Robotics 2

July 8, 2022

Exercise #1

A generic 3R spatial manipulator, which is self-balanced with respect to gravity, is driven by three actuators that deliver the torques $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix}^T$. When using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$, the robot dynamic model is expressed in compact form as

$$\boldsymbol{M_q}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c_q}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau_q}, \tag{1}$$

where

$$\boldsymbol{M_q} = \begin{pmatrix} m_{11}(q_2, q_3) & 0 & 0 \\ 0 & m_{22}(q_3) & m_{23}(q_3) \\ 0 & m_{23}(q_3) & m_{33} \end{pmatrix}, \quad \boldsymbol{c_q} = \begin{pmatrix} c_1(q_2, q_3, \dot{q}_2, \dot{q}_3) \\ c_2(q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \\ c_3(q_2, q_3, \dot{q}_1, \dot{q}_2) \end{pmatrix}, \quad \boldsymbol{\tau_q} = \begin{pmatrix} \tau_1 \\ \tau_2 + \tau_3 \\ \tau_3 \end{pmatrix}.$$

- Find the set of coordinates $\boldsymbol{p} \in \mathbb{R}^3$ on which the torque vector $\boldsymbol{\tau} \in \mathbb{R}^3$ produces work component-wise, and give the coordinate transformation between $\boldsymbol{q}$ and $\boldsymbol{p}$.
- Write the dynamic model in the coordinates $\boldsymbol{p}$, expressing the elements of the inertia matrix $\boldsymbol{M_p}$ and of the Coriolis and centrifugal vector $\boldsymbol{c_p}$ in terms of the elements m_{ij} and c_i of model (1). For compactness, there is no need to replace the dependences on $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ by those on $(\boldsymbol{p}, \dot{\boldsymbol{p}})$ within these terms.

Exercise #2

The dynamic model of a serial manipulator with n revolute joints can always be written as

$$\begin{pmatrix} m_{11}(\boldsymbol{q}) & \boldsymbol{m}_{12}^T(\boldsymbol{q}) \\ \boldsymbol{m}_{12}(\boldsymbol{q}) & \boldsymbol{M}_{22}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{\boldsymbol{q}}_2 \end{pmatrix} + \begin{pmatrix} n_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ \boldsymbol{n}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \boldsymbol{\tau}_2 \end{pmatrix}, \tag{2}$$

where the joint variables $\boldsymbol{q} \in \mathbb{R}^n$ are partitioned in $q_1 \in \mathbb{R}$ and $\boldsymbol{q}_2 \in \mathbb{R}^{n-1}$ and, similarly, the joint torques $\boldsymbol{\tau} \in \mathbb{R}^n$ in $\tau_1 \in \mathbb{R}$ and $\boldsymbol{\tau}_2 \in \mathbb{R}^{n-1}$. The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the dynamic terms $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$ in (2) have been partitioned accordingly. Suppose that a constraint is imposed on the first joint, so that $q_1(t) = k$ (an arbitrary constant value).

- Derive the explicit form of the $(n-1)$-dimensional *reduced dynamics* of the constrained robot.
- Provide the corresponding expression of the force multiplier $\lambda \in \mathbb{R}$ that arises when attempting to violate the constraint during a generic robot motion.
- Define control laws for τ_1 and for $\boldsymbol{\tau}_2$ that regulate the robot to a desired configuration $\boldsymbol{q}_d$, which is feasible (i.e., such that $q_{1d} = k$), while keeping $\lambda(t) = 0$ at all times.

Exercise #3

With reference to Fig. 1, consider a Cartesian (PP) robot with links of mass m_1 and m_2, moving in a vertical plane. The end-effector should transfer from rest to rest between two generic points P_s and P_g in *minimum time*, with the two input force commands being bounded as

$$|u_i| \leq U_{i,max}, \qquad i = x, y.$$

The robot starts at an equilibrium and should remain in equilibrium when the motion ends.

