Robotics 2

Remote Midterm Test – April 14, 2021

Exercise #1

The 2R robot in Fig. 1 moves in a vertical plane. The two links have, respectively, kinematic lengths l_1 and l_2, masses m_1 and m_2, and barycentric inertias I_1 and I_2 (around the axis normal to the motion plane). The position of the center of mass (CoM) of each link with respect to the attached link frame is given by $\boldsymbol{r}_{ci} = \begin{pmatrix} r_{ci,x} & r_{ci,y} & 0 \end{pmatrix}^T$, with $r_{ci,x} \neq -l_i$ and $r_{ci,y} \neq 0$, for $i = 1, 2$.

A) Determine the robot dynamic model, $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}$, neglecting dissipative effects.

B) Provide a linear parametrization of the model, $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{u}$, in terms of a regressor matrix $\boldsymbol{Y} \in \mathbb{R}^{2\times p}$ and a vector $\boldsymbol{a} \in \mathbb{R}^p$ of dynamic coefficients.

Exercise #2

Consider the planar 3R robot with links of unitary length in Fig. 2 and use the absolute coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The robot is commanded with the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Starting from the configuration $\boldsymbol{q}(0) = (\pi/4, 0, 0)$, the robot should simultaneously move its end-effector point P along a vertical line parallel to $\boldsymbol{y}_0$ with a constant speed $v > 0$, while keeping the second link horizontal. Determine the first encountered configuration $\boldsymbol{q}_s$ at which these two tasks run into an algorithmic singularity. In $\boldsymbol{q} = \boldsymbol{q}_s$ and for $v = 1$ [m/s], compute the following three commands:

a) $\dot{\boldsymbol{q}}_{PS}$ using pseudoinversion of the extended Jacobian of the two tasks;

b) $\dot{\boldsymbol{q}}_{DLS}$ using damped least squares on the extended Jacobian, with damping parameter $\mu^2 = 0.25$;

c) $\dot{\boldsymbol{q}}_{TP}$ using the task priority method, with the end-effector task having the highest priority.

Compare in the three cases the norm of the resulting joint velocity, the norm of the end-effector velocity error $\dot{\boldsymbol{e}}_P \in \mathbb{R}^2$, and the absolute value of the second joint velocity error $\dot{e}_{q_2} \in \mathbb{R}$.

Exercise #3

The dynamic model of a PR robot moving on a horizontal plane is given in the lecture slides[1]. The end-effector point P should trace in minimum time a circular path of radius $R = l_2$

$$\boldsymbol{p}(s) = \begin{pmatrix} k + R\cos(s-\alpha) \\ R\sin(s-\alpha) \end{pmatrix}, \qquad s = [0, 2\alpha] \qquad (0 < \alpha < \frac{\pi}{2}),$$

from rest to rest between P_i and P_f (see Fig. 3), under bounded input force/torque $|u_i| \leq U_{i,max,}$, $i = 1, 2$. Assuming that the second bound $U_{2,max}$ is the only limiting factor, provide the expression of the needed input $\boldsymbol{u}_d(t) \in \mathbb{R}^2$ and of the minimum time T. Sketch the time profiles of the inputs.

[180 minutes (3 hours); open books]

[1] Block `03_LagrangianDynamics_1.pdf`, slide #25.

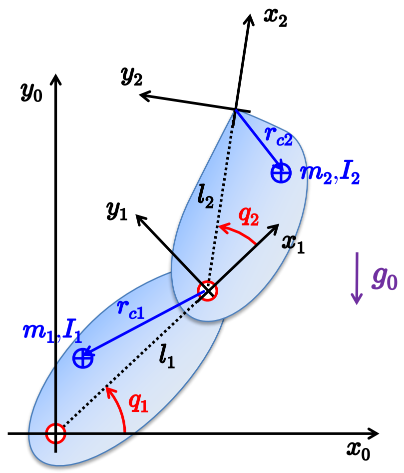

Figure 1: A planar 2R robot having the link CoMs in generic positions.

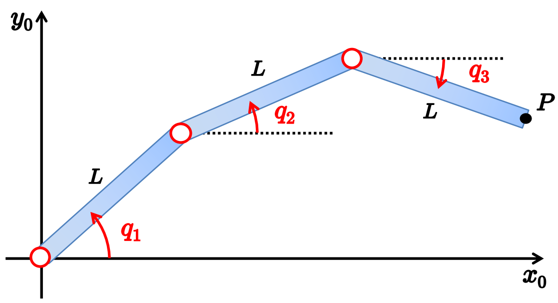

Figure 2: A planar 3R robot, with absolute coordinates $\boldsymbol{q}$ and equal links of length $L = 1$ m.

