Robotics 2

June 12, 2023

Exercise 1

Consider the 4R planar robot in Fig. 1, with generic lengths, masses and inertias of the links but with the center of mass of each link placed on its kinematic axis. As shown in the figure, the absolute angles of the links with respect to the axis $\boldsymbol{x}_0$ must be used as generalized coordinates $\boldsymbol{q}$.

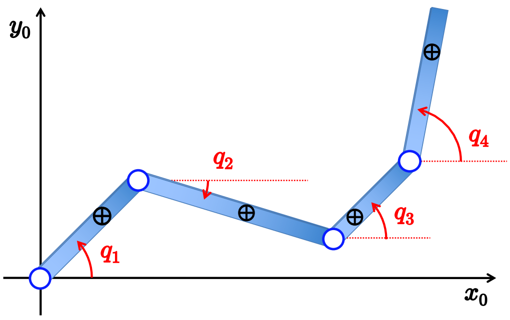

Figure 1: A 4R planar robot.

- Compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of this robot.
- From the elements of $\boldsymbol{M}(\boldsymbol{q})$, derive the expression of the robot inertia matrix when using instead the Denavit-Hartenberg joint angles $\boldsymbol{\theta}$ as generalized coordinates.
- With the experience gained for the case $n = 4$, provide the general expression of the kinetic energy $T_i(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of link i in a nR planar robot using the generalized coordinates $\boldsymbol{q}$ and under similar assumptions.

Exercise 2

Let the robot of Fig. 1 have all four links of unitary length, and suppose that we can command the robot using the joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^4$. With the robot in the configuration

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 & \frac{\pi}{6} & -\frac{\pi}{3} & -\frac{\pi}{3} \end{pmatrix}^T,$$

consider two (alternative or simultaneous) tasks: *(i)* the end-effector should move with a velocity $\boldsymbol{v}_e \in \mathbb{R}^2$; and *(ii)* the tip of the second link should move with a velocity $\boldsymbol{v}_t \in \mathbb{R}^2$. Determine the joint velocity commands $\dot{\boldsymbol{q}}$ for the following problems:

a. execute at best the end-effector task $\boldsymbol{v}_e = (0.4330, -0.75)$, while minimizing the norm of $\dot{\boldsymbol{q}}$;
b. execute at best the second link tip task $\boldsymbol{v}_t = (-0.5, 0.8660)$, while minimizing the norm of $\dot{\boldsymbol{q}}$;
c. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ simultaneously;
d. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the end-effector task $\boldsymbol{v}_e$;
e. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the second link tip task $\boldsymbol{v}_t$.

For each case, provide also the obtained velocity errors $\boldsymbol{e}_e$ and $\boldsymbol{e}_t$ on both tasks (whether assigned or not) and their norm.

Exercise 3

For regulating the PRR planar robot shown in Fig. 2 to a desired configuration $\boldsymbol{q}_d$, the PD+gravity compensation torque $\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$ is being used, with diagonal gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$. For this control law, provide the symbolic expression of the feedforward term $\boldsymbol{g}(\boldsymbol{q}_d)$ and of the minimum constant value for the elements of $\boldsymbol{K}_P$ that guarantees global asymptotic stabilization of any desired equilibrium configuration $\boldsymbol{q}_d$.

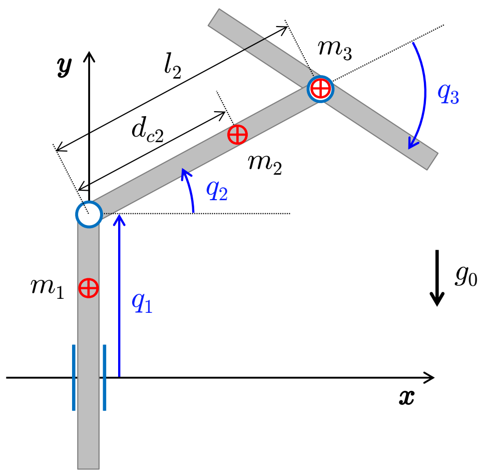

Figure 2: A PRR planar robot under gravity.

