Robotics 2

Remote Midterm Test – April 15, 2020

The test has the form of a Questionnaire. Please answer with texts and formulas and write clearly. You may also use the 'Reply Sheet' in the `Exam.net` *environment to type in some answers. Take pictures of each page of your handwritten answers and upload them in the system before submitting. Try to follow the same order of the questions. Number your replies accordingly.*

Question #1

When and why is it convenient to choose a two-stage calibration procedure for the uncertain Denavit-Hartenberg parameters in the kinematic model of a manipulator?

Question #2

The position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin O_n of the last frame of a n-dof serial manipulator is computed in homogeneous coordinates through the direct kinematics as

$$\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1\,{}^1\boldsymbol{A}_2 \ldots {}^{i-1}\boldsymbol{A}_i \ldots {}^{n-2}\boldsymbol{A}_{n-1}\,{}^{n-1}\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where 4×4 Denavit-Hartenberg homogeneous transformation matrices are used. Suppose that the only uncertainty in the kinematic model is on the value of the twist angle α_i of the ith homogeneous matrix around its nominal value $\alpha_i^{nom} = \pi/2$. Write the expression of the 3×1 regressor matrix $\boldsymbol{\Phi}$ in the basic equation $\Delta \boldsymbol{p} = \boldsymbol{\Phi}\, \Delta\alpha_i$ that is used for calibration at a generic configuration $\boldsymbol{q} \in \mathbb{R}^n$.

Question #3

The differential kinematics of a 3-dof robot performing a two-dimensional task $\boldsymbol{x}$ is expressed by $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$. Suppose that, in a given configuration $\boldsymbol{q} \in \mathbb{R}^3$, we have the following values for the task Jacobian $\boldsymbol{J}$ and the desired task velocity $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \begin{pmatrix} 3 & 1 & 2 \\ 1.5 & 0.5 & 1 \end{pmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the joint velocity $\dot{\boldsymbol{q}}^*$ of minimum norm that realizes at best the desired instantaneous task. Does the task velocity error vanish or not? Find another $\dot{\boldsymbol{q}}' \neq \dot{\boldsymbol{q}}^*$ providing the same task velocity error and show that $\|\dot{\boldsymbol{q}}^*\| < \|\dot{\boldsymbol{q}}'\|$.

Question #4

A 3R planar robot with links of unitary length moving in a vertical plane has to perform two tasks: *i)* follow a trajectory with its end-effector position, and *ii)* keep its last link upwards. At $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$ [rad], the desired end-effector linear velocity is $\boldsymbol{v}_p = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ [m/s]. Does there exist a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that executes both tasks simultaneously? If not, find a joint velocity $\dot{\boldsymbol{q}}_{TP}$ with the Task Priority method, giving higher priority to the last link orientation task.

Question #5

A 3R planar robot is moving on a horizontal plane. At a given instant of time t, the robot is in the configuration $\boldsymbol{q}(t) = \begin{pmatrix} 0 & \pi/2 & \pi/4 \end{pmatrix}^T$ [rad], with velocity $\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T$ [rad/s]. If the applied torque is $\boldsymbol{u}(t) = \begin{pmatrix} 1.5 & 0 & -4 \end{pmatrix}^T$ [Nm], will the instantaneous total energy E of the robot increase, stay the same, or decrease? And what about the Lagrangian function L?

Question #6

Given the inertia matrix of a 2R polar robot

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 + a_3 \cos^2 q_2 & 0 \\ 0 & a_4 \end{pmatrix},$$

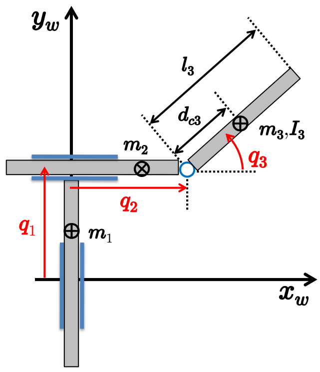
find two factorizations of the associated Coriolis/centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ is skew symmetric, while the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not.

Question #7

Consider the PPR planar robot in the figure below. Using the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined therein, determine the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Provide then a linear parametrization only of the inertial terms in the dynamic model, i.e., such that

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\, \boldsymbol{a}_M,$$

where the $3 \times p$ regressor matrix $\boldsymbol{Y}_M$ and the vector of dynamic coefficients $\boldsymbol{a}_M \in \mathbb{R}^p$ have the least possible dimension p.