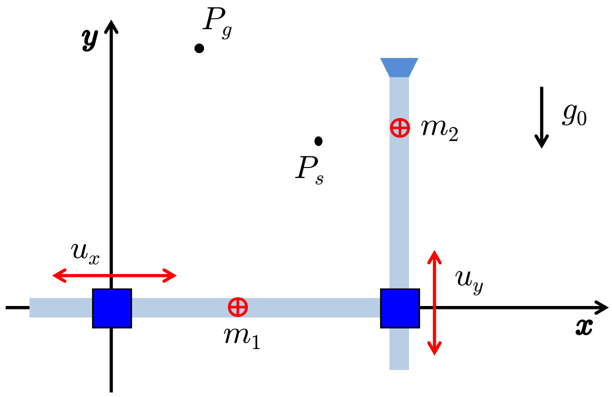

Figure 1: A Cartesian robot in a point-to-point task in the vertical plane.

- Determine the minimum feasible value T^* of the transfer time in a parametric form with respect to the problem data.
- For the numerical values

$$P_s = (1, 0.3), \quad P_g = (0.6, 0.7) \ \text{[m]}, \quad m_1 = 5, \ m_2 = 3 \ \text{[kg]}, \quad U_{x,max} = U_{y,max} = 40 \ \text{[N]},$$

  evaluate time T^* and sketch the optimal profiles of force, acceleration, velocity, and position of the two robot joints.
- Is the end-effector path associated to this time-optimal trajectory a straight line segment between P_s and P_g? (Support your answer with an argument: a simple 'yes' or 'no' does not count!).

[180 minutes; open books]

Solution

July 8, 2022

Exercise #1

The objective is to obtain the robot dynamic equations in the transformed coordinates $\boldsymbol{p} = \boldsymbol{t}(\boldsymbol{q})$ such that

$$\boldsymbol{M_p}(\boldsymbol{p})\ddot{\boldsymbol{p}} + \boldsymbol{c_p}(\boldsymbol{p}, \dot{\boldsymbol{p}}) = \boldsymbol{\tau_p} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}, \tag{3}$$

i.e., in the right-hand side of eq. (3) the three available actuators torques $\boldsymbol{\tau}$ are those performing work of the coordinates $\boldsymbol{p}$.
Since the following holds by duality

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{t}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{J_t}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Longleftrightarrow \qquad \boldsymbol{\tau_q} = \boldsymbol{J}_t^T(\boldsymbol{q})\boldsymbol{\tau_p},$$

we extract from the right-hand side of (1) the required Jacobian of the transformation,

$$\boldsymbol{\tau_q} = \begin{pmatrix} \tau_1 \\ \tau_2 + \tau_3 \\ \tau_3 \end{pmatrix} = \boldsymbol{J_t^T}\boldsymbol{\tau_p} \qquad \Rightarrow \qquad \boldsymbol{J_t^T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which turns out to be constant. Therefore, the change of coordinates is linear

$$\boldsymbol{p} = \boldsymbol{J_t}\,\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_2 + q_3 \end{pmatrix}$$

and its inverse is

$$\boldsymbol{q} = \boldsymbol{J_t^{-1}}\boldsymbol{p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 - p_2 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{J_t^{-1}}\dot{\boldsymbol{p}}, \quad \ddot{\boldsymbol{q}} = \boldsymbol{J_t^{-1}}\ddot{\boldsymbol{p}}.$$

Plugging these into (1) yields finally (3) with

$$\begin{aligned} \boldsymbol{M_p}(\boldsymbol{p}) &= \boldsymbol{J_t^{-T}}\,\boldsymbol{M_q}(\boldsymbol{q})\,\boldsymbol{J_t^{-1}} \\ &= \begin{pmatrix} m_{11}(p_2, p_3 - p_2) & 0 & 0 \\ 0 & m_{22}(p_3 - p_2) + m_{33} - 2m_{23}(p_3 - p_2) & m_{23}(p_3 - p_2) - m_{33} \\ 0 & m_{23}(p_3 - p_2) - m_{33} & m_{33} \end{pmatrix} \end{aligned} \tag{4}$$