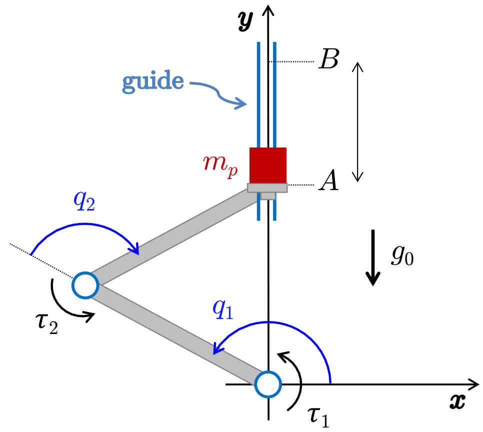

Figure 3: The mechanism with a constrained 2R robot for elevating payloads.

Exercise 4

The mechanism in Fig. 3 elevates payloads by means of a 2R robot, which is constrained at its end effector by a vertical guide. The robot has unitary link lengths. The extension of the vertical motion is limited between points $A = (0, 0.95)$ and $B = (0, 1.45)$ —the robot is never in a singularity. When including also the payload m_p, the inertia matrix of the unconstrained 2R robot is parametrized as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2\, c_2 & a_3 + a_2\, c_2 \\ a_3 + a_2\, c_2 & a_3 \end{pmatrix}.$$

Derive all the individual elements of the reduced dynamic model of this constrained robotic system. In particular, provide:

- the remaining elements $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{g}(\boldsymbol{q})$ of the 2R robot dynamic model;
- the 1×2 Jacobian of the constraint $\boldsymbol{A}(\boldsymbol{q})$ and a 1×2 completion matrix $\boldsymbol{D}(\boldsymbol{q})$ that guarantees non-singularity in the operating region, together with their time derivatives $\dot{\boldsymbol{A}}(\boldsymbol{q})$ and $\dot{\boldsymbol{D}}(\boldsymbol{q})$;
- a physical interpretation of the pseudo-velocity $v \in \mathbb{R}$;
- the (scalar) reduced inertia $\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})$.

Design then a suitable motion control law for the torque $\boldsymbol{\tau} \in \mathbb{R}^2$ that should impose a desired cyclic motion from A to B and vice versa in a total motion time T. The joint motion of the 2R robot should have a continuous acceleration profile at all times. Moreover, no reaction force $\lambda \in \mathbb{R}$ should be exerted on the end-effector by the constraining guide.

[240 minutes; open books]

Solution

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Exercise 1

We compute the kinetic energy of this 4R planar robot, taking advantage of the absolute coordinates q_i, $i = 1, \ldots, 4$, as shown in Fig. 1. Let m_i be the mass and l_i the kinematic length of link i, d_{ci} the distance along the link axis of the center of mass (CoM) of link i from the previous joint, and I_{ci} the barycentric inertia of link i around the axis normal to the plane of motion. The position $\boldsymbol{p}_{ci}$ and the velocity $\boldsymbol{v}_{ci}$ of the CoM of link i are two-dimensional vectors in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

For the first link, we have

$$T_1 = \frac{1}{2}\left(I_{c1} + m_1 d_{c1}^2\right)\dot{q}_1^2.$$

For the second link, being

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 + d_{c2} c_2 \\ l_1 s_1 + d_{c2} s_2 \end{pmatrix} = \begin{pmatrix} -\left(l_1 s_1 \dot{q}_1 + d_{c2} s_2 \dot{q}_2\right) \\ l_1 c_1 \dot{q}_1 + d_{c2} c_2 \dot{q}_2 \end{pmatrix},$$

it is

$$T_2 = \frac{1}{2}\, I_{c2}\dot{q}_2^2 + \frac{1}{2}\, m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{2}\left(I_{c2} + m_2 d_{c2}^2\right)\dot{q}_2^2 + m_2 l_1 d_{c2} c_{2-1} \dot{q}_1 \dot{q}_2,$$

where $c_{2-1} = \cos(q_2 - q_1)$.