Question #8

Provide the inertia matrix $\boldsymbol{M}_p(\boldsymbol{p})$ of the robot considered in Question **#7** when using for the Lagrangian dynamic modeling the new set of coordinates $\boldsymbol{p} = \begin{pmatrix} x & y & \alpha \end{pmatrix}^T \in \mathbb{R}^3$, where (x, y) are the components of the Cartesian position of the robot end-effector in world coordinates and α is the angle of the last link w.r.t. the $\boldsymbol{x}_w$ axis of the world frame.

Question #9

A single link moving under gravity is modeled by the differential equation $I\ddot{\theta} + mg_0 d \sin\theta = u$, with $m = 3$ [kg], $d = 0.5$ [m], $I = 1$ [kgm^2], and $g_0 = 9.81$ [m/s^2]. The motor torque is bounded by $|u| \leq U = 25$ [Nm]. The desired task is a rest-to-rest swing-up maneuver from $\theta(0) = 0$ to $\theta(T) = \pi$ [rad] in $T = 1$ [s], to be done with a bang-bang acceleration profile. Is the torque bound satisfied? If not, find the minimum uniform time scaling to execute the task in a feasible way.

Question #10

Assume that we have available the Newton-Euler routine $NE_\alpha(\text{arg}_1, \text{arg}_2, \text{arg}_3)$, equipped with the kinematic and dynamic data of a n-dof serial manipulator. How can we compute the kinetic energy T in a generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of this robot by just one call of this routine and one scalar product?

[180 minutes (3 hours); open books]

Solution

April 15, 2020

Question #1

When and why is it convenient to choose a two-stage calibration procedure for the uncertain Denavit-Hartenberg parameters in the kinematic model of a manipulator?

Reply #1

When it is expected that subsets of Denavit-Hartenberg parameters will have a very different uncertainty range (some with large, some with small uncertainty), the calibration procedure is performed in a first stage only for the set of parameters with large uncertainty, holding the others at their nominal values. In a second stage, calibration is completed for all parameters at the same time. In this stage, one starts with the nominal values for the original parameters with small uncertainty and with the updated values for those that have been partially calibrated in the first stage (and thus have now also a small residual uncertainty). This two-stage procedure improves the accuracy of the pseudoinverse solution of the regressor equation by equalizing the numerical conditioning of the regressor matrix. Normalizing a set of equations in this way is very common in optimization and in engineering practice. ■

Question #2

The position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin O_n of the last frame of a n-dof serial manipulator is computed in homogeneous coordinates through the direct kinematics as

$$\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1\, {}^1\boldsymbol{A}_2 \ldots {}^{i-1}\boldsymbol{A}_i \ldots {}^{n-2}\boldsymbol{A}_{n-1}\, {}^{n-1}\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where 4×4 Denavit-Hartenberg homogeneous transformation matrices are used. Suppose that the only uncertainty in the kinematic model is on the value of the twist angle α_i of the ith homogeneous matrix around its nominal value $\alpha_i^{nom} = \pi/2$. Write the expression of the 3×1 regressor matrix $\boldsymbol{\Phi}$ in the basic equation $\Delta\boldsymbol{p} = \boldsymbol{\Phi}\,\Delta\alpha_i$ that is used for calibration at a generic configuration $\boldsymbol{q} \in \mathbb{R}^n$.

Reply #2

We need to evaluate the sensitivity of $\boldsymbol{p} \in \mathbb{R}^3$ with respect to the single scalar parameter α_i, which appears only in the ith Denavit-Hartenberg (DH) homogeneous transformation matrix ${}^{i-1}\boldsymbol{A}_i$. Therefore, by rewriting the direct kinematics in compact form, we have

$$\frac{\partial \boldsymbol{p}_{hom}}{\partial \alpha_i} = \begin{pmatrix} \dfrac{\partial \boldsymbol{p}}{\partial \alpha_i} \\ 0 \end{pmatrix} = {}^0\boldsymbol{A}_{i-1} \frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\, {}^i\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where the sensitivity of the i-th DH matrix is

$$\begin{aligned} \frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i} &= \frac{\partial}{\partial \alpha_i} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\theta_i \sin\alpha_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & -\cos\theta_i \sin\alpha_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sin\theta_i \sin\alpha_i & \sin\theta_i \cos\alpha_i & 0 \\ 0 & -\cos\theta_i \sin\alpha_i & -\cos\theta_i \cos\alpha_i & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The first-order Taylor expansion of the direct kinematics around the nominal value of α_i is

$$\begin{aligned} & \boldsymbol{p}_{hom}^{nom} + \left.\frac{\partial \boldsymbol{p}_{hom}}{\partial \alpha_i}\right|_{\alpha_i=\alpha_i^{nom}} (\alpha_i - \alpha_i^{nom}) \\ & \qquad = {}^0\boldsymbol{A}_{i-1}^{nom} \left({}^{i-1}\boldsymbol{A}_i^{nom} + \left.\frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\right|_{DH_i=DH_i^{nom}} (\alpha_i - \alpha_i^{nom}) \right) {}^i\boldsymbol{A}_n^{nom} \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}. \end{aligned}$$

