Robotics 2

September 9, 2022

Exercise 1

Consider a planar 3R robot with equal links of length $L > 0$ that is commanded with joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Each joint has the same available motion range

$$q_i \in [q_{i,min}, q_{i,max}] = [-\Delta, +\Delta], \qquad \text{with } \Delta = \frac{3\pi}{4} \text{ [rad]}, \qquad \text{for } i = 1, 2, 3.$$

The joint range function is defined in general as

$$H(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{i,max} - q_{i,min}} \right)^2, \qquad \text{with } \bar{q}_i = \frac{q_{i,min} + q_{i,max}}{2},$$

where N is the number of robot joints. The end-effector position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ of this robot is constrained to be at $\boldsymbol{p} = \boldsymbol{0}$. Define a kinematic control law for $\dot{\boldsymbol{q}}$ that will always satisfy this task constraint and the joint range limits, when starting from an initial feasible configuration $\boldsymbol{q}(0)$ and while attempting to minimize the function $H(\boldsymbol{q})$. Show that the robot converges to a unique configuration $\bar{\boldsymbol{q}}$ such that $\nabla H(\bar{\boldsymbol{q}}) \neq 0$ but $\dot{\boldsymbol{q}} = \boldsymbol{0}$. Provide the values of $\bar{\boldsymbol{q}}$, $H(\bar{\boldsymbol{q}})$ and $\nabla H(\bar{\boldsymbol{q}})$.

Exercise 2

The large RP robot in Fig. 1, with coordinates $\boldsymbol{q} = (q_1, q_2)$ and dynamic parameters m_2, d_{c2}, I_1 and I_2 defined therein, moves on a horizontal plane. The robot is controlled by the generalized force $\boldsymbol{u} = (\tau, F)$ [Nm, N] at the joints, while dissipative effects can be neglected. The robot end-effector point P should execute a linear rest-to-rest trajectory from $P_i = (2, 3)$ [m] to $P_f = (-2, 0)$ [m] in $T = 2$ [s], with bang-coast-bang symmetric acceleration profile and cruising speed $V = 3$ [m/s]. The robot is initially at rest in $\boldsymbol{q}(0) = (0, 2)$ [rad,m].

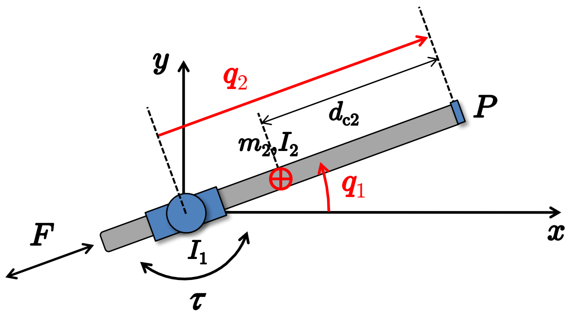

Figure 1: A RP planar robot with the relevant parameters and variables.

Define a control law for $\boldsymbol{u}$ such that the following performance is obtained:

- the Cartesian trajectory error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ goes to zero exponentially;
- the trajectory error components e_t and e_n, respectively along the tangent and the normal directions to the path, have a decoupled dynamics, governed by the differential equations

$$\ddot{e}_t + 4\,\dot{e}_t + 4\,e_t = 0, \qquad \ddot{e}_n + 8\,\dot{e}_n + 16\,e_n = 0.$$

Provide the explicit expression of all needed terms in the resulting control law, and in particular the analytic expression of generalized force $\boldsymbol{u}(t)$ at the initial time $t = 0$. Compute then its numerical value $\boldsymbol{u}(0) = (\tau(0), F(0))$ when the dynamic parameters are

$$m_2 = 10 \text{ [kg]}, \qquad d_{c2} = 2.5 \text{ [m]}, \qquad I_1 = I_2 = 20 \text{ [kg}\cdot\text{m}^2\text{]}.$$

Compute also the numerical value of the actual position $\boldsymbol{p}(t)$ of the end-effector at the half-time $t = T/2 = 1$ [s] of the robot motion.

Exercise 3

Consider the simple mechanical system with one degree of freedom in Fig. 2(a). The actuated mass $B > 0$ moves on a frictionless horizontal plane and is driven by an input force F. Its position is θ. The mass is connected through a pulley and a rigid cable to a second mass M that is suspended vertically under gravity.

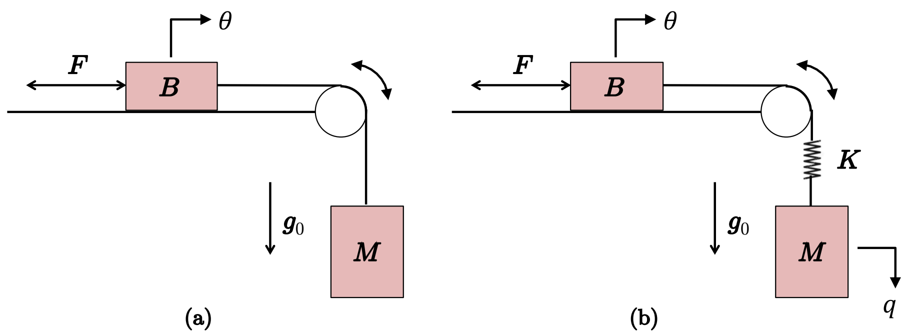

Figure 2: A mass B driven by an input force F and connected to a second mass M: the cable is rigid in (a) and flexible with stiffness K in (b).