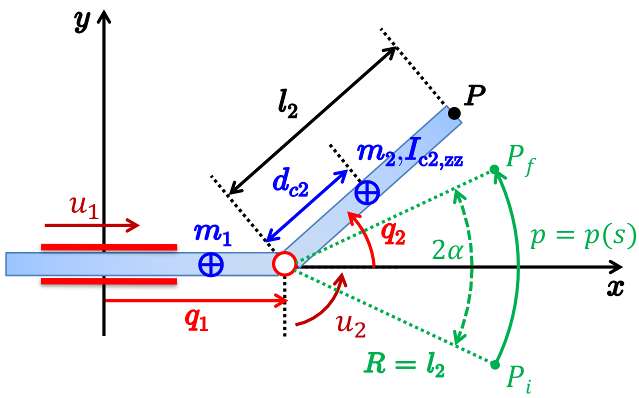

Figure 3: The assigned motion task for a planar PR robot.

Solution

April 14, 2021

Exercise #1

The special feature of this planar 2R arm is the generic location of the CoM of the links, not necessarily placed on the kinematic link axis. A Lagrangian approach is followed for dynamic modeling. We can work either with vectors in 3D or in 2D, considering the planar nature of the problem[2]. In the first case, we may also use the recursive algorithm with moving frames (the result is indeed the same).

Kinetic energy. For the first link, we have

$$T_1 = \frac{1}{2}\, m_1 \|\boldsymbol{v}_{c1}\|^2 + \frac{1}{2}\, \boldsymbol{\omega}_1^T \boldsymbol{I}_1\, \boldsymbol{\omega}_1 = \frac{1}{2}\, m_1 \left((l_1 + r_{c1,x})^2 + r_{c1,y}^2 \right) \dot{q}_1^2 + \frac{1}{2}\, I_1 \dot{q}_1^2,$$

where the coefficient in parentheses multiplying m_1 is the squared distance of the CoM of link 1 from the axis of joint 1. For the second link, we have

$$T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} \boldsymbol{\omega}_2^T \boldsymbol{I}_2\, \boldsymbol{\omega}_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2.$$

The velocity of the CoM of link 2 can be computed in two alternative ways. We can start from the absolute position of the CoM in 2D,

$${}^0\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + {}^0\bar{\boldsymbol{R}}_2(q_1, q_2) \begin{pmatrix} l_2 + r_{c2,x} \\ r_{c2,y} \end{pmatrix}, \qquad {}^0\bar{\boldsymbol{R}}_2(q_1, q_2) = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix},$$

leading to

$${}^0\boldsymbol{v}_{c2} = {}^0\dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -l_1 s_1 \\ l_1 c_1 \end{pmatrix} \dot{q}_1 + \begin{pmatrix} -\left(l_2 + r_{c2,x}\right) s_{12} - r_{c2,y}\, c_{12} \\ \left(l_2 + r_{c2,x}\right) c_{12} - r_{c2,y}\, s_{12} \end{pmatrix} (\dot{q}_1 + \dot{q}_2).$$

Or we can work with velocities in 3D and rely on moving frames; in this case, using

$${}^1\boldsymbol{\omega}_2 = {}^2\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\begin{aligned}
{}^2\boldsymbol{v}_2 &= {}^1\boldsymbol{R}_2^T(q_2) \left({}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{r}_{12}\right) = {}^1\boldsymbol{R}_2^T(q_2) \begin{pmatrix} 0 \\ l_1 \dot{q}_1 \\ 0 \end{pmatrix} + {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{r}_{12} \\
&= {}^2\boldsymbol{v}_1 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{12} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 \\ l_1 c_2\, \dot{q}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 \\ l_1 c_2\, \dot{q}_1 + l_2 \left(\dot{q}_1 + \dot{q}_2\right) \\ 0 \end{pmatrix},
\end{aligned}$$

and then

$${}^2\boldsymbol{v}_{c2} = {}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{c2} = {}^2\boldsymbol{v}_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} r_{c2,x} \\ r_{c2,y} \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 - r_{c2,y} \left(\dot{q}_1 + \dot{q}_2\right) \\ l_1 c_2\, \dot{q}_1 + \left(l_2 + r_{c2,x}\right) \left(\dot{q}_1 + \dot{q}_2\right) \\ 0 \end{pmatrix}.$$

[2] We use the compact trigonometric notation throughout this exercise, e.g., $c_{12} = \cos(q_1 + q_2)$.

As a result

$$\left\|{}^2\boldsymbol{v}_{c2}\right\|^2 = l_1^2\,\dot q_1^2 + \left((l_2 + r_{c2,x})^2 + r_{c2,y}^2\right)(\dot q_1 + \dot q_2)^2 + 2l_1\left((l_2 + r_{c2,x})\,c_2 - r_{c2,y}s_2\right)\dot q_1\,(\dot q_1 + \dot q_2)\,.$$