Eliminating the nominal identities on the left and right side, we obtain the regressor matrix $\boldsymbol{\Phi}$ (actually, a vector here) as

$$\begin{pmatrix} \boldsymbol{\Phi} \\ 0 \end{pmatrix} = {}^0\boldsymbol{A}_{i-1}^{nom} \left.\frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\right|_{DH_i=DH_i^{nom}} {}^i\boldsymbol{A}_n^{nom} \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

Being $\alpha_i^{nom} = \pi/2$, the evaluation of the sensitivity matrix $\left(\partial^{i-1}\boldsymbol{A}_i\right)/\partial\alpha_i$ in nominal conditions yields

$$\left.\begin{pmatrix} 0 & \sin\theta_i \sin\alpha_i & \sin\theta_i \cos\alpha_i & 0 \\ 0 & -\cos\theta_i \sin\alpha_i & -\cos\theta_i \cos\alpha_i & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|_{DH_i=DH_i^{nom}} = \begin{pmatrix} 0 & \sin\theta_i^{nom} & 0 & 0 \\ 0 & -\cos\theta_i^{nom} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the regressor equation is

$$\boldsymbol{\Phi}\, \Delta\alpha_i = \Delta\boldsymbol{p},$$

with $\Delta\alpha_i = \alpha_i - \alpha_i^{nom} \in \mathbb{R}$ and $\Delta\boldsymbol{p} \in \mathbb{R}^3$ being the end-effector position error measured in a generic experiment. ■

Question #3

The differential kinematics of a 3-dof robot performing a two-dimensional task $\boldsymbol{x}$ is expressed by $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$. Suppose that, in a given configuration $\boldsymbol{q} \in \mathbb{R}^3$, we have the following values for the task Jacobian $\boldsymbol{J}$ and the desired task velocity $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \begin{pmatrix} 3 & 1 & 2 \\ 1.5 & 0.5 & 1 \end{pmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the joint velocity $\dot{\boldsymbol{q}}^*$ of minimum norm that realizes at best the desired instantaneous task. Does the task velocity error vanish or not? Find another $\dot{\boldsymbol{q}}' \neq \dot{\boldsymbol{q}}^*$ providing the same task velocity error and show that $\|\dot{\boldsymbol{q}}^*\| < \|\dot{\boldsymbol{q}}'\|$.

Reply #3

It is easy to see that $\text{rank}(\boldsymbol{J}) = 1$, but also that $\dot{\boldsymbol{x}} \in \text{range}\{\boldsymbol{J}\}$ so that we can find (∞^2!) solutions to this underdetermined system of linear equations. The minimum norm solution $\dot{\boldsymbol{q}}^*$ is the one based on the pseudoinverse of $\boldsymbol{J}$, i.e., $\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\dot{\boldsymbol{x}}$, and will yield in this case zero task velocity error (i.e., $\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}^{PS} = \boldsymbol{0}$). Since we can discard one of the two equations in $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$ (because of their linear dependence and consistency), the pseudoinverse solution is easily computed from

$$\boldsymbol{J}_1\dot{\boldsymbol{q}} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}\dot{\boldsymbol{q}} = 2 = \dot{x}_1 \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}_1^{\#}\dot{x}_1 = \frac{1}{14}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot 2 = \begin{pmatrix} 3/7 \\ 1/7 \\ 2/7 \end{pmatrix} = \begin{pmatrix} 0.4286 \\ 0.1429 \\ 0.2857 \end{pmatrix}.$$

Another solution is found by simple inspection. For instance, being the third column of $\boldsymbol{J}$ equal to $\dot{\boldsymbol{x}}$, the joint velocity $\dot{\boldsymbol{q}}' = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ is also a solution. Indeed, $\|\dot{\boldsymbol{q}}_{PS}\| = 0.5345 < 1 = \|\dot{\boldsymbol{q}}'\|$. ■