- Derive the dynamic model of this system.
- Prove that the PID control law

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + K_I \int (\theta_d - \theta(\tau))\,d\tau,$$

  with suitable choices of the gains K_P, K_D and K_I, will asymptotically stabilize the desired equilibrium state $(\theta, \dot{\theta}) = (\theta_d, 0)$ of the closed-loop system. Will also exponential stabilization be achieved in this case?
- Consider the PD control law with iterative feedforward

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + v_{i-1},$$

  where the feedforward is updated at successive steady states $(\theta, \dot{\theta}) = (\theta_i, 0)$, $i = 1, 2, \dots,$ as

$$v_i = v_{i-1} + K_P\,(\theta_d - \theta_i)\,, \qquad \text{(with } v_0 = 0\text{)}.$$

  Prove that, with suitable gains K_P and K_D, this iterative learning control will globally, asymptotically stabilize the desired state $(\theta, \dot{\theta}) = (\theta_d, 0)$ of the closed-loop system. In particular, show that the convergence of the sequence $\{\theta_i\}$ exactly to θ_d occurs in a finite number of iterations.
- Assuming now, as in Fig. 2(b), that the cable is elastic with finite stiffness $K > 0$, derive the new dynamic model using the additional coordinate q for the position of the mass M.

[210 minutes; open books]

Solution

September 9, 2022

Exercise 1

The end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ of the considered planar 3R robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = L \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

with the usual shorthand notation for trigonometric functions, e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$. The associated 2×3 Jacobian matrix in $\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} = \begin{pmatrix} -p_y & -p_y + L\, s_1 & -p_y + L\,(s_1 + s_{12}) \\ p_x & p_x - L\, c_1 & p_x - L\,(c_1 + c_{12}) \end{pmatrix},$$

where the last equivalent expression will be convenient for what follows. Further, from the given joint limits, it is $\bar{q}_i = 0$, for $i = 1, 2, 3$, and the specific joint range function of this robot is

$$H(\boldsymbol{q}) = \frac{1}{6} \sum_{i=1}^{3} \left(\frac{q_i}{2\Delta}\right)^2 = \frac{1}{24\,\Delta^2} \left(q_1^2 + q_2^2 + q_3^2\right), \qquad \Delta = \frac{3\pi}{4}\ \text{[rad]},$$

with gradient

$$\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \frac{1}{12\,\Delta^2} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

To minimize this function without moving the end-effector ($\dot{\boldsymbol{p}} = \boldsymbol{0}$), the joint velocity should be chosen as

$$\dot{\boldsymbol{q}} = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right) \nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = -\alpha\, \boldsymbol{P}(\boldsymbol{q}) \nabla_{\boldsymbol{q}} H(\boldsymbol{q}), \tag{1}$$

where $\boldsymbol{J}^{\#}$ is the pseudoinverse of matrix $\boldsymbol{J}$, $\boldsymbol{P}$ is the projection matrix in its null space, and $\alpha > 0$ is a suitable scalar step. When starting from an initial feasible configuration $\boldsymbol{q}(0)$ such that $\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{0}$, the command (1) will keep satisfying the task constraint on the position of the robot end-effector. However, one should also check that the configuration $\boldsymbol{q}$ remains always within the hard limits of the joint ranges.

The problem is largely simplified by the requirement that the end-effector should be kept in particular at the origin. This implies that the three links of the robot will always form an equilateral triangle (with sides L), whose orientation is parametrized by the first joint angle q_1. Thus, only one of the two classes of configurations are feasible:

$$\boldsymbol{q}_{right} = (q_1, 2\pi/3, 2\pi/3) \qquad \text{or} \qquad \boldsymbol{q}_{left} = (q_1, -2\pi/3, -2\pi/3).$$

Consider then the right-arm class (the following treatment is analogous for the left-arm case). With $p_x = p_y = 0$ and $q_2 (= q_3) = 2\pi/3$, the Jacobian becomes after trigonometric simplifications

$$\boldsymbol{J}(q_1) = \begin{pmatrix} 0 & L \sin q_1 & L \sin(q_1 + \pi/3) \\ 0 & -L \cos q_1 & -L \cos(q_1 + \pi/3) \end{pmatrix}.$$

