p}} = -\dot{\boldsymbol{p}} = -\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Using (14) and (17) in eq. (15) and solving again for $\ddot{\boldsymbol{q}}_{12}$ yields

$$\ddot{\boldsymbol{q}}_{12} = \boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\left(\boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_3(\boldsymbol{q})\,(k_{P,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right). \tag{18}$$

We can also combine (14) and (18) in a single formula as

$$\boldsymbol{a} = \begin{pmatrix} \ddot{\boldsymbol{q}}_{12} \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\,\boldsymbol{J}_3(\boldsymbol{q}) \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \left(\boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}} \\ k_{P,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3 \end{pmatrix}. \tag{19}$$

Finally, plugging (19) into (10) yields the torque control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\, \boldsymbol{J}_{3}(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right) - \left(\boldsymbol{K}_D\, \boldsymbol{J}(\boldsymbol{q}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\right) \dot{\boldsymbol{q}} \\ k_{P,3} \left(q_{3,fin} - q_3\right) - k_{d,3}\, \dot{q}_3 \end{pmatrix} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \tag{20}$$

The last thing to check is the absence of singularities for $\boldsymbol{J}_{12}(\boldsymbol{q})$ during the self-motion under the control law (20), or simply the acceleration law (19). It can be shown that $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = 0$ if and only if the end-effector of the 3R robot finds itself aligned with the first link of the structure. From the illustration in Fig. 8, it is rather evident that such condition is not encountered in this task.

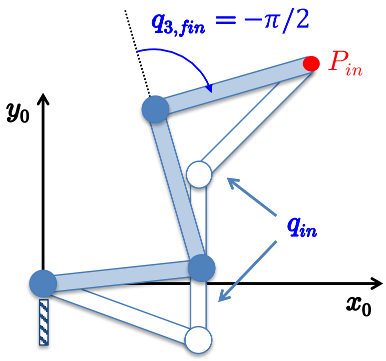

Figure 8: Configuration reached by the 3R robot at the end of the controlled self-motion task.

Exercise #4

The dynamics of the system represented in Fig. 4 is

$$m_r \ddot{x}_r + (b_r + b_s)\, \dot{x}_r + k_s x_r = F, \tag{21}$$

with the contact force measured by the sensor given by

$$F_c = k_s x_r.$$

Since we deal with a linear dynamics, one can also transform (21) in the Laplace domain and represent the system by its transfer function from the control input F to the controlled output F_c as

$$P(s) = \frac{F_c(s)}{F(s)} = \frac{k_s x_r(s)}{F(s)} = \frac{k_s}{m_r s^2 + (b_r + b_s)\, s + k_s}. \tag{22}$$

Since the physical parameters m_r, b_r, b_s and k_s are all positive, $P(s)$ has two poles with negative real part, and the open-loop system is thus asymptotically stable (with a unitary steady-state gain, $P(0) = 1$). The simplest feedback controller $C(s)$ that tries to regulate the contact force to a (constant, but arbitrary) desired value F_d is a proportional law to the force error $F_e = F_d - F_c$,

$$F = K_P\, (F_d - F_c) = K_P\, F_e \qquad \Longleftrightarrow \qquad C(s) = \frac{F(s)}{F_e(s)} = K_P > 0.$$

The input-output transfer function of the closed-loop system would then be

$$W(s) = \frac{F(s)}{F_d(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{k_s K_P}{m_r s^2 + (b_r + b_s)\, s + k_s\, (1 + K_P)},$$

which is still asymptotically stable, but with a non-unitary gain $W(0) = K_P/(1 + K_P) \neq 1$. This means that the steady-state output response to a desired step input F_d would have an error (unless $K_P \to \infty$, which is impossible). The value of this force error can also be found from the input-error transfer function,

$$W_e(s) = \frac{F_e(s)}{F_d(s)} = \frac{F_d(s) - F(s)}{F_d(s)} = 1 - W(s) = \frac{1}{1 + P(s)C(s)} = \frac{m_r s^2 + (b_r + b_s)\, s + k_s}{m_r s^2 + (b_r + b_s)\, s + k_s\,(1 + K_P)}.$$