Question #4

A 3R planar robot with links of unitary length moving in a vertical plane has to perform two tasks: i) *follow a trajectory with its end-effector position, and* ii) *keep its last link upwards. At* $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$ *[rad], the desired end-effector linear velocity is* $\boldsymbol{v}_p = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ *[m/s]. Does there exist a joint velocity* $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ *that executes both tasks simultaneously? If not, find a joint velocity* $\dot{\boldsymbol{q}}_{TP}$ *with the Task Priority method, giving higher priority to the last link orientation task.*

Reply #4

Since

$$\alpha = f_1(\boldsymbol{q}) = q_1 + q_2 + q_3, \qquad \boldsymbol{p} = \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \\ \sin q_1 + \sin(q_1+q_2) + \sin(q_1+q_2+q_3) \end{pmatrix},$$

the two Jacobians of the link orientation task and, respectively, of the position task are

$$\boldsymbol{J}_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

with the usual shorthand notation for trigonometric quantities (e.g., $s_{12} = \sin q_1 + \sin(q_1 + q_2)$). At $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$, the orientation of the third link is already upwards ($\alpha = \pi/2$), and this would means that no task velocity is needed for keeping the correct link orientation, or $v_\alpha = 0$. The complete task Jacobian matrix and the associated task velocity vector are thus

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2.4142 & -1.7071 & -1 \\ 1.4142 & 0.7071 & 0 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} v_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} v_\alpha \\ \boldsymbol{v}_p \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

The Jacobian $\boldsymbol{J}$ is singular (the sum of its rows is zero), while $\boldsymbol{v} \notin \text{range}\{\boldsymbol{J}\}$ (in fact, the sum of the scalar components of $\boldsymbol{v}$ is 1). This means that the two tasks are in conflict and cannot be executed simultaneously without an error. If attempting a solution with, for instance, the pseudoinverse of $\boldsymbol{J}$ (rather than with the forbidden inverse), we would get

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{v} = \begin{pmatrix} -0.8873 \\ -0.1111 \\ 0.6650 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e}_{v,PS} = \boldsymbol{v} - \boldsymbol{J}^{\#}\dot{\boldsymbol{q}}_{PS} = \begin{pmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{pmatrix},$$

spamming equally the error on all components of both velocity tasks. Instead, consider the Task Priority (TP) method for the two tasks, each with its assigned priority. For the highest priority task, we have

$$\boldsymbol{J}_1^{\#} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{P}_1 = \boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Considering that $v_1 = 0$, the TP method simplifies to

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#} v_1 + (\boldsymbol{J}_2\boldsymbol{P}_1)^{\#}\left(\boldsymbol{v}_2 - \boldsymbol{J}_2\boldsymbol{J}_1^{\#} v_1\right) = (\boldsymbol{J}_2\boldsymbol{P}_1)^{\#}\,\boldsymbol{v}_2,$$

yielding

$$\dot{\boldsymbol{q}}_{TP} = \left(\begin{pmatrix} -2.4142 & -1.7071 & 1 \\ 1.4142 & 0.7071 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \right)^{\#} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & -0.7071 \end{pmatrix}^{\#} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.3536 & 0.3536 \\ 0 & 0 \\ 0.3536 & -0.3536 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1.0607 \\ 0 \\ 1.0607 \end{pmatrix}.$$

Thus, joints 1 and 3 will move with the same speed but in opposite directions so as to satisfy the first task, pushing the error only to the second task. In fact, we have

$$\boldsymbol{e}_{v,TP} = \boldsymbol{v} - \boldsymbol{J}^{\#}\dot{\boldsymbol{q}}_{TP} = -\begin{pmatrix} 0 \\ 0.5 \\ 0.5 \end{pmatrix},$$

with $e_{v_1,TP} = 0$. Note, however, that the TP method leads to a larger norm of the error on the linear velocity task than the PS method: $\|\boldsymbol{e}_{v_2,TP}\| = 0.7071 > 0.4243 = \|\boldsymbol{e}_{v_2,PS}\|$. ■

Question #5

A 3R planar robot is moving on a horizontal plane. At a given instant of time t, the robot is in the configuration $\boldsymbol{q}(t) = \begin{pmatrix} 0 & \pi/2 & \pi/4 \end{pmatrix}^T$ [rad], with velocity $\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T$ [rad/s]. If the applied torque is $\boldsymbol{u}(t) = \begin{pmatrix} 1.5 & 0 & -4 \end{pmatrix}^T$ [Nm], will the instantaneous total energy E of the robot increase, stay the same, or decrease? And what about the Lagrangian function L?