Any configuration in the $\boldsymbol{q}_{right}$ class is clearly regular: in fact, $\det(\boldsymbol{J}(q_1)\boldsymbol{J}^T(q_1)) = 0.75L^4 \neq 0$. Then, the pseudoinverse $\boldsymbol{J}^{\#}(q_1)$ can be computed as

$$\boldsymbol{J}^{\#}(q_1) = \boldsymbol{J}^T(q_1)\left(\boldsymbol{J}(q_1)\boldsymbol{J}^T(q_1)\right)^{-1} = \frac{2\sqrt{3}}{3L}\begin{pmatrix} 0 & 0 \\ \cos(q_1+\pi/3) & \sin(q_1+\pi/3) \\ \cos q_1 & \sin q_1 \end{pmatrix},$$

and the projection matrix $\boldsymbol{P}$ becomes the constant matrix

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This result should not come unexpected: no non-zero velocity of joints 2 or 3 will be admissible if the end-effector should keep the position $\boldsymbol{p} = \boldsymbol{0}$. Conversely, any velocity of joint 1 does move the end-effector and in fact is left unchanged when multiplied by the projector $\boldsymbol{P}$.

Therefore, starting from $\boldsymbol{q}(0) = (q_1(0), 2\pi/3, 2\pi/3)$, with $q_1(0) \in [-\Delta, +\Delta]$, eq. (1) becomes

$$\dot{\boldsymbol{q}} = -\alpha\, \boldsymbol{P}(\boldsymbol{q})\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = -\frac{\alpha}{12\,\Delta^2}\begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}.$$

Accordingly, if $q_1(0) > 0$ then $\dot{q}_1 < 0$ and, vice versa, if $q_1(0) < 0$ then $\dot{q}_1 > 0$. This guarantees that the robot motion will execute correctly the Cartesian task, while remaining always within the joint limits. The motion will stop when q_1 reaches zero. Thus, the final reached configuration is in any case

$$\bar{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix},$$

with

$$H(\bar{\boldsymbol{q}}) = \frac{1}{24\,\Delta^2}\left(\left(\frac{2\pi}{3}\right)^2 + \left(\frac{2\pi}{3}\right)^2\right) = 0.0658,$$

and

$$\nabla H(\bar{\boldsymbol{q}}) = \frac{1}{12\,\Delta^2}\begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0314 \\ 0.0314 \end{pmatrix} \neq 0.$$

Exercise 2

The problem is solved by a feedback linearization control law that imposes a prescribed linear and decoupled dynamics to the trajectory errors in the task space. The task space for this planar robot is a two-dimensional Cartesian space for the end-effector, which is rotated so as to align its axes with the tangent and normal directions to the desired linear path.

We define first the desired task trajectory for the robot end-effector. The path is a segment from $P_i = (2, 3)$ to $P_f = (-2, 0)$, having length $L = \|P_f - P_i\| = 5$ [m] and being expressed in terms of its arc length σ as

$$\boldsymbol{p}(\sigma) = P_i + \frac{P_f - P_i}{L}\,\sigma = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}\sigma, \qquad \sigma \in [0, L] = [0, 5].$$

One can associate to the path (see Fig. 3) a rotated right-handed planar frame $\Sigma_R = (\boldsymbol{t}, \boldsymbol{n})$, having axis $\boldsymbol{t}$ tangent to the path and axis $\boldsymbol{n}$ normal to it. The orientation of this frame is given by the 2×2 matrix

$$\boldsymbol{R}(\alpha) = \begin{pmatrix} \boldsymbol{t} & \boldsymbol{n} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \text{with } \alpha = \operatorname{atan2}\left\{-\frac{3}{5}, -\frac{4}{5}\right\} = -2.4981 \text{ [rad]} = -143.13°,$$

and thus

$$\boldsymbol{R}(\alpha) = \begin{pmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{pmatrix}. \tag{2}$$

The task frame Σ_R will be used to conveniently express the dynamically decoupled error components $e_t(t)$ and $e_n(t)$ along the trajectory.

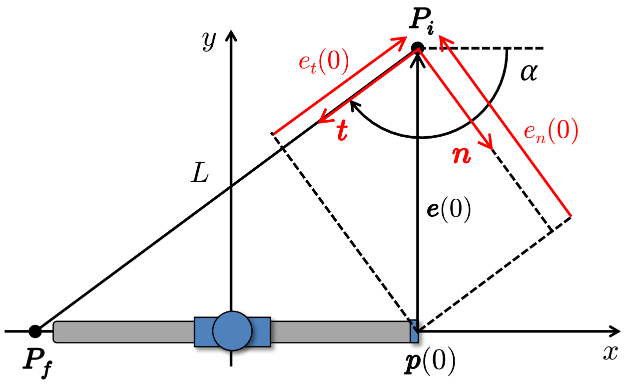

Figure 3: The desired linear path, with the definition of the task frame $\Sigma_R = (\boldsymbol{t}, \boldsymbol{n})$ whose constant orientation is given by matrix $\boldsymbol{R}(\alpha)$. The initial position of the robot end-effector $\boldsymbol{p}(0)$ is also shown, with the Cartesian initial error $\boldsymbol{e}(0) \in \mathbb{R}^2$ and the (rotated) initial position error components $e_t(0)$ and $e_n(0)$.