Indeed, it is $\left\|{}^2\boldsymbol{v}_{c2}\right\|^2 = \left\|{}^0\boldsymbol{v}_{c2}\right\|^2$. But computations (and simplifications) are easier when using the moving frames. The total kinetic energy is thus

$$\begin{aligned}
T = T_1 + T_2 = {}& \frac{1}{2}\Big(I_1 + m_1\left((l_1 + r_{c1,x})^2 + r_{c1,y}^2\right) + m_2 l_1^2 + I_2 + m_2\left((l_2 + r_{c2,x})^2 + r_{c2,y}^2\right) \\
&\qquad + 2m_2 l_1\left((l_2 + r_{c2,x})\,c_2 - r_{c2,y}s_2\right)\Big)\dot q_1^2 + \frac{1}{2}\left(I_2 + m_2\left((l_2 + r_{c2,x})^2 + r_{c2,y}^2\right)\right)\dot q_2^2 \\
&+ \Big(I_2 + m_2\left((l_2 + r_{c2,x})^2 + r_{c2,y}^2\right) + m_2 l_1\left((l_2 + r_{c2,x})\,c_2 - r_{c2,y}s_2\right)\Big)\dot q_1\dot q_2 \\
= {}& \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}.
\end{aligned}$$

Inertia matrix. One can organize the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ in a compact way, by introducing the following dynamic coefficients:

$$\begin{aligned}
a_1 &= I_1 + m_1\big((l_1 + r_{c1,x})^2 + r_{c1,y}^2\big) + m_2 l_1^2 + I_2 + m_2\big((l_2 + r_{c2,x})^2 + r_{c2,y}^2\big) \\
a_2 &= m_2 l_1 (l_2 + r_{c2,x}) \\
a_3 &= -m_2 l_1 r_{c2,y} \\
a_4 &= I_2 + m_2\big((l_2 + r_{c2,x})^2 + r_{c2,y}^2\big)
\end{aligned} \tag{1}$$

As a result,

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2c_2 + 2a_3s_2 & a_4 + a_2c_2 + a_3s_2 \\ a_4 + a_2c_2 + a_3s_2 & a_4 \end{pmatrix} \tag{2}$$

This compact form is useful for the following derivation of the velocity terms in the dynamic model. Note that the asymmetric location of the CoM of link 2 w.r.t. the link axis $\boldsymbol{x}_2$ (i.e., $r_{c2,y} \neq 0$) has introduced the extra dynamic coefficient a_3 and modified the two coefficients a_1 and a_4. On the other hand, asymmetry in the CoM of link 1 (i.e., $r_{c1,y} \neq 0$) modifies only a_1.

Coriolis and centrifugal terms. Denoting by $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, we compute the components of the Coriolis/centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ using the Christoffel symbols:

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{M}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}}{\partial q_i}\right), \qquad i = 1, 2.$$

We obtain

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -2a_2s_2 + 2a_3c_2 \\ 0 & -a_2s_2 + a_3c_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2s_2 + 2a_3c_2 & -a_2s_2 + a_3c_2 \end{pmatrix} - \boldsymbol{O}\right) \\
&= \begin{pmatrix} 0 & -a_2s_2 + a_3c_2 \\ -a_2s_2 + a_3c_2 & -a_2s_2 + a_3c_2 \end{pmatrix} \;\Rightarrow\; \begin{aligned} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= (a_3c_2 - a_2s_2)\,(2\dot q_1 + \dot q_2)\,\dot q_2 \\ &= -m_2 l_1\left(r_{c2,y}c_2 + (l_2 + r_{c2,x})s_2\right)(2\dot q_1 + \dot q_2)\,\dot q_2 \end{aligned}
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_2s_2 + a_3c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -a_2s_2 + a_3c_2 & 0 \end{pmatrix} - \begin{pmatrix} 2a_3c_2 - 2a_2s_2 & a_3c_2 - a_2s_2 \\ a_3c_2 - a_2s_2 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} -(a_3c_2 - a_2s_2) & 0 \\ 0 & 0 \end{pmatrix} \;\Rightarrow\; \begin{aligned} c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -\,(a_3c_2 - a_2s_2)\,\dot q_1^2 \\ &= m_2 l_1\left(r_{c2,y}c_2 + (l_2 + r_{c2,x})s_2\right)\dot q_1^2 \end{aligned}
\end{aligned}$$

Thus, the final expression of the quadratic velocity terms in the model is

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (a_3 c_2 - a_2 s_2) \begin{pmatrix} \dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2 \\ -\dot{q}_1^2 \end{pmatrix}. \tag{3}$$

Potential energy and gravity term. From the expression of the potential energy of a generic link i in the serial chain,

$$U_i = -m_i \, \boldsymbol{g}_0^T \boldsymbol{r}_{0,ci},$$

we obtain for the first link

$$U_1 = -m_1 \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \begin{pmatrix} * \\ (l_1 + r_{c1,x})\, s_1 + r_{c1,y} c_1 \\ * \end{pmatrix} = m_1 g_0 \big((l_1 + r_{c1,x})\, s_1 + r_{c1,y} c_1 \big),$$

and for the second link

$$U_2 = -m_2 \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \begin{pmatrix} * \\ l_1 s_1 + (l_2 + r_{c2,x})\, s_{12} + r_{c2,y} c_{12} \\ * \end{pmatrix} = m_2 g_0 \big(l_1 s_1 + (l_2 + r_{c2,x})\, s_{12} + r_{c2,y} c_{12} \big).$$