For the third link, being

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 + l_2 c_2 + d_{c3} c_3 \\ l_1 s_1 + l_2 s_2 + d_{c3} s_3 \end{pmatrix} = \begin{pmatrix} -\left(l_1 s_1 \dot{q}_1 + l_2 s_2 \dot{q}_2 + d_{c3} s_3 \dot{q}_3\right) \\ l_1 c_1 \dot{q}_1 + l_2 c_2 \dot{q}_2 + d_{c3} c_3 \dot{q}_3 \end{pmatrix},$$

it is

$$\begin{aligned} T_3 &= \frac{1}{2}\, I_{c3}\dot{q}_3^2 + \frac{1}{2}\, m_3 \|\boldsymbol{v}_{c3}\|^2 \\ &= \frac{1}{2} m_3 \left(l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2\right) + \frac{1}{2}\left(I_{c3} + m_3 d_{c3}^2\right)\dot{q}_3^2 \\ &\quad + m_3 \left(l_1 l_2 c_{2-1} \dot{q}_1 \dot{q}_2 + l_1 d_{c3} c_{3-1} \dot{q}_1 \dot{q}_3 + l_2 d_{c3} c_{3-2} \dot{q}_2 \dot{q}_3\right), \end{aligned}$$

where $c_{3-1} = \cos(q_3 - q_1)$ and $c_{3-2} = \cos(q_3 - q_2)$.

For the fourth and last link, we follow the same pattern and obtain

$$\begin{aligned} T_4 &= \frac{1}{2}\, I_{c4}\dot{q}_4^2 + \frac{1}{2}\, m_4 \|\boldsymbol{v}_{c4}\|^2 \\ &= \frac{1}{2} m_4 \left(l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2 + l_3^2 \dot{q}_3^2\right) + \frac{1}{2}\left(I_{c4} + m_4 d_{c4}^2\right)\dot{q}_4^2 \\ &\quad + m_4 \big(l_1 l_2 c_{2-1} \dot{q}_1 \dot{q}_2 + l_1 l_3 c_{3-1} \dot{q}_1 \dot{q}_3 + l_2 l_3 c_{3-2} \dot{q}_2 \dot{q}_3 \\ &\qquad + \left(l_1 c_{4-1} \dot{q}_1 + l_2 c_{4-2} \dot{q}_2 + l_3 c_{4-3} \dot{q}_3\right) d_{c4} \dot{q}_4\big), \end{aligned}$$

where $c_{4-1} = \cos(q_4 - q_1)$, $c_{4-2} = \cos(q_4 - q_2)$ and $c_{4-3} = \cos(q_4 - q_4)$.

Finally,

$$T = T_1 + T_2 + T_3 + T_4 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and the robot inertia matrix is given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1} + m_1 d_{c1}^2 + (m_2+m_3+m_4)\, l_1^2 & & & symm \\ (m_2 d_{c2} + (m_3+m_4)\, l_2)\, l_1 c_{2-1} & I_{c2} + m_2 d_{c2}^2 + (m_3+m_4)\, l_2^2 & & \\ (m_3 d_{c3} + m_4 l_3)\, l_1 c_{3-1} & (m_3 d_{c3} + m_4 l_3)\, l_2 c_{3-2} & I_{c3} + m_3 d_{c3}^2 + m_4 l_3^2 & \\ m_4 d_{c4} l_1 c_{4-1} & m_4 d_{c4} l_2 c_{4-2} & m_4 d_{c4} l_3 c_{4-3} & I_{c4} + m_4 d_{c4}^2 \end{pmatrix}.$$

The coordinate transformation between the $\boldsymbol{\theta}$ variables of Denavit-Hartenberg and the generalized coordinates $\boldsymbol{q}$ is linear and is given by

$$\boldsymbol{q} = \boldsymbol{T}\boldsymbol{\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} \qquad \Longleftrightarrow \qquad \boldsymbol{\theta} = \boldsymbol{T}^{-1}\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \boldsymbol{q}.$$