The timing law $\sigma = \sigma_d(t)$ on the reference path is a rest-to-rest motion with a symmetric bang-coast-bang acceleration profile. For such a profile, the maximum acceleration A along the path is computed from the total motion time T, the path length L, and the cruise speed V as

$$T = \frac{LA + V^2}{AV} \qquad \Rightarrow \qquad A = \frac{V^2}{TV - L} = \frac{9}{2 \cdot 3 - 5} = 9 \text{ [m/s}^2\text{]},$$

whereas the acceleration and deceleration phases last each $T_s = V/A = 0.333$ [s]. The assumed existence of a cruising velocity at $V = 3$ [m/s] is confirmed by the obtained value $T_s < T/2 = 1$. Thus, the initial acceleration of the timing law is $\ddot{\sigma}_d(0) = A = 9$ [m/s^2]. Accordingly, the initial desired end-effector velocity and acceleration in the base frame are

$$\dot{\boldsymbol{p}}_d(0) = \boldsymbol{0}, \qquad \ddot{\boldsymbol{p}}_d(0) = \frac{P_f - P_i}{L}\,\ddot{\sigma}_d(0) = -\begin{pmatrix} 7.2 \\ 5.4 \end{pmatrix} \text{ [m/s}^2\text{]},$$

while the same quantities expressed in the rotated task frame are indeed

$${}^R\dot{\boldsymbol{p}}_d(0) = \boldsymbol{0}, \qquad {}^R\ddot{\boldsymbol{p}}_d(0) = \boldsymbol{R}^T(\alpha)\,\ddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

Next, define the initial position and velocity error of the robot end-effector with respect to the desired trajectory. The direct kinematics of the RP robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}. \tag{3}$$

The robot is initially at rest in $\boldsymbol{q}(0) = (0, 2)$. Thus,

$$\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ [m]}, \qquad \dot{\boldsymbol{q}}(0) = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{p}}(0) = \boldsymbol{0},$$

and so

$$\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \begin{pmatrix} e_x(0) \\ e_y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad \dot{\boldsymbol{e}}(0) = \dot{\boldsymbol{p}}_d(0) - \dot{\boldsymbol{p}}(0) = \boldsymbol{0}.$$

In the rotated frame Σ_R, we have

$${}^R\boldsymbol{e}(0) = \boldsymbol{R}^T(\alpha)\, \boldsymbol{e}(0) = \begin{pmatrix} e_t(0) \\ e_n(0) \end{pmatrix} = \begin{pmatrix} -1.8 \\ -2.4 \end{pmatrix}, \qquad {}^R\dot{\boldsymbol{e}}(0) = \boldsymbol{R}^T(\alpha)\, \dot{\boldsymbol{e}}(0) = \boldsymbol{0}.$$

Consider now the robot dynamics. Since the RP robot moves on the horizontal plane, its dynamic model has $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u}.$$

Applying the feedback linearizing control

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a_q} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad \text{with} \quad \boldsymbol{a_q} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{R}(\alpha)\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right), \tag{4}$$

leads to the Cartesian acceleration of the end-effector in the task frame

$$\ddot{\boldsymbol{p}} = \boldsymbol{R}(\alpha)\, \boldsymbol{a} \qquad \Rightarrow \qquad {}^R\ddot{\boldsymbol{p}} = \boldsymbol{R}^T(\alpha)\, \ddot{\boldsymbol{p}} = \boldsymbol{a}.$$

Choosing then

$$\boldsymbol{a} = {}^R\ddot{\boldsymbol{p}}_d + {}^R\boldsymbol{K}_D \left({}^R\dot{\boldsymbol{p}}_d - {}^R\dot{\boldsymbol{p}} \right) + {}^R\boldsymbol{K}_P \left({}^R\boldsymbol{p}_d - {}^R\boldsymbol{p} \right) \tag{5}$$

with

$${}^R\boldsymbol{K}_P = \begin{pmatrix} k_{P,t} & 0 \\ 0 & k_{P,n} \end{pmatrix} \qquad \text{and} \qquad {}^R\boldsymbol{K}_D = \begin{pmatrix} k_{D,t} & 0 \\ 0 & k_{D,n} \end{pmatrix},$$

yields for the dynamics of the task trajectory error ${}^R\boldsymbol{e} = {}^R\boldsymbol{p}_d - {}^R\boldsymbol{p}$

$${}^R\ddot{\boldsymbol{e}} + {}^R\boldsymbol{K}_D {}^R\dot{\boldsymbol{e}} + {}^R\boldsymbol{K}_P {}^R\boldsymbol{e} = \boldsymbol{0},$$

or, componentwise,

$$\ddot{e}_t + k_{D,t}\, \dot{e}_t + k_{P,t}\, e_t = 0, \qquad \ddot{e}_n + k_{D,n}\, \dot{e}_n + k_{P,n}\, e_n = 0. \tag{6}$$

Accordingly, by choosing the scalar control gains

$$k_{P,t} = 4, \qquad k_{D,t} = 4, \qquad k_{P,n} = 16, \qquad k_{D,n} = 8, \tag{7}$$

we obtain the prescribed dynamic behavior for the transient errors along the tangent and normal directions to the desired trajectory.