From $U = U_1 + U_2$, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} g_0 \left((m_1 (l_1 + r_{c1,x}) + m_2 l_1)\, c_1 - m_1 r_{c1,y} s_1 + m_2 (l_2 + r_{c2,x})\, c_{12} - m_2 r_{c2,y} s_{12} \right) \\ m_2 g_0 \big((l_2 + r_{c2,x})\, c_{12} - r_{c2,y} s_{12} \big) \end{pmatrix}.$$

The gravity vector in the dynamic model is rewritten more compactly as

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_5 c_1 + a_6 s_1 + a_7 c_{12} + a_8 s_{12} \\ a_7 c_{12} + a_8 s_{12} \end{pmatrix} \tag{4}$$

by introducing the additional dynamic coefficients

$$\begin{aligned} a_5 &= g_0 \big(m_1 (l_1 + r_{c1,x}) + m_2 l_1 \big) \\ a_6 &= -m_1 g_0 r_{c1,y} \\ a_7 &= m_2 g_0 (l_2 + r_{c2,x}) \\ a_8 &= -m_2 g_0 r_{c2,y}. \end{aligned} \tag{5}$$

In the gravity term, the asymmetric location of the CoM of each link introduces a single extra dynamic coefficient, namely a_6 for the first link and a_8 for the second. Instead, the two other gravity coefficients of a 2R robot with symmetric CoMs are not modified.

Linear parametrization. The complete dynamic model of the considered 2R robot,

$$\boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}, \tag{6}$$

is obtained by using (2), (3), and (4). We have already introduced the inertia-related dynamic coefficients in (1) and the gravity-related ones in (5). The linear factorization of the model (6),

$$\boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a}, \tag{7}$$

is immediately obtained in terms of the coefficient vector $\boldsymbol{a} \in \mathbb{R}^8$. The regressor matrix in (7) is

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & \begin{matrix} c_2 (2\ddot{q}_1 + \ddot{q}_2) \\ -\, s_2 \dot{q}_2 (2\dot{q}_1 + \dot{q}_2) \end{matrix} & \begin{matrix} s_2 (2\ddot{q}_1 + \ddot{q}_2) \\ +\, c_2 \dot{q}_2 (2\dot{q}_1 + \dot{q}_2) \end{matrix} & \ddot{q}_2 & c_1 & s_1 & c_{12} & s_{12} \\ 0 & c_2 \ddot{q}_1 + s_2 \dot{q}_1^2 & s_2 \ddot{q}_1 - c_2 \dot{q}_1^2 & \ddot{q}_1 + \ddot{q}_2 & 0 & 0 & c_{12} & s_{12} \end{pmatrix}. \tag{8}$$

We finally note that, assuming both the link length l_1 and the gravity acceleration g_0 to be known, the number of independent dynamic coefficients reduces from $p = 8$ to $p = 6$. In facts, two pairs of dynamic coefficients collapse:

$$\left.\begin{array}{l} a_2 = m_2\, l_1\, (l_2 + r_{c2,x}) = l_1\, a_2' \\ a_7 = m_2\, g_0\, (l_2 + r_{c2,x}) = g_0\, a_2' \end{array}\right\} \iff a_2' = m_2\, (l_2 + r_{c2,x})\,,$$

$$\left.\begin{array}{l} a_3 = -m_2\, l_1\, r_{c2,y} = l_1\, a_3' \\ a_8 = -m_2\, g_0\, r_{c2,y} = g_0\, a_3' \end{array}\right\} \iff a_3' = -m_2\, r_{c2,y}.$$

The first merging is present also in the 2R robot with symmetric CoMs. The second is related to the asymmetric case only.

Exercise #2

Taking into account the use of absolute coordinates (link angles w.r.t. the $\boldsymbol{x}_0$ axis) for this planar robot with $n = 3$ and unitary link lengths, the kinematics of the first task (of dimension $m_1 = 2$) involving the position $\boldsymbol{p}$ of the end-effector point P is

$$\boldsymbol{r}_1 = \boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos q_1 + \cos q_2 + \cos q_3 \\ \sin q_1 + \sin q_2 + \sin q_3 \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q}), \tag{9}$$

with associated Jacobian

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \end{pmatrix}. \tag{10}$$

From (9), the end-effector position in the initial configuration $\boldsymbol{q}(0) = (\pi/4, 0, 0)$ is

$$\boldsymbol{p}(0) = \boldsymbol{f}_1(\boldsymbol{q}(0)) = \begin{pmatrix} 2 + \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 2.7071 \\ 0.7071 \end{pmatrix}.$$

The desired behavior is to move the point P from $\boldsymbol{p}(0)$ along a vertical line parallel to $\boldsymbol{y}_0$ with a constant speed $v > 0$. Thus

$$\boldsymbol{r}_{1d}(t) = \boldsymbol{p}(0) + \begin{pmatrix} 0 \\ vt \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{r}}_{1d} = \dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v.$$