$$\boldsymbol{c}_p(\boldsymbol{p}, \dot{\boldsymbol{p}}) = \boldsymbol{J}_t^{-T}\,\boldsymbol{c}_q(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(p_2, p_3 - p_2, \dot{p}_2, \dot{p}_3 - \dot{p}_2) \\ c_2(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2, \dot{p}_3 - \dot{p}_2) - c_3(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2) \\ c_3(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2) \end{pmatrix}, \tag{5}$$

where the arguments in the functions on the right-hand sides of (4) and (5) have been substituted with the inverse mappings $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \left(\boldsymbol{J_t^{-1}}\boldsymbol{p}, \boldsymbol{J_t^{-1}}\dot{\boldsymbol{p}}\right)$. This is not strictly needed in general (nor required by the text), but is particularly simple here because of the linearity of the transformation.

Exercise #2

We apply the standard procedure for obtaining the reduced dynamic model, which is particularly simple in this case.

The Jacobian of the scalar constraint $h(\boldsymbol{q}) = q_1(t) - k = 0$ is $\boldsymbol{A} = \partial h(\boldsymbol{q})/\partial \boldsymbol{q} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \end{pmatrix}$. Therefore, the obvious completion of $\boldsymbol{A}$ with a matrix $\boldsymbol{D}$ to obtain a square nonsingular matrix is

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{(n-1)\times(n-1)} \end{pmatrix} = \boldsymbol{I}_{n\times n},$$

and thus

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{(n-1)\times(n-1)} \end{pmatrix}.$$

As a result, the pseudo-velocity vector is

$$\boldsymbol{v} = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_2 \in \mathbb{R}^{n-1}.$$

Being all the defined transformation matrices constant, the reduced dynamics becomes

$$\left(\boldsymbol{F}^T \boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}\right)\dot{\boldsymbol{v}} = \boldsymbol{F}^T\left(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right),$$

or

$$\boldsymbol{M}_{22}(\boldsymbol{q})\,\ddot{\boldsymbol{q}}_2 = \boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}_2 - \bar{\boldsymbol{c}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) - \bar{\boldsymbol{g}}_2(\boldsymbol{q}), \tag{6}$$

where a 'bar' on a dynamic term means that:

- $\boldsymbol{M}_{22}$ is identical to the same block in (2), because q_1 does not appear in the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of any robot (a so-called *cyclic variable*);
- $\bar{\boldsymbol{c}}_2$ is evaluated at $\dot{q}_1 = 0$ while, as a result of the previous property, is also independent from q_1;
- $\bar{\boldsymbol{g}}_2$ is evaluated at $q_1 = q_{1d} = k$.

Similarly, the expression of the (scalar) force multiplier λ becomes

$$\begin{aligned} \lambda = \boldsymbol{E}^T\left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}\dot{\boldsymbol{v}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{\tau}\right) &= \boldsymbol{m}_{12}^T(\boldsymbol{q})\ddot{\boldsymbol{q}}_2 + \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) - \tau_1 \\ &= \boldsymbol{m}_{12}^T(\boldsymbol{q})\boldsymbol{M}_{22}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}})\right) + \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) - \tau_1, \end{aligned} \tag{7}$$

where eq. (6) has been used.

For any arbitrary choice of the torque $\boldsymbol{\tau}_2$, the control law applied at joint 1 to make sure that $\lambda(t) \equiv 0$ is then

$$\tau_1 = \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{m}_{12}^T(\boldsymbol{q})\boldsymbol{M}_{22}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}})\right). \tag{8}$$

Note that gravity effects acting on joint 1 are also cancelled at rest by the $\bar{g}_1$ torque within $\bar{n}_1$. Furthermore, in order to achieve regulation to a desired $\boldsymbol{q}_{2d}$, one can use a feedback linearization approach yielding

$$\boldsymbol{\tau}_2 = \boldsymbol{M}_{22}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{q}_{2d} - \boldsymbol{q}_2) - \boldsymbol{K}_D\dot{\boldsymbol{q}}_2\right) + \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \boldsymbol{K}_D > 0, \tag{9}$$

which provides exponential and decoupled stabilization of the error $\boldsymbol{e}_2 = \boldsymbol{q}_{d2} - \boldsymbol{q}_2$ to zero. In alternative, one can design a simpler PD regulator with gravity cancellation

$$\boldsymbol{\tau}_2 = \boldsymbol{K}_P(\boldsymbol{q}_{2d} - \boldsymbol{q}_2) - \boldsymbol{K}_D\dot{\boldsymbol{q}}_2 + \bar{\boldsymbol{g}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \boldsymbol{K}_D > 0. \tag{10}$$