To obtain the inertia matrix in the new coordinates $\boldsymbol{\theta}$, we first replace the arguments of the cosine functions inside the elements $m_{ij}(q)$ of $\boldsymbol{M}(\boldsymbol{q})$ as follows:

$$c_{2-1} = \cos\theta_2, \qquad c_{3-1} = \cos(\theta_2+\theta_3), \qquad c_{4-1} = \cos(\theta_2+\theta_3+\theta_4),$$
$$c_{3-2} = \cos\theta_3 \qquad c_{4-2} = \cos(\theta_3+\theta_4), \qquad c_{4-3} = \cos\theta_4.$$

Applying then the transformation rule, one obtains

$$\widetilde{\boldsymbol{M}}(\boldsymbol{\theta}) = \boldsymbol{T}^T \boldsymbol{M}(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{T}\boldsymbol{\theta}}\, \boldsymbol{T}$$
$$= \begin{pmatrix} \begin{matrix} m_{11} + 2m_{12} + 2m_{13} + 2m_{14} \\ + m_{22} + 2m_{23} + 2m_{24} \\ + m_{33} + 2m_{34} + m_{44} \end{matrix} & & & symm \\ \begin{matrix} m_{12} + m_{13} + m_{14} \\ + m_{22} + 2m_{23} + 2m_{24} \\ + m_{33} + 2m_{34} + m_{44} \end{matrix} & \begin{matrix} m_{22} + 2m_{23} + 2m_{24} \\ + m_{33} + 2m_{34} + m_{44} \end{matrix} & & \\ \begin{matrix} m_{13} + m_{14} + m_{23} + m_{24} \\ + m_{33} + 2m_{34} + m_{44} \end{matrix} & \begin{matrix} m_{23} + m_{24} \\ + m_{33} + 2m_{34} + m_{44} \end{matrix} & m_{33} + 2m_{34} + m_{44} & \\ m_{14} + m_{24} + m_{34} + m_{44} & m_{24} + m_{34} + m_{44} & m_{34} + m_{44} & m_{44} \end{pmatrix}_{|\boldsymbol{q}=\boldsymbol{T}\boldsymbol{\theta}},$$

which clearly shows how more cumbersome would be the explicit expression of the robot inertia matrix for this robot when using the DH (relative) angles $\boldsymbol{\theta}$.

For the nR planar robot, based on the previous derivations and under the same assumptions, it is easy to find the general expression for the kinetic energy of link i, for $i = 1, \ldots, n$:

$$T_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\, I_{ci} \dot{q}_i^2 + \frac{1}{2}\, m_i \|\boldsymbol{v}_{ci}\|^2,$$

with

$$\|\boldsymbol{v}_{ci}\|^2 = \sum_{j=1}^{i-1} l_j^2 \dot{q}_j^2 + d_{ci}^2 \dot{q}_i^2 + 2 \sum_{j=1}^{i-1} \left(\sum_{k=j+1}^{i-1} l_k c_{k-j} \dot{q}_k \right) l_j \dot{q}_j + 2 \left(\sum_{j=1}^{i-1} l_j c_{i-j} \dot{q}_j \right) d_{ci} \dot{q}_i.$$

Exercise 2

With all links of the 4R planar robot being of unitary length and using again the absolute joint variables of Fig. 1, the Jacobians for the two considered tasks are

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_2 & -s_3 & -s_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \qquad \boldsymbol{v}_e = \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and

$$\boldsymbol{J}_t(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{v}_t = \boldsymbol{J}_t(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