Putting together (4) and (5), the final control law is thus

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{R}(\alpha) \left({}^R\ddot{\boldsymbol{p}}_d + {}^R\boldsymbol{K}_D {}^R\dot{\boldsymbol{e}} + {}^R\boldsymbol{K}_P {}^R\boldsymbol{e} \right) - \boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{8}$$

Beside (2) and (3), the other kinematic terms needed in the control law (8) are the Jacobian matrix and its inverse

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}^{-1}(\boldsymbol{q}) = -\frac{1}{q_2} \begin{pmatrix} \sin q_1 & -\cos q_1 \\ -q_2 \cos q_1 & -q_2 \sin q_1 \end{pmatrix},$$

and the time derivative of the Jacobian

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_1 q_2 \cos q_1 - \dot{q}_2 \sin q_1 & -\dot{q}_1 \sin q_1 \\ -\dot{q}_1 q_2 \sin q_1 + \dot{q}_2 \cos q_1 & \dot{q}_1 \cos q_1 \end{pmatrix}.$$

From these, it follows also

$$\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -\dot{q}_1^2\, q_2 \cos q_1 - 2\,\dot{q}_1\dot{q}_2 \sin q_1 \\ -\dot{q}_1^2\, q_2 \sin q_1 + 2\,\dot{q}_1\dot{q}_2 \cos q_1 \end{pmatrix} \quad \Rightarrow \quad -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{q_2} \begin{pmatrix} -2\,\dot{q}_1\dot{q}_2 \\ \dot{q}_1^2\, q_2^2 \end{pmatrix}.$$

Moreover, the dynamic terms used in (8) are the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$. These are obtained as usual from the following steps.

Kinetic energy

$$T = T_1 + T_2$$

with

$$T_1 = \frac{1}{2} I_1\, \dot{q}_1^2, \qquad T_2 = \frac{1}{2} I_2\, \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2,$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt} \left((q_2 - d_{c2}) \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} \right) = \dot{q}_2 \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} + (q_2 - d_{c2})\, \dot{q}_1 \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ (q_2 - d_{c2})\, \dot{q}_1 \end{pmatrix},$$

and so

$$T = \frac{1}{2} (I_1 + I_2)\, \dot{q}_1^2 + \frac{1}{2} m_2 \left(\dot{q}_2^2 + (q_2 - d_{c2})^2\, \dot{q}_1^2 \right).$$

Inertia matrix

$$T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} I_1 + I_2 + m_2\, (q_2 - d_{c2})^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Coriolis and centrifugal terms

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\, \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})\, \dot{\boldsymbol{q}} \end{pmatrix}, \quad \text{with } \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \quad i = 1, 2.$$

Since

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2\, (q_2 - d_{c2}) \\ m_2\, (q_2 - d_{c2}) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = -\begin{pmatrix} m_2\, (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2\,(q_2 - d_{c2})\,\dot{q}_1\dot{q}_2 \\ -m_2\,(q_2 - d_{c2})\,\dot{q}_1^2 \end{pmatrix}.$$

Moreover, the analytic expression of the generalized force $\boldsymbol{u}(t)$ at the initial time $t = 0$ is obtained by particularizing (8) at the initial robot state $(\boldsymbol{q},\dot{\boldsymbol{q}}) = (\boldsymbol{q}(0),\boldsymbol{0})$ and with the initial task position error ${}^R\boldsymbol{e}(0)$ and task velocity error ${}^R\dot{\boldsymbol{e}}(0) = \boldsymbol{0}$. We have

$$\boldsymbol{u}(0) = \boldsymbol{M}(\boldsymbol{q}(0))\boldsymbol{J}^{-1}(\boldsymbol{q}(0))\boldsymbol{R}(\alpha)\left({}^R\ddot{\boldsymbol{p}}_d(0) + {}^R\boldsymbol{K}_P\,{}^R\boldsymbol{e}(0)\right). \tag{9}$$

With the given dynamic data, the numerical value of the initial control (9) is

$$\boldsymbol{u}(0) = \begin{pmatrix} \tau(0) \\ F(0) \end{pmatrix} = \begin{pmatrix} 629.8 \\ -244.8 \end{pmatrix} \text{ [Nm, N]}.$$

Finally, the numerical value of the position $\boldsymbol{p}(t)$ of the end-effector at the half-time $t = T/2 = 1$ of the motion is obtained as $\boldsymbol{p}(T/2) = \boldsymbol{p}_d(T/2) - \boldsymbol{e}(T/2)$, from the knowledge of the desired trajectory $\boldsymbol{p}_d(t)$ and using the position error $\boldsymbol{e}(t)$ resulting from the feedback linearization control law (8) —thus, analytically and without the need of a numerical simulation of the robot dynamics!