It is straightforward to prove asymptotic stability of the closed-loop system with (10), using a Lyapunov/LaSalle argument on the reduced dynamics (6).

Exercise #3

The task does not require any coordination between the two joints, nor there is a velocity bound. Thus, each joint will move as fast as possible with a bang-bang force profile. The minimum transfer time will be given by the slowest joint completing its motion (while the fastest joint remains at rest for some interval).

The two scalar problems are however different because of the presence of gravity on the vertical (second) joint, which offsets its feasible acceleration range. With $\boldsymbol{q} = (x, y)$, the dynamic model of this PP robot is

$$\begin{aligned} (m_1 + m_2)\,\ddot{x} &= u_x \\ m_2\,\ddot{y} + m_2\,g_0 &= u_y, \end{aligned} \tag{11}$$

being $g_0 = 9.81$ [m/s^2]. As a result

$$\begin{aligned} |u_x| \le U_{x,max} \quad &\Rightarrow \quad |\ddot{x}| \le \frac{U_{x,max}}{m_1 + m_2} \\ |u_y| \le U_{y,max} \quad &\Rightarrow \quad -\left(\frac{U_{y,max}}{m_2} + g_0\right) \le \ddot{y} \le \frac{U_{y,max}}{m_2} - g_0, \end{aligned}$$

with an asymmetric feasible range for $\ddot{y}$. Moreover, let $P_g - P_s = (\Delta x, \Delta y)$ be the required displacement.

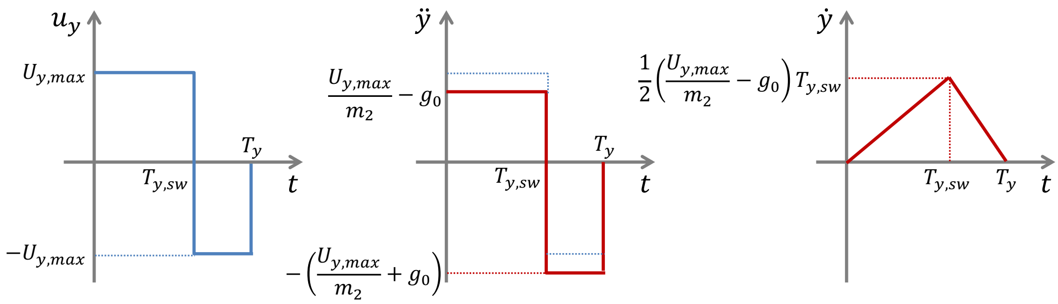

Figure 2: Force, acceleration and velocity profiles for the joint moving under gravity by $\Delta y > 0$.

Consider first the time-optimal motion for the y-axis under gravity. With reference to Fig. 2, which shows only the case of $\Delta y > 0$, the acceleration is bang-bang and the velocity is triangular, both switching at $T_{y,sw}$ and being typically asymmetric in time with respect to the total interval T_y. Two relations are then obtained from these behaviors: *i*) the area with sign covered by the acceleration profile (i.e., its integral) should be zero in order to obtain a rest-to-rest motion, i.e.,

$$\left(\operatorname{sign}(\Delta y)\,\frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw} - \left(\operatorname{sign}(\Delta y)\,\frac{U_{y,max}}{m_2} + g_0\right)(T_y - T_{y,sw}) = 0;$$

ii) the area with sign covered by the velocity profile should be equal to the required displacement Δy of the joint, i.e.,

$$\frac{1}{2}\left(\operatorname{sign}(\Delta y)\,\frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw}^2 + \frac{1}{2}\left(\operatorname{sign}(\Delta y)\,\frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw}\,(T_y - T_{y,sw}) = \Delta y.$$