Reply #5

Since the robot moves with constant potential energy U, we have $\dot{U} = 0$. Then, the instantaneous variation $\dot{E}$ of the total energy $E = T + U$ and the instantaneous variation $\dot{L}$ of the Lagrangian function $L = T - U$ will be the same. At the time instant t, we have

$$\dot{E}(t) = \dot{L}(t)\,(= \dot{T}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{u}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T \begin{pmatrix} 1.5 \\ 0 \\ -4 \end{pmatrix} = 0.7854 > 0.$$

Thus, the total energy of the robot and its Lagrangian will instantaneously increase. ■

Question #6

Given the inertia matrix of a 2R polar robot

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 + a_3 \cos^2 q_2 & 0 \\ 0 & a_4 \end{pmatrix},$$

find two factorizations of the associated Coriolis/centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ is skew symmetric, while the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not.

Reply #6

We compute the velocity terms using the matrices $\boldsymbol{C}_i$ of Christoffel's symbols. These are also helpful for defining a factorization that satisfies the requested skew-symmetric property. For the components of vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, we have:

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \quad \text{for } i = 1, 2,$$

being $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\begin{pmatrix} 0 & 2(a_2-a_3)\sin q_2\cos q_2 \\ 2(a_2-a_3)\sin q_2\cos q_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2}(a_2-a_3)\sin(2q_2) \\ \frac{1}{2}(a_2-a_3)\sin(2q_2) & 0 \end{pmatrix}$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = -\frac{1}{2}\begin{pmatrix} 2(a_2-a_3)\sin q_2\cos q_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(a_2-a_3)\sin(2q_2) & 0 \\ 0 & 0 \end{pmatrix},$$

leading to

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} (a_2-a_3)\sin(2q_2)\dot{q}_1\dot{q}_2 \\ -\frac{1}{2}(a_2-a_3)\sin(2q_2)\dot{q}_1^2 \end{pmatrix}.$$

We need at this point the time derivative of the inertia matrix, i.e.,

$$\dot{\boldsymbol{M}} = \begin{pmatrix} (a_2-a_3)\sin(2q_2)\dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

A factorization $\boldsymbol{S}'$ that satisfies the skew-symmetric property is then given by

$$\boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T\boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T\boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_2-a_3)\sin(2q_2)\,\dot{q}_2 & \frac{1}{2}(a_2-a_3)\sin(2q_2)\,\dot{q}_1 \\ -\frac{1}{2}(a_2-a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix},$$

being $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ and

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \begin{pmatrix} 0 & -(a_2-a_3)\sin(2q_2)\,\dot{q}_1 \\ (a_2-a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix}.$$

A possible factorization $\boldsymbol{S}''$ that, on the contrary, fails to satisfy the skew-symmetric property is

$$\boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} (a_2-a_3)\sin(2q_2)\,\dot{q}_2 & 0 \\ -\frac{1}{2}(a_2-a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix}.$$

In fact, one can verify that $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, but the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}'' = \begin{pmatrix} -(a_2-a_3)\sin(2q_2)\dot{q}_1 & 0 \\ (a_2-a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix},$$

is not skew-symmetric. ■

Question #7

Consider the PPR planar robot in the figure below. Using the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined therein, determine the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Provide then a linear parametrization only of the inertial terms in the dynamic model, i.e., such that

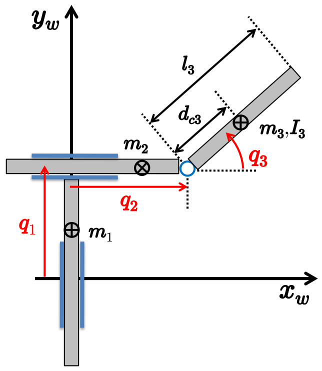
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_M,$$

where the $3 \times p$ regressor matrix $\boldsymbol{Y}_M$ and the vector of dynamic coefficients $\boldsymbol{a}_M \in \mathbb{R}^p$ have the least possible dimension p.

Reply #7

The first two simple contributions to the robot kinetic energy are

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2 \qquad \text{and} \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2\right).$$



For T_3, we compute first (in the plane)

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_2 + d_{c3}\cos q_3 \\ q_1 + d_{c3}\sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_2 - d_{c3}\sin q_3\,\dot{q}_3 \\ \dot{q}_1 + d_{c3}\cos q_3\,\dot{q}_3 \end{pmatrix}$$

$$\Rightarrow \quad \|\boldsymbol{v}_{c3}\|^2 = \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\dot{q}_3^2 + 2d_{c3}\left(\cos q_3\,\dot{q}_1 - \sin q_3\,\dot{q}_2\right)\dot{q}_3$$

and then

$$T_3 = \frac{1}{2} I_3 \dot{q}_3^2 + \frac{1}{2} m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\dot{q}_3^2 + 2d_{c3}\left(\cos q_3\,\dot{q}_1 - \sin q_3\,\dot{q}_2\right)\dot{q}_3\right).$$