For the desired trajectory position, one has

$$\boldsymbol{p}_d(1) = P_i + \frac{P_f - P_i}{2} = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \text{ [m]},$$

namely, the midpoint on the linear path. On the other hand, the closed-form solutions of the linear differential equations in (6) for the initial values $e_t(0) = 1.8$, $e_n(0) = 2.4$, $\dot{e}_t(0) = \dot{e}_n(0) = 0$, and for the specific numerical gains in (7) are[1]

$$e_t(t) = -1.8\,(1 + 2t)\exp(-2t), \qquad e_n(t) = -2.4\,(1 + 4t)\exp(-4t). \tag{10}$$

Thus

$$\boldsymbol{e}(1) = \boldsymbol{R}(\alpha)\,{}^R\boldsymbol{e}(1) = \boldsymbol{R}(\alpha)\begin{pmatrix} -5.4\,\exp(-2) \\ -12\,\exp(-4) \end{pmatrix} = \begin{pmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{pmatrix}\begin{pmatrix} -0.7308 \\ -0.2198 \end{pmatrix} = \begin{pmatrix} 0.4528 \\ 0.6143 \end{pmatrix},$$

and therefore

$$\boldsymbol{p}(1) = \boldsymbol{p}_d(1) - \boldsymbol{e}(1) = \begin{pmatrix} -0.4528 \\ 0.8857 \end{pmatrix} \text{ [m]}.$$

Figure 4 shows the evolution in time of the relevant variables concerning the robot end-effector: on the left, the tracking errors $e_t(t)$ and $e_n(t)$ along the tangential and normal directions to the path; at the center, the components $e_x(t)$ and $e_y(t)$ of the tracking error in the base frame; on the right, the coordinates $p_x(t)$ and $p_y(t)$ of the actual position of the end-effector. One can see that the chosen PD gains of the linear part of the control design are not sufficient large to fully recover the tracking error before the end of the trajectory. Indeed, these residual errors will approach exponentially zero after $T = 5$ [s], when the desired reference motion has ended and the trajectory tracking problem has become a regulation problem.

[1] The two differential equations (6) are written in the Laplace domain as

$$\left(s^2 + 4s + 4\right)e_t(s) = (s+2)^2\,e_t(s) = 0, \qquad \left(s^2 + 4s + 16\right)e_n(s) = (s+4)^2\,e_n(s) = 0,$$

i.e., they have two real and coincident roots, respectively in -2 and -4. Accordingly, the time solutions for arbitrary initial conditions are

$$e_t(t) = e_t(0)\exp(-2t) + (\dot{e}_t(0) + 2e_t(0))\,t\exp(-2t), \qquad e_n(t) = e_n(0)\exp(-4t) + (\dot{e}_n(0) + 4e_n(0))\,t\exp(-4t).$$

The results in (10) follow for the considered case with $\dot{e}_t(0) = \dot{e}_n(0) = 0$.

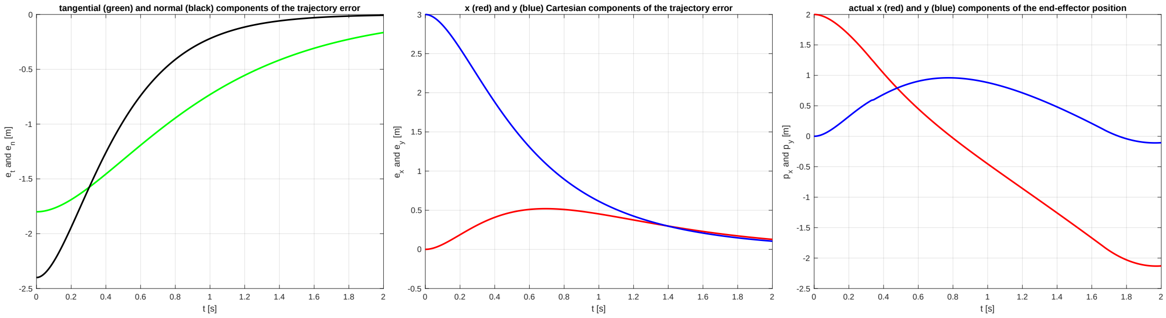

Figure 4: Time evolution of the tracking errors $e_t(t)$ and $e_n(t)$ in the task frame [left] and of the tracking errors $e_x(t)$ and $e_y(t)$ in the base frame [center]; coordinates $p_x(t)$ and $p_y(t)$ of the actual end-effector position [right].

Extra comments to the solution of Exercise 2

The evolutions of the Cartesian errors and positions, respectively in the center and right plots of Fig. 4, deserve a special comment. While the initial errors $e_t(0)$ and $e_n(0)$ are both non-zero, the position component $p_x(0)$ is matched with its desired value $p_{d,x}(0)$ and the initial Cartesian position error limited to the y-direction. However, the reaction to trajectory errors is designed to be decoupled in the $\boldsymbol{t}$ and $\boldsymbol{n}$ (task) directions, not in the $\boldsymbol{x}$ and $\boldsymbol{y}$ (Cartesian) directions. Therefore, the presence of an initial error along $\boldsymbol{y}$ will induce later on also an error along $\boldsymbol{x}$, because the control action is designed to reduce separately the two error components in the task directions (see Fig. 4 [left]). After both these task errors have been sufficiently reduced, also the two Cartesian errors will eventually be reduced in a monotonic way. Note also that the y-position of the end-effector (the blue plot in Fig. 4 [right]) increases first in order to approach the planned path, but then reduces because the desired trajectory has reduced as well its y-component (see the path in Fig. 3).