The second task (of dimension $m_2 = 1$) is to keep the second link always horizontal (as in the initial configuration $\boldsymbol{q}(0)$). It is described by

$$r_2 = q_2 = f_2(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{J}_2 = \frac{\partial f_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \qquad r_{2d}(t) = 0 \quad \Rightarrow \quad \dot{r}_{2d} = \dot{q}_{2d} = 0. \tag{11}$$

The two tasks are simultaneously executed using the extended Jacobian $\boldsymbol{J}_E(\boldsymbol{q})$ (a square matrix of size $m = m_1 + m_2 = 3 = n$) and the extended task velocity defined by

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \dot{\boldsymbol{r}}_{E,d} = \begin{pmatrix} \dot{\boldsymbol{r}}_{1d} \\ \dot{r}_{2d} \end{pmatrix} \in \mathbb{R}^3. \tag{12}$$

Therefore, out of singularities, the joint velocity will be commanded in an unique way by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{r}}_{E,d} = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}. \tag{13}$$

In the initial configuration $\boldsymbol{q}(0)$, the extended Jacobian $\boldsymbol{J}_E(\boldsymbol{q}(0))$ has full rank. From (13), we obtain $\dot{\boldsymbol{q}}(0) = (0, 0, 1)$ [rad/s], with only the third link rotating counterclockwise. It is rather intuitive that, when the robot moves its end effector upwards vertically and keeps its second link horizontal ($q_2 = 0$), the first link will rotate clockwise (decreasing its orientation from $\pi/4$) and the third link counterclockwise (increasing its absolute orientation from 0). This motion will continue until a singular configuration $\boldsymbol{q}_s$ is first encountered for the extended Jacobian in (12). To determine $\boldsymbol{q}_s$, we impose at the same time

$$\det \boldsymbol{J}_E(\boldsymbol{q}_s) = \sin(q_{s3} - q_{s1}) = 0$$

and that the end effector is still on the initial vertical path (with the second link horizontal), or

$$p_x(\boldsymbol{q}_s)|_{q_{s2}=0} = \cos q_{s1} + \cos q_{s2}|_{q_{s2}=0} + \cos q_{s3} = \cos q_{s1} + 1 + \cos q_{s3} = 2 + \frac{\sqrt{2}}{2} = p_x(\boldsymbol{q}(0)).$$

These two equations are solved as

$$q_{s3} = q_{s1} \quad\Rightarrow\quad 2\cos q_{s1} = 1 + \frac{\sqrt{2}}{2} \quad\Rightarrow\quad q_{s1} = \arccos\left(\frac{2+\sqrt{2}}{4}\right) = 0.5480 \text{ [rad]}.$$

The singular configuration and the associated end-effector position are thus (see Fig. 4)

$$\boldsymbol{q}_s = \begin{pmatrix} 0.5480 \\ 0 \\ 0.5480 \end{pmatrix} \text{[rad]} \qquad\Rightarrow\qquad \boldsymbol{p}_s = \boldsymbol{f}(\boldsymbol{q}_s) = \begin{pmatrix} 2.7071 \\ 1.0420 \end{pmatrix} \text{[m]}.$$

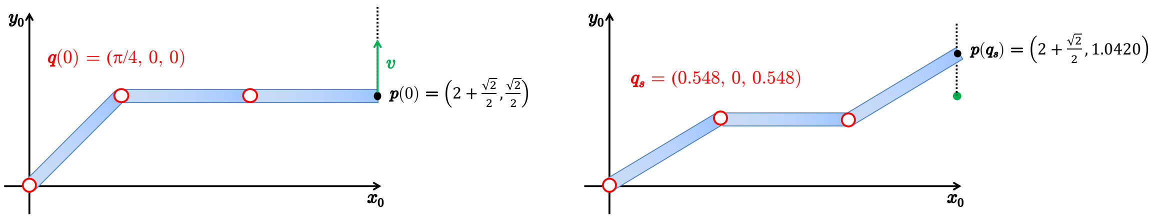

Figure 4: The 3R robot in its initial configuration $\boldsymbol{q}(0)$ and in the singular configuration $\boldsymbol{q}_s$.

The extended Jacobian is evaluated as

$$\boldsymbol{J}_E(\boldsymbol{q}_s) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}_s) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} -0.5210 & 0 & -0.5210 \\ 0.8536 & 1 & 0.8536 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\operatorname{rank} \boldsymbol{J}_1(\boldsymbol{q}_s) = 2 = m_1, \quad \operatorname{rank} \boldsymbol{J}_2 = 1 = m_2, \qquad \text{but} \qquad \operatorname{rank} \boldsymbol{J}_E(\boldsymbol{q}_s) = 2 < 3 = m = m_1 + m_2,$$

the configuration $\boldsymbol{q}_s$ is a true *algorithmic* singularity. Moreover, the extended task cannot be realized in this configuration, since

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}_E(\boldsymbol{q}_s)\right\}, \qquad \forall v \neq 0.$$

We evaluate then the three requested joint velocity commands, setting in particular $v = 1$ [m/s]. Using the pseudoinverse method, we have