In the configuration $\boldsymbol{q}_0$ (see Fig. 4), we have

$$\boldsymbol{J}_e(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -0.5 & 0.8660 & 0.8660 \\ 1 & 0.8660 & 0.5 & 0.5 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\, \boldsymbol{J}_e(\boldsymbol{q}_0) = 2$$

and

$$\boldsymbol{J}_t(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0.8660 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\, \boldsymbol{J}_t(\boldsymbol{q}_0) = 2,$$

showing that both tasks can certainly be executed separately, no matter which are the values of $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$. However, the complete Jacobian for the two simultaneous tasks is singular,

$$\text{rank}\, \boldsymbol{J}(\boldsymbol{q}_0) = \text{rank} \begin{pmatrix} \boldsymbol{J}_e(\boldsymbol{q}_0) \\ \boldsymbol{J}_t(\boldsymbol{q}_0) \end{pmatrix} = 3 < 4,$$

and thus the robot is in an algorithmic singularity.

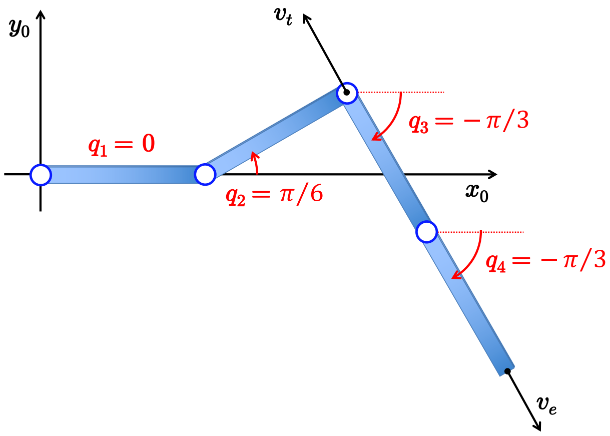

Figure 4: The 4R planar robot in the configuration $\boldsymbol{q}_0$, with the two assigned tasks.

With the above in mind, we set for $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ the given numerical values (see again Fig. 4), and solve the stated problems as follows.

a. Execute at best the end-effector task $\boldsymbol{v}_e = (0.4330, -0.75)$, while minimizing the norm of $\dot{\boldsymbol{q}}$:

$$\dot{\boldsymbol{q}}_\text{a} = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e = \begin{pmatrix} -0.4 \\ -0.5196 \\ 0.1 \\ 0.1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{e}_e &= \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_\text{a} = \boldsymbol{0} \\ \boldsymbol{e}_t &= \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_\text{a} = \begin{pmatrix} -0.7598 \\ 1.7160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}_t\| = 1.8767. \end{aligned}$$

b. Execute at best the second link tip task $\boldsymbol{v}_t = (-0.5, 0.8660)$, while minimizing the norm of $\dot{\boldsymbol{q}}$:

$$\dot{\boldsymbol{q}}_{\text{b}} = \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{b}} = \boldsymbol{0} \\ \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{b}} = \begin{pmatrix} 0.9330 \\ -1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}_e\| = 1.8660. \end{array}$$

c. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ simultaneously:

$$\dot{\boldsymbol{q}}_{\text{c}} = \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\begin{pmatrix} \boldsymbol{v}_e \\ \boldsymbol{v}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0670 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e} = \begin{pmatrix} \boldsymbol{v}_e \\ \boldsymbol{v}_t \end{pmatrix} - \boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{c}} = \begin{pmatrix} 0.4665 \\ -0.8080 \\ -0.4665 \\ 0.8080 \end{pmatrix}$$

$$\Rightarrow \quad \|\boldsymbol{e}\| = 1.3195.$$

d. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the end-effector task $\boldsymbol{v}_e$:

$$\dot{\boldsymbol{q}}_{\text{d}} = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e + \left(\boldsymbol{J}_t(\boldsymbol{q}_0)\boldsymbol{P}_e(\boldsymbol{q}_0)\right)^{\#}\left(\boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e\right) = \begin{pmatrix} 0 \\ -0.8660 \\ 0 \\ 0 \end{pmatrix}$$

with $\boldsymbol{P}_e(\boldsymbol{q}_0) = \boldsymbol{I} - \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{J}_e(\boldsymbol{q}_0)$.