Solving these two equations for T_y and $T_{y,sw}$ gives

$$T_y = 2\sqrt{\frac{m_2\,|\Delta y|\,U_{y,max}}{U_{y,max}^2 - (m_2 g_0)^2}} \tag{12}$$

and

$$T_{y,sw} = \frac{T_y}{2}\left(1 + \text{sign}(\Delta y)\,\frac{m_2 g_0}{U_{y,max}}\right) \neq \frac{T_y}{2}. \tag{13}$$

If $\Delta y > 0$ then $T_{y,sw} > T_y/2$ and, viceversa, if $\Delta y < 0$ then $T_{y,sw} < T_y/2$.

The time-optimal motion for the x-axis without gravity is a sub-case of the formulas (12) and (13), obtained by setting $g_0 = 0$ and replacing m_2 with the total mass $m_1 + m_2$ driven by this joint. Thus,

$$T_x = 2\sqrt{\frac{(m_1 + m_2)\,|\Delta x|}{U_{x,max}}} \qquad \text{and} \qquad T_{x,sw} = \frac{T_x}{2}. \tag{14}$$

Therefore,

$$T^* = \max\{T_x, T_y\}. \tag{15}$$

Note that the joint that arrives first should remain then at rest, waiting for the slower joint to reach its goal. Indeed, the faster joint could also remain at rest at the beginning and then start moving at an instant such that task completion occurs simultaneously at T^* for both joints. In any event, to stay at rest at steady state, the horizontal joint does not require any force ($u_{x,ss} = 0$), whereas the vertical joint should sustain gravity ($u_{y,ss} = m_2 g_0$). Except for very special combinations of problem data, the above minimum-time control strategy will not result in a coordinated robot motion (i.e., all joints start and end their motion at the same instant, without intermediate stops).

With the problem data, it is $\Delta x = -0.4$, $\Delta y = 0.4$ [m] and we obtain the following motion times (in seconds):

$$T_x = 0.5657, \quad T_{x,sw} = 0.2828, \quad T_y = 0.5115, \quad T_{y,sw} = 0.4439 \qquad \Rightarrow \qquad T^* = 0.5657.$$

The results are reported in Fig. 3. We note that, since in this case the y-axis is faster, when this joint reaches its goal (at $t = T_y < T^*$), the control input switches to the steady-state equilibrium force $u_{y,ss} = 20.43$ [N].

The resulting Cartesian path of the robot end-effector is not a straight line segment between P_s and P_g —see Fig. 4. Rather, the initial part of the path, between $t = 0$ and $t = T_{x,sw}$, is linear since

$$\frac{dy}{dx} = \frac{\dot y}{\dot x} = \frac{\dfrac{U_{y,max}}{m_2}\,t}{\dfrac{U_{x,max}}{m_1 + m_2}\,t} = \frac{U_{y,max}}{U_{x,max}}\,\frac{m_2}{m_1 + m_2} = k;$$

it is followed then by two different curvilinear parts[1], between $t = T_{x,sw}$ and $t = T_{y,sw}$ and between $t = T_{y,sw}$ and $t = T_y$; the last part, between $t = T_y$ and $t = T_x = T^*$, is again a (very short) linear segment.

It can be shown[2] that even if the problem data were such that the two joints complete their task at the same instant (i.e., $T^* = T_x = T_y$), the resulting path would still not be a linear segment between P_s and P_g.

[1] These parts have no easy geometric expressions. In fact, the tangent to each curve is a rational function given by the ratio of linear polynomials in time t.

[2] If you write a code for this problem, try out $P_g = (0.673, 0.7)$ with all the rest being the same. This will result in a coordinated joint motion, but still without a resulting linear Cartesian path. Try out also a motion task in favor of gravity ($\Delta y < 0$), in order to better understand the need of the absolute value and sign of Δy in eqs. (12) and (13).