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}_E^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_d = \begin{pmatrix} -0.4098 & 0.3357 & -0.3357 \\ 0.3498 & 0.2135 & 0.7865 \\ -0.4098 & 0.3357 & -0.3357 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3357 \\ 0.2135 \\ 0.3357 \end{pmatrix}. \tag{14}$$

Using instead the damped least squares method, with damping parameter $\mu^2 = 0.25$, we obtain

$$\dot{\boldsymbol{q}}_{DLS} = \boldsymbol{J}_E^T(\boldsymbol{q}_s)\left(\mu^2\boldsymbol{I} + \boldsymbol{J}_E(\boldsymbol{q}_s)\boldsymbol{J}_E^T(\boldsymbol{q}_s)\right)^{-1}\dot{\boldsymbol{r}}_d = \begin{pmatrix} -0.3251 & 0.2959 & -0.2367 \\ 0.2467 & 0.2199 & 0.6241 \\ -0.3251 & 0.2959 & -0.2367 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2959 \\ 0.2199 \\ 0.2959 \end{pmatrix}. \tag{15}$$

Note that the two velocities $\dot{\boldsymbol{q}}_{PS}$ and $\dot{\boldsymbol{q}}_{DLS}$ are quite similar. In particular, in both commands the first and the third joint move with the same speed (different in the two methods). Finally, to apply the task priority method when the end-effector task is given the highest priority, we need to compute[3]

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_{d1} + (\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s))^{\#}\left(\dot{r}_{d2} - \boldsymbol{J}_2\boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\dot{\boldsymbol{r}}_{d1}\right), \tag{16}$$

where, being $\boldsymbol{J}_1(\boldsymbol{q}_s)$ a full rank matrix, the pseudoinverse of the first task Jacobian is evaluated as

$$\boldsymbol{J}_1^{\#}(\boldsymbol{q}_s) = \boldsymbol{J}_1^T(\boldsymbol{q}_s)\left(\boldsymbol{J}_1(\boldsymbol{q}_s)\boldsymbol{J}_1^T(\boldsymbol{q}_s)\right)^{-1} = \begin{pmatrix} -0.9597 & 0 \\ 1.6383 & 1 \\ -0.9597 & 0 \end{pmatrix},$$

and the associated projector in the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ becomes

$$\boldsymbol{P}_1(\boldsymbol{q}_s) = \boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\boldsymbol{J}_1(\boldsymbol{q}_s) = \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0.5 \end{pmatrix}.$$

It is easy then to see the vanishing of the term

$$\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad (\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s))^{\#} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \tag{17}$$

As a result, the task priority method (16) collapses here into the simple use of the pseudoinverse of the first task Jacobian

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_{d1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{18}$$

[3] Here, we set $\dot{r}_{d2} = 0$. However, one should not care too much about the terms inside the last parenthesis in (16): these will be premultiplied anyway by zero —see eq. (17).

Only the second link rotates, fully violating the second task but perfectly realizing the first one. The task executions obtained with the three methods (14), (15), and (18) are

$$
\left.
\begin{array}{l}
\dot{\boldsymbol{r}}_{PS} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{PS} = \begin{pmatrix} -0.3498 \\ 0.7865 \\ 0.2135 \end{pmatrix} \\[2ex]
\dot{\boldsymbol{r}}_{DLS} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{DLS} = \begin{pmatrix} -0.3084 \\ 0.7251 \\ 0.2199 \end{pmatrix} \\[2ex]
\dot{\boldsymbol{r}}_{TP} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{TP} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\end{array}
\right\} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{r}}_{E,d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
$$

For comparison, the norms of the joint velocity $\dot{\boldsymbol{q}}_{method}$ obtained with the three methods, together with the norms of the end-effector velocity error $\dot{\boldsymbol{e}}_P = \dot{\boldsymbol{p}}_d - \boldsymbol{J}_1(\boldsymbol{q})\dot{\boldsymbol{q}}_{method}$ (task 1), and the absolute value of the velocity error of the second joint $\dot{e}_{q_2} = \dot{q}_{2,method}$ (task 2) are reported in Table 1.

Table 1: Comparison of results with the three methods.

method	$\lVert\dot{\boldsymbol{q}}_{method}\rVert$ [rad/s]	$\lVert\dot{\boldsymbol{e}}_P\rVert$ [m/s]	$\lvert\dot{e}_{q_2}\rvert$ [rad/s]
PS	0.5205	0.4098	0.2135
DLS	0.4728	0.4131	0.2199
TP	1	0	1

Exercise #3

The dynamic model of the PR robot in Fig. 3 is

$$(m_1 + m_2)\,\ddot{q}_1 - m_2 d_{c2} \sin q_2\, \ddot{q}_2 - m_2 d_{c2} \cos q_2\, \dot{q}_2^2 \;=\; u_1, \tag{19}$$

$$-m_2 d_{c2} \sin q_2\, \ddot{q}_1 + \left(I_{c2,zz} + m_2 d_{c2}^2\right) \ddot{q}_2 \;=\; u_2. \tag{20}$$