$$\Rightarrow \quad \begin{array}{l} \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{d}} = \boldsymbol{0} \\ \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{d}} = \begin{pmatrix} -0.9330 \\ 1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}\| = \|\boldsymbol{e}_t\| = 1.8660. \end{array}$$

e. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the second link tip task $\boldsymbol{v}_t$:

$$\dot{\boldsymbol{q}}_{\text{e}} = \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t + \left(\boldsymbol{J}_e(\boldsymbol{q}_0)\boldsymbol{P}_t(\boldsymbol{q}_0)\right)^{\#}\left(\boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with $\boldsymbol{P}_t(\boldsymbol{q}_0) = \boldsymbol{I} - \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{J}_t(\boldsymbol{q}_0)$.

$$\Rightarrow \quad \begin{array}{l} \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{e}} = \boldsymbol{0} \\ \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\text{e}} = \begin{pmatrix} 0.9330 \\ -1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}\| = \|\boldsymbol{e}_e\| = 1.8660. \end{array}$$

Summarizing, one can observe that:

- As expected, the attempt to execute both tasks simultaneously without the use of any priority (case c.) produces errors on all tasks components, due to the algorithmic singularity. On the other hand, the introduction of priority preserves the correct execution of one of the two tasks. Nonetheless, the norm of the error on the complete set of tasks in the first case is smaller ($\|\boldsymbol{e}\| = 1.3195$) than when using priorities ($\|\boldsymbol{e}\| = 1.8660$ in both cases d. and e.).
- In the specific situation considered, the independent execution of either of the two tasks without care of the other (cases a. and b.) and their simultaneous execution with priority given to either of the two tasks (cases d. and e., respectively) produce exactly the same error in norm for the discarded or the lower priority task.

- The two chosen velocity tasks for the tip of the second link and for the end-effector are highly conflicting (as apparent also from the geometry in Fig. 4). In fact, the two velocity vectors $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ have a common direction, but opposite orientations. Moreover, they lie along the Cartesian direction where the third and fourth link of the robot are stretched: as a consequence, joints 3 and 4 cannot contribute to their simultaneous execution. Note also that these two task velocities are slightly different in norm ($\|\boldsymbol{v}_e\| = 0.8660$, $\|\boldsymbol{v}_t\| = 1$): if we had chosen still the same common direction, but exactly opposite values (i.e., $\boldsymbol{v}_e = -\boldsymbol{v}_t$), the best solution in case c. would have been $\dot{\boldsymbol{q}}_c = \boldsymbol{0}$ (the robot does not move!).

Exercise 3

With reference to Fig. 2, the potential energy due to gravity is computed (up to constants) for each link as

$$U_1(q_1) = m_1 g_0 q_1 \qquad U_2(q_1, q_2) = m_2 g_0 \left(q_1 + d_{c2} \sin q_2\right) \qquad U_3(q_1, q_2) = m_3 g_0 \left(q_1 + l_2 \sin q_2\right),$$

where $g_0 = 9.81$. Thus, from $U = U_1 + U_2 + U_3 = U(\boldsymbol{q})$, one has

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \left(m_1 + m_2 + m_3\right) \\ g_0 \cos q_2 \left(m_2 d_{c2} + m_3 l_2\right) \\ 0 \end{pmatrix}, \tag{1}$$

from which the feedforward term $\boldsymbol{g}(\boldsymbol{q}_d)$ in the control law follows by direct substitution of $q_2 = q_{2,d}$.

A well known sufficient condition for the global asymptotic stability of a desired equilibrium configuration $\boldsymbol{q}_d$ in a robot controlled by PD + gravity compensation is that $\boldsymbol{K}_{P,m} > \alpha$. For a diagonal gain matrix $\boldsymbol{K}_P$, $\boldsymbol{K}_{P,m}$ is the smallest diagonal element of the matrix. The constant α is defined as a value that bounds the norm of the Hessian matrix of the gravitational potential energy in all configurations, or

$$\left\|\frac{\partial^2 U}{\partial \boldsymbol{q}^2}\right\| = \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right\| = \sqrt{\lambda_{max}\left\{\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)\right\}} \leq \alpha, \qquad \forall \boldsymbol{q}. \tag{2}$$