The total kinetic energy of the robot is thus

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} m_1 + m_2 + m_3 & 0 & m_3 d_{c3}\cos q_3 \\ 0 & m_2 + m_3 & -m_3 d_{c3}\sin q_3 \\ m_3 d_{c3}\cos q_3 & -m_3 d_{c3}\sin q_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix} \dot{\boldsymbol{q}}.$$

By introducing a vector $\boldsymbol{a}_M \in \mathbb{R}^4$ of dynamic coefficients, the inertia matrix can be rewritten as

$$\boldsymbol{a}_M = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 + m_3 \\ m_2 + m_3 \\ I_3 + m_3 d_{c3}^2 \\ m_3 d_{c3} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & 0 & a_4\cos q_3 \\ 0 & a_2 & -a_4\sin q_3 \\ a_4\cos q_3 & -a_4\sin q_3 & a_3 \end{pmatrix}.$$

Clearly, $p = 4$ is the minimum number of dynamic coefficients for this robot. The linear parametrization of the inertial terms is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 & 0 & 0 & \cos q_3\,\ddot{q}_3 \\ 0 & \ddot{q}_2 & 0 & -\sin q_3\,\ddot{q}_3 \\ 0 & 0 & \ddot{q}_3 & \cos q_3\,\ddot{q}_1 - \sin q_3\,\ddot{q}_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}_M. \qquad \blacksquare$$

Question #8

Provide the inertia matrix $\boldsymbol{M}_p(\boldsymbol{p})$ of the robot considered in Question **#7** *when using for the Lagrangian dynamic modeling the new set of coordinates $\boldsymbol{p} = \begin{pmatrix} x & y & \alpha \end{pmatrix}^T \in \mathbb{R}^3$, where (x, y) are the components of the Cartesian position of the robot end-effector in world coordinates and α is the angle of the last link w.r.t. the $\boldsymbol{x}_w$ axis of the world frame.*

Reply #8

The change of coordinates

$$\boldsymbol{p} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 + l_3 \cos q_3 \\ q_1 + l_3 \sin q_3 \\ q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q})$$

represents the desire to use Cartesian variables for describing the dynamics of a PPR robot. The change of coordinates is here a diffeomorphism (i.e., a differentiable mapping with a unique and differentiable inverse) in $\mathbb{R}_2 \times SO(1)$. Its inverse is

$$\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p}) = \begin{pmatrix} p_2 - l_3 \sin p_3 \\ p_1 - l_3 \cos p_3 \\ p_3 \end{pmatrix},$$

while the Jacobian matrix of the transformation (and its inverse) takes the form

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & -l_3 \sin q_3 \\ 1 & 0 & l_3 \cos q_3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}^{-1}(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & -l_3 \cos q_3 \\ 1 & 0 & l_3 \sin q_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The inertia matrix of the PPR robot in the new coordinates is obtained as

$$\boldsymbol{M}_p(\boldsymbol{p}) = \left(\boldsymbol{J}^{-T}(\boldsymbol{q}) \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{J}^{-1}(\boldsymbol{q}) \right) \Big|_{\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p})}$$

in which all quantities have been already defined. ■

Additional remark: Further elaboration of the above expression is straightforward but lengthy (and beyond the scope of the present question). Nonetheless, using the Matlab Symbolic Toolbox, it can be shown that the explicit expression of $\boldsymbol{M}_p$ as a function of $\boldsymbol{p}$ only can be rewritten as

$$\boldsymbol{M}_p(\boldsymbol{p}) = \begin{pmatrix} a_{p2} & 0 & -a_{p4} \sin p_3 \\ 0 & a_{p1} & a_{p5} \cos p_3 \\ -a_{p4} \sin p_3 & a_{p5} \cos p_3 & a_{p3} + (a_{p1} - a_{p2})\, l_3^2 \cos^2 p_3 \end{pmatrix},$$

where a new set of dynamic coefficients $\boldsymbol{a}_p \in \mathbb{R}^5$ has been introduced for compactness, defined in terms of the dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^4$ already present in $\boldsymbol{M}(\boldsymbol{q})$. These new dynamic coefficients are

$$\boldsymbol{a}_p = \begin{pmatrix} a_1 \\ a_2 \\ a_3 + a_2 l_3^2 - 2 a_4 l_3 \\ a_4 - a_2 l_3 \\ a_4 - a_1 l_3 \end{pmatrix}.$$

Indeed, note that the 5 coefficients in $\boldsymbol{a}_p$ are not a minimal dynamic set: they can be expressed in fact as linear combinations of the 4 coefficients in $\boldsymbol{a}$, provided that the length l_3 of the third link (a kinematic quantity) is known.