The geometry of the desired Cartesian motion task is very peculiar to this robot: an arc of a circle with center C on the joint axis 2 and radius R equal to the length l_2 of the second link. This requires simply no motion for the first joint, namely

$$q_{1d} = k \quad \text{(this value is irrelevant)}, \qquad \dot{q}_{1d} = \ddot{q}_{1d} = 0.$$

Accordingly, the inverse dynamics obtained from (19–20) yields

$$u_{1d} \;=\; -m_2 d_{c2} \sin q_{2d}\, \ddot{q}_{2d} - m_2 d_{c2} \cos q_{2d}\, \dot{q}_{2d}^2, \tag{21}$$

$$u_{2d} \;=\; I\, \ddot{q}_{2d}, \tag{22}$$

where we set for compactness $I = I_{c2,zz} + m_2 d_{c2}^2 > 0$. Therefore, in order to trace the arc of the circle from rest to rest ($\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \boldsymbol{0}$) in minimum time, equation (22) implies that a bang-bang torque $\pm U_{2,max}$ will be applied at the second (revolute) joint, with switching at the middle time $t = T/2$ of the motion. The force u_{1d} in (21) at the prismatic joint is needed to keep the first link at rest. The assumption that the bound $U_{2,max}$ is the only limiting factor when reducing as much as possible the motion time implies that the input force $|u_{1d}(t)|$ will never exceed $U_{1,max}$.

The motion profile $q_{2d}(t)$ of the second joint is then easily defined. Setting $A_{2,max} = U_{2,max}/I$ as the maximum acceleration of the second joint, and taking into account that $q_{2d}(0) = -\alpha$ and $\dot{q}_{2d}(0) = 0$, by successive integration and boundary condition satisfaction we get

$$
\begin{aligned}
\ddot{q}_{2d}(t) &= \begin{cases} A_{2,max}, & t \in \left[0, \frac{T}{2}\right] \\ -A_{2,max}, & t \in \left[\frac{T}{2}, T\right] \end{cases} \\
\dot{q}_{2d}(t) &= \begin{cases} A_{2,max}\, t, & t \in \left[0, \frac{T}{2}\right] \\ A_{2,max}\,(T-t)\,, & t \in \left[\frac{T}{2}, T\right] \end{cases} \\
q_{2d}(t) &= \begin{cases} -\alpha + \frac{1}{2}\, A_{2,max}\, t^2, & t \in \left[0, \frac{T}{2}\right] \\ -\alpha + A_{2,max} \left(\frac{T}{2}\right)^2 - \frac{1}{2}\, A_{2,max}\,(T-t)^2\,, & t \in \left[\frac{T}{2}, T\right]. \end{cases}
\end{aligned}
\tag{23}
$$

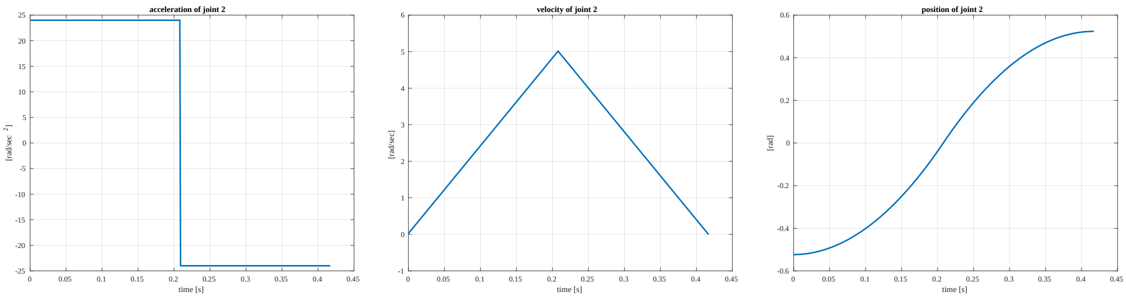

Figure 5: Time-optimal acceleration, velocity, and position profiles for the second joint.

The motion time T is determined by imposing that the area of the (triangular and symmetric) velocity profile $\dot{q}_{2d}(t)$ in $[0, T]$ is equal to the required joint displacement $\Delta q_2 = 2\alpha$. Thus

$$
\dot{q}_{2d}\,(T/2) \cdot \frac{T}{2} = A_{2,max}\,\frac{T}{2} \cdot \frac{T}{2} = 2\alpha \qquad \Rightarrow \qquad T = \sqrt{\frac{8\alpha}{A_{2,max}}}. \tag{24}
$$

Figure 5 shows representative kinematic profiles of the second joint motion[4]. As for the first input, we have from eqs. (21) and (23)

$$
\begin{aligned}
u_{1d}(t) &= -m_2 d_{c2} \sin q_{2d}(t)\, \ddot{q}_{2d}(t) - m_2 d_{c2} \cos q_{2d}(t)\, \dot{q}_{2d}^2(t) \\
&= u_{1d,acceleration}(t) + u_{1d,centripetal}(t) \\
&= -m_2 d_{c2}\, A_{2,max} \left(\sin q_{2d}(t) + \cos q_{2d}(t) A_{2,max}\, t^2\right),
\end{aligned}
\tag{25}
$$

where the last identity holds for the first half of the motion, i.e., for $t \in \left[0, \frac{T}{2}\right]$. The behavior in the second half of the motion, for $t \in \left[\frac{T}{2}, T\right]$, is perfectly specular.