Therefore, from

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -g_0 \sin q_2 \left(m_2 d_{c2} + m_3 l_2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the only positive eigenvalue of the symmetric, positive semi-definite matrix $\left\{\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)\right\}$ is

$$\lambda_{max}(q_2) = g_0^2 \left(m_2 d_{c2} + m_3 l_2\right)^2 \sin^2 q_2$$

and so the minimum value for α that globally satisfies (2) is

$$\alpha = g_0 \left(m_2 d_{c2} + m_3 l_2\right) > 0.$$

Indeed, due to the structure of the gravity vector (1) for this robot, with a first constant component and a zero third component, it is easy to see that the sufficient condition is simplified to $\boldsymbol{K}_{P,2} > \alpha$ while the other two diagonal gains $\boldsymbol{K}_{P,1}$ and $\boldsymbol{K}_{P,3}$ only need to be positive.

Exercise 4

Given the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the unconstrained 2R planar robot, the Coriolis and centrifugal terms are derived using the Christoffel symbols (without the need of knowing the actual expressions of the dynamic coefficients). One obtains

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 s_2 \left(2\dot{q}_1 + \dot{q}_2\right) \dot{q}_2 \\ a_2 s_2 \dot{q}_1^2 \end{pmatrix}.$$

As for the gravity term, the potential energy should include also the payload. Therefore, being the link lengths $l_1 = l_2 = 1$, we have

$$U_1(q_1) = m_1 g_0 d_{c1} s_1 \qquad U_2(q_1, q_2) = m_2 g_0 \left(s_1 + d_{c2} s_{12}\right) \qquad U_p(q_1, q_2) = m_p g_0 \left(s_1 + s_{12}\right),$$

where $g_0 = 9.81$. From $U = U_1 + U_2 + U_p = U(\boldsymbol{q})$, one has

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \left(m_1 d_{c1} + m_2 + m_p\right) c_1 + g_0 \left(m_2 d_{c2} + m_p\right) c_{12} \\ g_0 \left(m_2 d_{c2} + m_p\right) c_{12} \end{pmatrix} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix}.$$

The guide constrains the motion of the robot end-effector to $h(\boldsymbol{q}) = p_x(\boldsymbol{q}) = 0$. Thus, from the direct kinematics, we have

$$h(\boldsymbol{q}) = c_1 + c_{12} = 0 \qquad \Rightarrow \qquad \boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \end{pmatrix}.$$

A convenient choice for completing a square nonsingular matrix is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} & c_{12} \end{pmatrix}. \tag{3}$$

In fact, the resulting matrix is nothing else than the robot Jacobian

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q}),$$

whose determinant $\det \boldsymbol{J}(\boldsymbol{q}) = s_2$ never vanishes in the operating region of the constrained mechanism. Therefore, we can safely invert this matrix and obtain

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \frac{1}{s_2} \begin{pmatrix} c_{12} & s_{12} \\ -\left(c_1 + c_{12}\right) & -\left(s_1 + s_{12}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.$$

Moreover, the following time derivatives are needed:

$$\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} -c_1 \dot{q}_1 - c_{12} \left(\dot{q}_1 + \dot{q}_2\right) & -c_{12} \left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix} \qquad \dot{\boldsymbol{D}}(\boldsymbol{q}) = \begin{pmatrix} -s_1 \dot{q}_1 - s_{12} \left(\dot{q}_1 + \dot{q}_2\right) & -s_{12} \left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix}.$$