In fact, from the final value theorem, the steady-state error for a constant F_d is computed as

$$F_{e,\infty} = \lim_{t\to\infty} F_e(t) = \lim_{s\to 0} F_e(s) = \lim_{s\to 0} W_e(s) F_d(s) = W_e(0)\, F_d = \frac{1}{1 + K_P}\, F_d \neq 0.$$

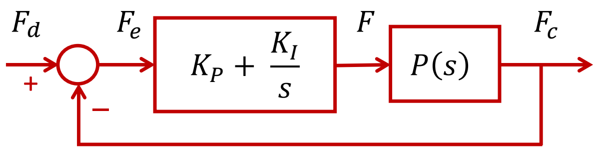

Figure 9: The closed-loop scheme of the robot-environment system under PI force control.

In order to eliminate this steady-state error in a robust way (i.e., using feedback), we need an integral action[2] (a pole in $s = 0$) in the controller $C(s)$. With reference to Fig. 9, we consider then a proportional-integral (PI) controller on the force error $F_e = F_d - F_c$, or

$$F(t) = K_P\, F_e(t) + K_I \int_0^t F_e(\tau) d\tau \qquad \Longleftrightarrow \qquad C(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} \tag{23}$$

Combining (23) with (22) gives the closed-loop system

$$W(s) = \frac{F(s)}{F_d(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{k_s\,(K_P s + K_I)}{m_r s^3 + (b_r + b_s)\, s^2 + k_s\,(1 + K_P)\, s + k_s K_I}, \tag{24}$$

with gain $W(0) = 1$. To check the conditions under which the three poles of $W(s)$ will all have negative real part, we apply the Routh criterion. From the Routh table built for the polynomial denominator of $W(s)$ in (24)

$$\begin{array}{c|cc} 3 & m_r & k_s\,(1 + K_P) \\ 2 & b_r + b_s & k_s K_I \\ 1 & k_s\,(1 + K_P) - \dfrac{m_r k_s K_I}{b_r + b_s} & \\ 0 & k_s K_I & \end{array}$$

we see that the elements in the first column have the same (here, positive) sign iff

$$K_I > 0, \qquad (1 + K_P) - \frac{m_r K_I}{b_r + b_s} > 0.$$

[2]From the elementary feedback theory, in order to guarantee zero error at steady state in the step response, the control system in Fig. 9 should be asymptotically stable and have (at least) a pole in $s = 0$ in the forward path (type I). If the process $P(s)$ does not have already such a pole, it should be introduced in the controller.

Therefore, choosing the two gains K_P and K_I in the ranges

$$K_I \geq \frac{b_r + b_s}{m_r} > 0, \qquad K_P > \frac{m_r K_I - (b_r + b_s)}{b_r + b_s} \geq 0$$

will ensure asymptotic stability of the closed-loop system (in a robust way with respect to uncertainties in system parameters —there is only a need to enforce inequalities that are simple to overbound). Moreover, the force error at steady state will be zero as expected, since the input-error transfer function

$$W_e(s) = 1 - W(s) = \frac{\left(m_r s^2 + (b_r + b_s)\, s + k_s\right) s}{m_r s^3 + (b_r + b_s)\, s^2 + k_s\,(1 + K_P)\, s + k_s K_I},$$

has a zero at $s = 0$, and thus

$$F_{e,\infty} = W_e(0)\, F_d = 0 \cdot F_d = 0.$$

As a result, at steady state

$$\bar{F}_c = F_d = k_s \bar{x}_r, \qquad \bar{x}_r = \frac{F_d}{k_s}, \qquad \bar{F} = K_I \int_0^\infty F_e(\tau) d\tau = F_d.$$

* * * * *