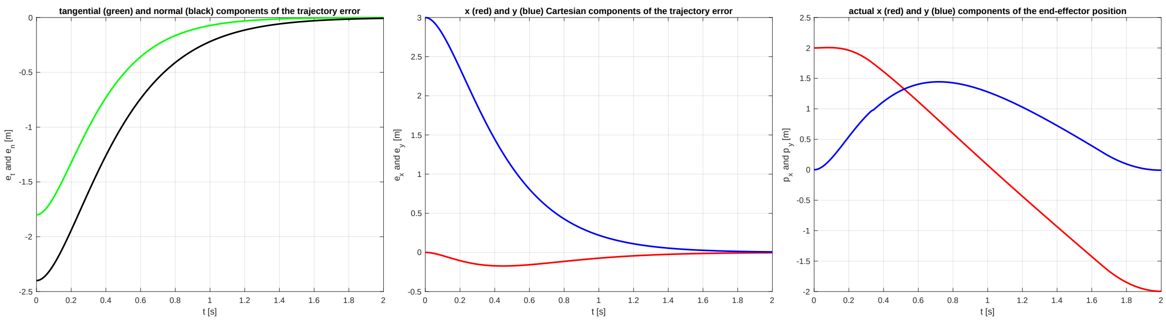

Figure 5: Improved evolution of the tracking errors in task frame [left] and in base frame [center] and actual position of the end-effector [right] when the double pole of the linear controller in the tangential direction is moved from -2 to -5.

A better, and perhaps clearer, behavior could be observed if the control gains in the tangential direction were properly increased. Choosing for instance

$$k_{P,t} = 25, \qquad k_{D,t} = 10$$

would lead in the Laplace domain to a double pole in $s = -5$ (rather than in $s = -2$) along the tangential direction, with the associated error decreasing at a faster exponential rate as

$$e_t(t) = -1.8\,(1 + 5t)\exp(-5t),$$

which is to be compared with $e_t(t)$ in (10). The control gains in the normal direction have remained unchanged. The obtained results are shown in Fig. 5, where the green profile in the left plot is the new $e_t(t)$. The faster reaction to this tangential error has a counter-effect on $e_x(t)$, which now becomes negative for some time during the transient, although with a much smaller magnitude than before. In this case, all errors have vanished by the end of the desired motion ($T = 5$ [s]).

Exercise 3

The dynamic model of the two-mass system with a rigid cable of Fig. 2(a) is given by

$$(B + M)\,\ddot{\theta} - M g_0 = F.$$

This can be obtained by simple balance of inertial, gravity and external forces or from a Lagrangian approach, with $L = T - U_g$ and where $T = \frac{1}{2}B\dot{\theta}^2 + \frac{1}{2}M\dot{\theta}^2$ and $U_g = -Mg_0\theta + U_0$. Note also that, being the gravity term $g = \partial U_g/\partial\theta = -Mg_0$ constant, its gradient $\partial g/\partial\theta = 0$ is upper bounded just by $\alpha = 0$; thus, to have a unique closed-loop equilibrium, we expect no positive lower bound larger than zero for the proportional gain K_P in order to contrast gravity.

As for the control design, both proposed control laws do not use the knowledge of the masses B and M. Thus, we shall assume in the following that their value is unknown, while only an upper bound is available on each, i.e., and $0 < B \le B_{max}$ and $0 < M \le M_{max}$.

Consider first the PID control law

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + K_I \int (\theta_d - \theta)\, d\tau, \tag{11}$$

with θ_d being constant. This leads to the following closed-loop equation for the error $e = \theta_d - \theta$

$$(B + M)\,\ddot{e} + K_D\dot{e} + K_P e + K_I \int_0^t e\, d\tau + M g_0 = 0.$$

In order to have the desired closed-loop equilibrium for $t \to \infty$, the error e should vanish and thus the integral term should balance at steady state the gravity term:

$$F_\infty = K_I \int_0^\infty e(\tau)\, d\tau = -M g_0.$$

To verify the asymptotic stability of the desired closed-loop equilibrium, we use the fact that the system dynamics is linear and transform the closed-loop equation in the Laplace domain. The gravity term $d = Mg_0$ is considered here as a constant external disturbance. We have

$$\left((B + M)\,s^2 + K_D s + K_P + \frac{K_I}{s} \right) \theta(s) = \left(K_P + \frac{K_I}{s} \right) \theta_d(s) + d(s),$$

or, multiplying by s and re-organizing terms,

$$\theta(s) = \frac{K_P s + K_I}{(B + M)\,s^3 + K_D s^2 + K_P s + K_I}\,\theta_d(s) + \frac{s}{(B + M)\,s^3 + K_D s^2 + K_P s + K_I}\,d(s).$$