The choice (3) leads also to a simple physical interpretation of the pseudo-velocity

$$v = \boldsymbol{D}(\boldsymbol{q}) \dot{\boldsymbol{q}} = c_1 \dot{q}_1 + c_{12} \left(\dot{q}_1 + \dot{q}_2\right)$$

as the end-effector velocity component along the $\boldsymbol{y}$ direction, i.e., $v = v_y = \dot{p}_y(\boldsymbol{q})$. Finally, the reduced inertia of the constrained robot is evaluated as

$$\boldsymbol{F}^T(\boldsymbol{q}) \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{F}(\boldsymbol{q}) = \frac{1}{s_2^2} \left(a_3 s_1^2 + \left(a_1 - a_3\right) s_{12}^2 - 2 a_2 s_1 s_{12} c_2\right) > 0.$$

The motion control law is computed by inverse dynamics and is given by

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{F}(\boldsymbol{q})\dot{v}_d - \left(\boldsymbol{E}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q}) \right) \dot{\boldsymbol{q}} \right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \tag{4}$$

where each individual term has already been defined, except for the desired pseudo-acceleration $\dot{v}_d$. It can be shown that the control law (4) provides

$$\dot{v} = \dot{v}_d, \qquad \lambda = 0,$$

as requested.

Multiple possibilities are available for the definition of a smooth cyclic motion with period T between points A and B. For instance, one can use two specular quintic polynomials, one for the elevation from A to B in the time interval $t \in [0, T/2]$, the other for returning from B to A with $t \in [T/2, T]$. By imposing zero boundary conditions on the first and second time derivatives at $A_y = 0.95$ and $B_y = 1.45$, we obtain

$$p_{y,d}(t) = \begin{cases} A_y + (B_y - A_y)\left(10\sigma^3 - 15\sigma^4 + 6\sigma^5\right), & \sigma = \dfrac{t}{T/2}, \quad t \in [0, T/2], \\ B_y + (A_y - B_y)\left(10\sigma^3 - 15\sigma^4 + 6\sigma^5\right), & \sigma = \dfrac{t - T/2}{T/2}, \quad t \in [T/2, T]. \end{cases}$$

From this, we get the pseudo-acceleration command

$$\dot{v}_d(t) = \ddot{p}_{y,d}(t) = \begin{cases} \dfrac{60\,(B_y - A_y)}{(T/2)^2}\left(\sigma - 3\sigma^2 + 2\sigma^3\right), & \sigma = \dfrac{t}{T/2}, \quad t \in [0, T/2], \\ \dfrac{60\,(A_y - B_y)}{(T/2)^2}\left(\sigma - 3\sigma^2 + 2\sigma^3\right), & \sigma = \dfrac{t - T/2}{T/2}, \quad t \in [T/2, T]. \end{cases}$$

The desired position profile $p_{y,d}(t)$ and the pseudo-acceleration command $\dot{v}_d(t)$ are shown in Fig. 5, for a chosen motion period of $T = 1$ s.

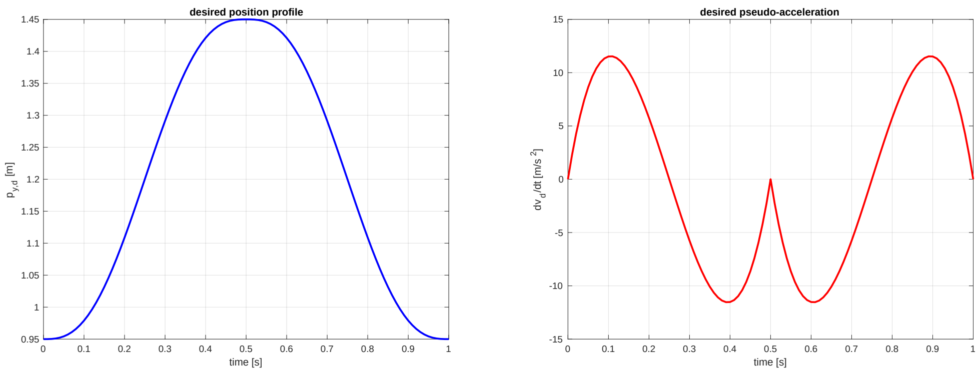

Figure 5: Periodic motion of the constrained robot end-effector: $p_{y,d}(t)$ [left] and $\dot{v}_d(t)$ [right].

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