Question #9

A single link moving under gravity is modeled by the differential equation $I\ddot{\theta} + mg_0 d \sin\theta = u$, *with* $m = 3$ *[kg],* $d = 0.5$ *[m],* $I = 1$ *[kgm*^{2}*], and* $g_0 = 9.81$ *[m/s*^{2}*]. The motor torque is bounded by* $|u| \leq U = 25$ *[Nm]. The desired task is a rest-to-rest swing-up maneuver from* $\theta(0) = 0$ *to* $\theta(T) = \pi$ *[rad] in* $T = 1$ *[s], to be done with a bang-bang acceleration profile. Is the torque bound satisfied? If not, find the minimum uniform time scaling to execute the task in a feasible way.*

Reply #9

We start by determining the value A of the piecewise constant (bang-bang) acceleration profile requested for executing the desired trajectory, given $\Delta\theta = \theta(T) - \theta(0) = \pi$ and $T = 1$. Starting at rest, the velocity $\dot{\theta}$ will grow linearly up to the midtime $t = T/2$, reaching a value $V = A \cdot T/2$, and then returning linearly to zero at $t = T$. The area underlying the triangular velocity profile is equal to the angular displacement $\Delta\theta$. Thus,

$$\Delta\theta = \int_0^T \dot{\theta}\, d\tau = \frac{V \cdot T}{2} = \frac{A \cdot T^2}{4} \qquad \Rightarrow \qquad A = \frac{4\Delta\theta}{T^2} = 4\pi \text{ [rad/s}^2\text{]}.$$

The inertial term in the dynamic model will have a constant value $u_i = I\ddot{\theta}(t) = IA = 4\pi$ in the first half of the motion, $t \in [0, T/2]$, until the link reaches the midpoint $\Delta/2 = \Pi/2$ of the motion trajectory; during the second half, $t \in (T/2, T]$, this inertial term will have the same amplitude but a negative sign. On the other hand, the gravitational torque $u_g = mg_0 d \sin\theta(t) = 14.715 \sin\theta(t)$ will grow from zero to its maximum at $t = T/2$, when $\theta(T/2) = \Delta\theta/2 = \pi/2$ and $u_{g,max} = 14.7150$, and return then symmetrically to zero. As a result, the maximum (positive) torque requested by the desired trajectory is attained at $t = T/2 = 1$ [s] and is equal to $u_{max} = u_i + u_{g,max} = 27.2814 > 25 = U$ [Nm], exceeding so the motor torque bound. The original trajectory is unfeasible. We need then to uniformly slow down motion by a factor $k > 1$, in order to reduce the inertial acceleration component of the inverse dynamics torque (the gravitational torque u_g is unaffected by any time scaling). Since the inertial torque scales with k^2 (quadratically), the minimum scaling factor k is computed as

$$k = \max\left\{1, \sqrt{\frac{u_{max} - u_{g,max}}{U - u_{g,max}}}\right\} = \max\left\{1, \sqrt{\frac{4\pi}{10.2850}}\right\} = 1.1054.$$

The new motion time will be $T_s = kT = 1.1054$ [s] and the peak of the total torque will be again assumed at $t = T_s/2$, where $\theta(T_s/2) = \pi/2$. Without the need of a new inverse dynamics analysis, this is computed as

$$u_{max,s} = \frac{u_{max} - u_{g,max}}{k^2} + u_{g,max} = \frac{u_i}{k^2} + u_{g,max} = (U - u_{g,max}) + u_{g,max} = U = 25 \text{ [Nm]}. \qquad \blacksquare$$

Additional material: Plots of the relevant quantities obtained using Matlab are reported at the end for the original trajectory (Figure 1) and for the scaled, feasible one (Figure 2).