The input-output and disturbance-output transfer functions, respectively $W(s) = \theta(s)/\theta_d(s)$ and $W_d(s) = \theta(s)/d(s)$, have a common polynomial at the denominator. The asymptotic stability of the closed-loop system depends on the localization of the three closed-loop poles, namely the three roots of the characteristic equation

$$(B + M)\,s^3 + K_D s^2 + K_P s + K_I = 0.$$

Using the Routh criterion, we build the Routh table

$$
\begin{array}{c|cc}
3 & B+M & K_P \\
2 & K_D & K_I \\
1 & K_P - \dfrac{(B+M)K_I}{K_D} & \\
0 & K_I &
\end{array}
$$

From this, we find that all three roots will be asymptotically stable (i.e., will have negative real parts) if and only if there is no change of sign in the first column of the table, namely

$$K_I > 0, \qquad K_D > 0, \qquad K_P > \frac{(B+M)K_I}{K_D} > 0.$$

Under this condition, the steady-state value of θ for the input-output relation will be

$$\theta_{ss} = \lim_{t\to\infty} \theta(t) = \lim_{s\to 0} W(s)\theta_d = \theta_d,$$

i.e., the desired one, while for the disturbance-output relation due to gravity it will be

$$\theta_{ss} = \lim_{t\to\infty} \theta(t) = \lim_{s\to 0} W_d(s)d = 0.$$

i.e., the effect of gravity is completely rejected. By superposition of the two effects, it is $\theta_{ss} = \theta_d$.

Summarizing, the PID control law (11) will achieve the desired objective, provided that positive gains are used for the integral and derivative action, and a positive and sufficiently large gain is used for the proportional action. In particular, choosing

$$K_P > \frac{(B_{max} + M_{max})K_I}{K_D} > 0.$$

will guarantee a robust performance in spite of uncertainties on the two masses. Finally, since the system has a linear dynamics, the obtained asymptotic stabilization will be both global and exponential.

Consider next the PD control law with feedforward

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + v_{i-1}, \tag{12}$$

at a generic iteration of the method. The following closed-loop equation holds for the error $e = \theta_d - \theta$

$$(B+M)\,\ddot{e} + K_D\dot{e} + K_P e + v_{i-1} + Mg_0 = 0. \tag{13}$$

In order for the error e to vanish at steady state, the feedforward should balance the gravity term:

$$v_{i-1} = -Mg_0.$$

If this is not the case, the steady-state error $e_i = \theta_d - \theta_i \neq 0$ at the end of iteration i will satisfy the equilibrium equation

$$K_P e_i + v_{i-1} = -Mg_0 \qquad \Rightarrow \qquad \theta_i = \theta_d + \frac{1}{K_P}\,(v_{i-1} + Mg_0) \neq \theta_d.$$

Analyzing eq. (13) as before, this (wrong) equilibrium will be asymptotically reached if and only if $K_P > 0$ and $K_D > 0$. The uniqueness of such equilibrium is again guaranteed without any further condition on the proportional gain, thanks to the linearity of the system. When the feedforward is updated at successive steady states $(\theta, \dot{\theta}) = (\theta_i, 0)$, $i = 1, 2, \ldots$, as

$$v_i = v_{i-1} + K_P\,(\theta_d - \theta_i)\,,$$

the desired closed-loop equilibrium state $(q_d, 0)$ will be reached by iteration. Remarkably, when initializing the feedforward term with $v_0 = 0$ (i.e., a pure PD is applied at the first iteration), convergence occurs in just two iterations. In fact, at the end of the first iteration, we will have

$$K_P e_1 = K_P(\theta_d - \theta_1) = -Mg_0$$

and the update is

$$v_1 = v_0 + K_P(\theta_d - \theta_1) = 0 - Mg_0 = -Mg_0.$$

Then, at the end of the second iteration, we will have

$$K_P e_2 + v_1 = K_P(\theta_d - \theta_2) - Mg_0 = -Mg_0 \qquad \Rightarrow \qquad \theta_2 = \theta_d.$$

Summarizing, the PD + iterative feedforward control (12) will achieve the desired objective, provided that positive gains are used for the proportional and derivative action; the convergence is achieved in two iterations without any further condition on the proportional gain.

Finally, if the cable is flexible (in the domain of linear elasticity) with a finite stiffness $K > 0$ as in Fig. 2(b), the dynamic model consists of two differential equations for the generalized coordinates θ and q. We have

$$B\ddot{\theta} + K\,(\theta - q) = F$$
$$M\ddot{q} + K\,(q - \theta) - Mg_0 = 0.$$

In a Lagrangian framework, these equations are obtained with the vector of generalized coordinates $Q = (\theta, q)$, having $L = T - (U_g + U_e)$, and where $T = \frac{1}{2}B\dot{\theta}^2 + \frac{1}{2}M\dot{q}^2$, $U_g = -Mg_0 q + U_0$, and the added elastic potential is $U_e = \frac{1}{2}K(\theta - q)^2$.

* * * * *