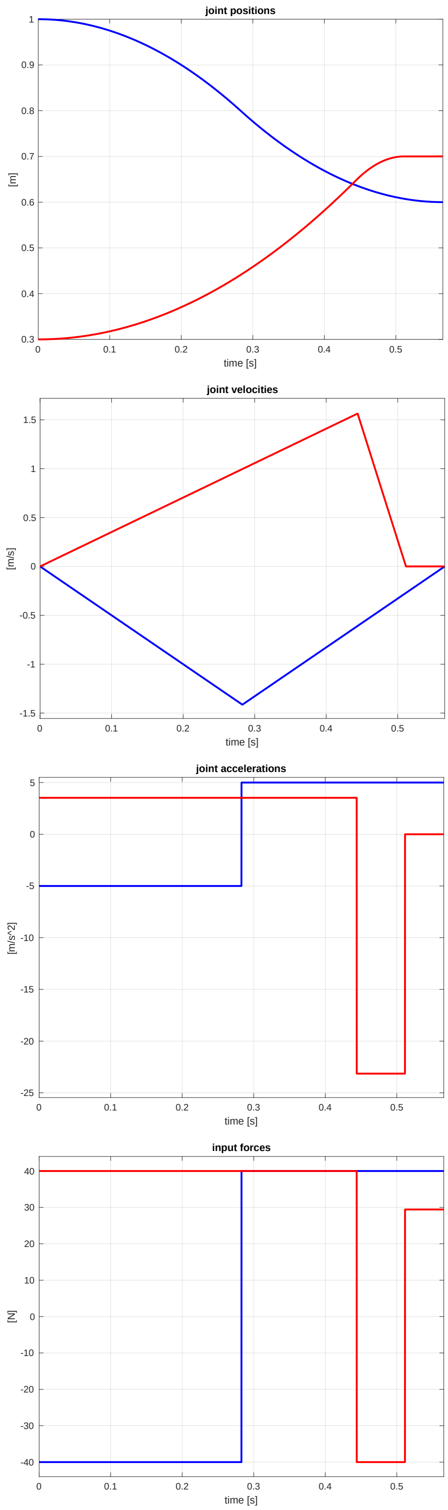

Figure 3: From top to bottom: Joint position, velocity, acceleration, and input force in the time-optimal solution (blue = x-axis, red = y-axis).

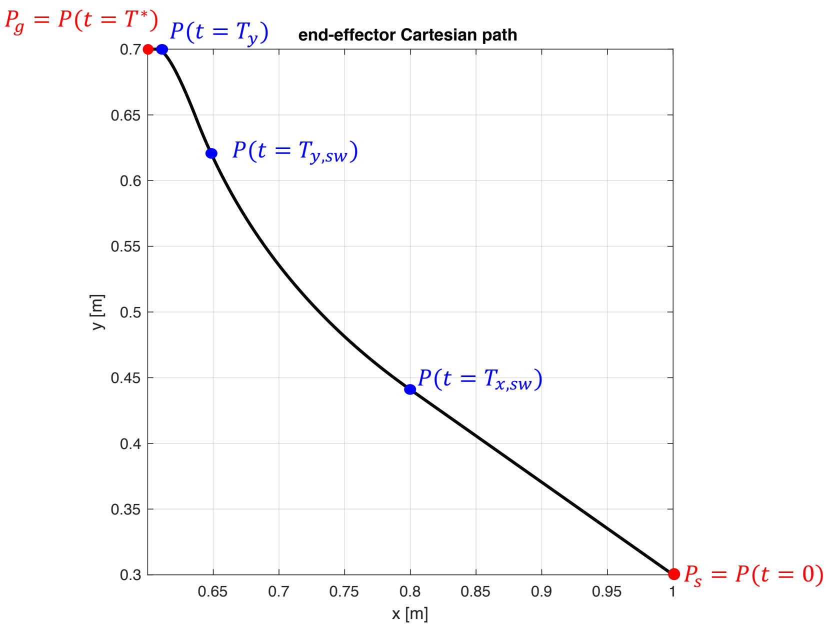

Figure 4: Cartesian path of the robot end-effector.

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