The analysis of the two contributions to $u_{1d}(t)$ in (25) is simple —see also Fig. 6. The acceleration term is always non-negative, with a sinusoidal profile that has its maximum at $t = 0$ and $t = T$, where

$$
u_{1d,acceleration}(0) = u_{1d,acceleration}(T) = m_2 d_{c2}\, A_{2,max} \cdot \sin\alpha, \tag{26}
$$

while $u_{1d,acceleration}(T/2) = 0$. Vice versa, the centripetal term is never positive, it is zero at $t = 0$ and $t = T$, and takes its maximum (negative) value at $t = T/2$, when $q_{2d}(T/2) = 0$, with

$$
u_{1d,centripetal}(T/2) = -m_2 d_{c2}\, A_{2,max}^2 \left(\frac{T}{2}\right)^2 = m_2 d_{c2}\, A_{2,max} \cdot 2\alpha, \tag{27}
$$

[4]To generate these plots, the data reported in (29–30) have been used.

where (24) has been used. It is easy to see that the maximum value in (27) always dominates (26). Therefore,

$$|u_{1d}(t)| \leq U_{1,max}, \quad \forall t \in [0, T] \quad \iff \quad \max_{t \in [0,T]} |u_{1d}(t)| = 2\alpha\, m_2 d_{c2}\, A_{2,max} \leq U_{1,max}.$$

For this inequality to be verified with the assumed time-optimal solution for joint 2 (i.e., for the assumption in the text to hold true), the bounds on the two inputs should satisfy

$$A_{2,max} = \frac{U_{2,max}}{I} \quad \Rightarrow \quad \frac{2\alpha\, m_2 d_{c2}}{I}\, U_{2,max} \leq U_{1,max}. \tag{28}$$

While it is straightforward to sketch the input profiles of u_{1d} (approximately) and u_{2d} (exactly, being this a bang-bang torque), we conclude instead with a numerical evaluation using MATLAB. Setting for the arc of the circle the value $\alpha = \pi/6 = 30°$ and using the robot data

$$m_1 = m_2 = 2 \text{ [kg]}, \quad l_2 = 0.5,\ d_{c2} = 0.25 \text{ [m]}, \quad I = 0.1667 \text{ [kg m}^2\text{]}, \tag{29}$$

with bounds

$$U_{1,max} = 14 \text{ [N]}, \qquad U_{2,max} = 4 \text{ [Nm]} \qquad \Rightarrow \qquad A_{2,max} = 24 \text{ [rad/s}^2\text{]}, \tag{30}$$

the minimum motion time is found by (24) as $T = 0.4178$ [s]. Note that the input bounds in (30) satisfy inequality (28), being $u_{1d}(T/2) = -12.587$ [N]. The associated profile of the force input on the prismatic joint is shown in Fig. 6, together with those of its two contributions. The two time-optimal inputs profiles and their assigned bounds are reported in Fig. 7.

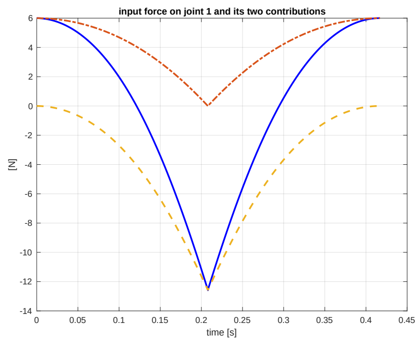

Figure 6: Input force $u_{1d}(t)$, with its acceleration (dotted-dashed) and centripetal (dashed) terms.

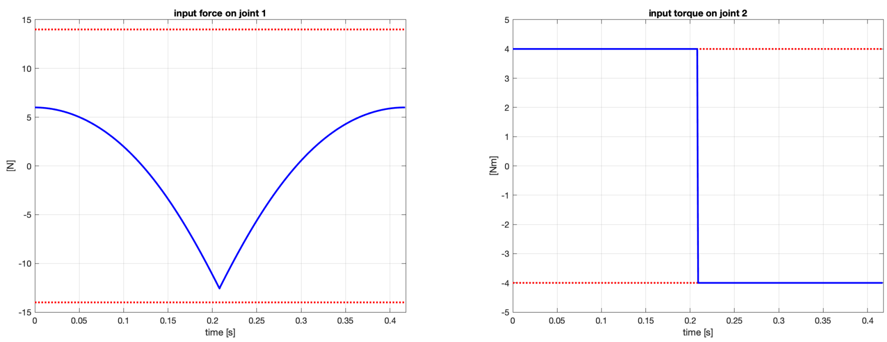

Figure 7: Time-optimal input profiles along the assigned path and their maximum bounds.

* * * * *