Question #10

Assume that we have available the Newton-Euler routine $\mathrm{NE}_\alpha(\mathrm{arg}_1, \mathrm{arg}_2, \mathrm{arg}_3)$, *equipped with the kinematic and dynamic data of a n-dof serial manipulator. How can we compute the kinetic energy* T *in a generic state* $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ *of this robot by just one call of this routine and one scalar product?*

Reply #10

We compute first the Newton-Euler routine output $\boldsymbol{y} = \frac{1}{2}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = NE_0\left(\boldsymbol{q}, \boldsymbol{0}, \frac{1}{2}\dot{\boldsymbol{q}}\right)$ and then obtain the result with a scalar product: $\dot{\boldsymbol{q}}^T\boldsymbol{y} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = T(\boldsymbol{q}, \dot{\boldsymbol{q}})$. $\blacksquare$

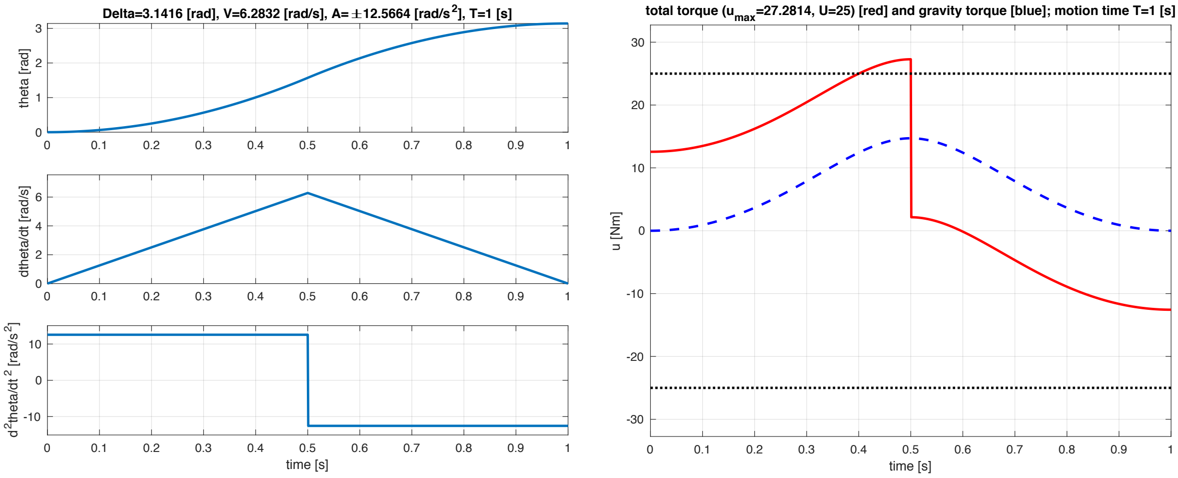


Figure 1: Position, velocity and acceleration profiles [left] and total and gravitational torques [right] for the original unfeasible trajectory with $T = 1$ [s].

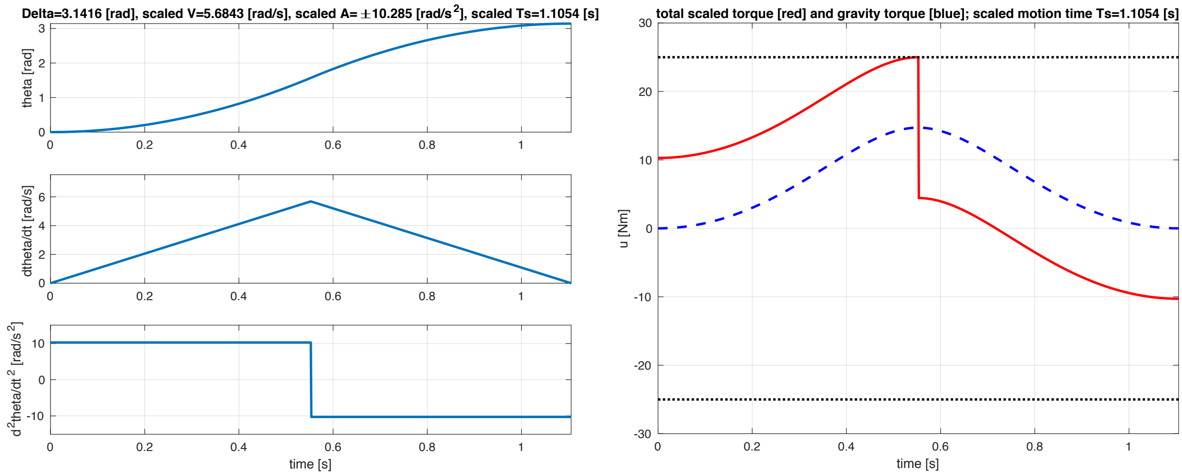


Figure 2: Position, velocity and acceleration profiles [left] and total and gravitational torques [right] for the scaled feasible trajectory with $T_s = 1.1054$ [s].

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