*Robotics 2*

# Dynamic model of robots:

## Analysis, properties, extensions, uses


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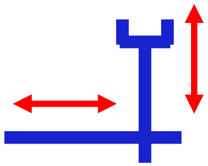
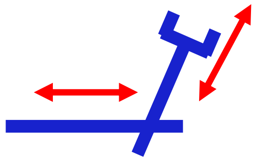
# Analysis of inertial couplings

- Cartesian robot



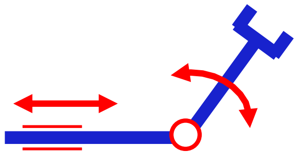
$$M = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}$$

- Cartesian "skew" robot



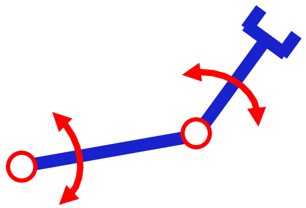
$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$$

- PR robot



$$M = \begin{pmatrix} m_{11} & m_{12}(q_2) \\ m_{12}(q_2) & m_{22} \end{pmatrix}$$

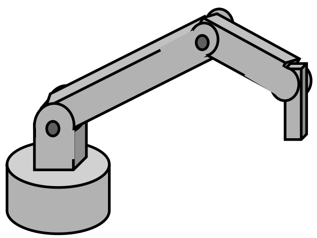
- 2R robot



$$M = \begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{12}(q_2) & m_{22} \end{pmatrix}$$

- 3R articulated robot

  (under simplifying assumptions on the CoMs)



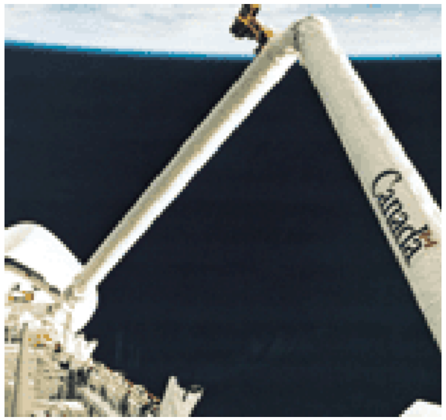
$$M = \begin{pmatrix} m_{11}(q_2, q_3) & 0 & 0 \\ 0 & m_{22}(q_3) & m_{23}(q_3) \\ 0 & m_{23}(q_3) & m_{33} \end{pmatrix}$$

# Analysis of gravity term

- absence of gravity
  - constant $U_g$ (motion on horizontal plane)
  - applications in remote space
- static balancing
  - distribution of masses (including motors)
- mechanical compensation
  - articulated system of springs
  - closed kinematic chains

$\Rightarrow$ $g(q) \approx 0$





# Bounds on dynamic terms

- for an open-chain (serial) manipulator, there always exist positive real constants $k_0$ to $k_7$ such that, for any value of $q$ and $\dot{q}$

$$k_0 \leq \|M(q)\| \leq k_1 + k_2\|q\| + k_3\|q\|^2$$
inertia matrix

$$\|S(q,\dot{q})\| \leq (k_4 + k_5\|q\|)\ \|\dot{q}\|$$
factorization matrix of Coriolis/centrifugal terms

$$\|g(q)\| \leq k_6 + k_7\|q\|$$
gravity vector

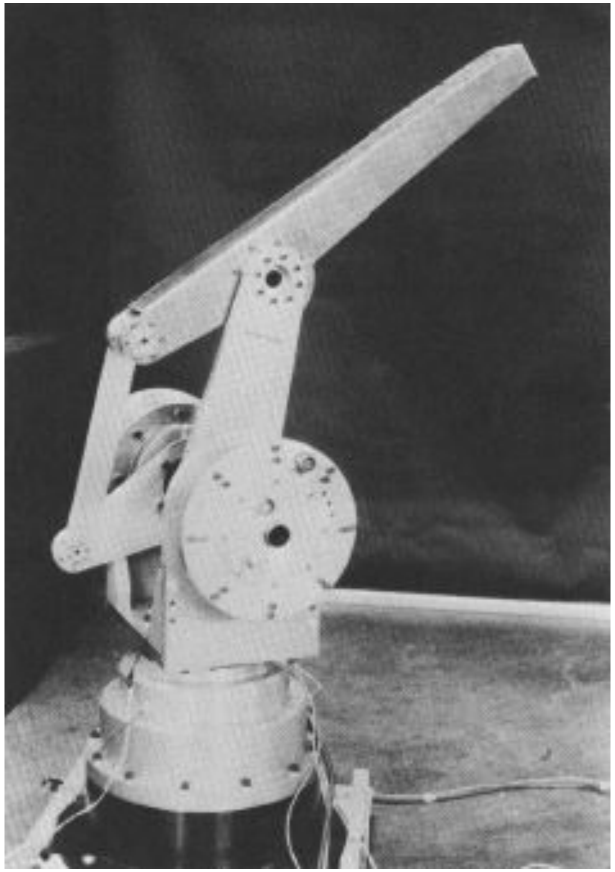
- if the robot has only revolute joints, these simplify to

$$k_0 \leq \|M(q)\| \leq k_1 \quad \|S(q,\dot{q})\| \leq k_4\|\dot{q}\| \quad \|g(q)\| \leq k_6$$

(the same holds true with bounds $q_{i,min} \leq q_i \leq q_{i,max}$ on prismatic joints)

NOTE: norms are either for vectors or for matrices (induced norms)

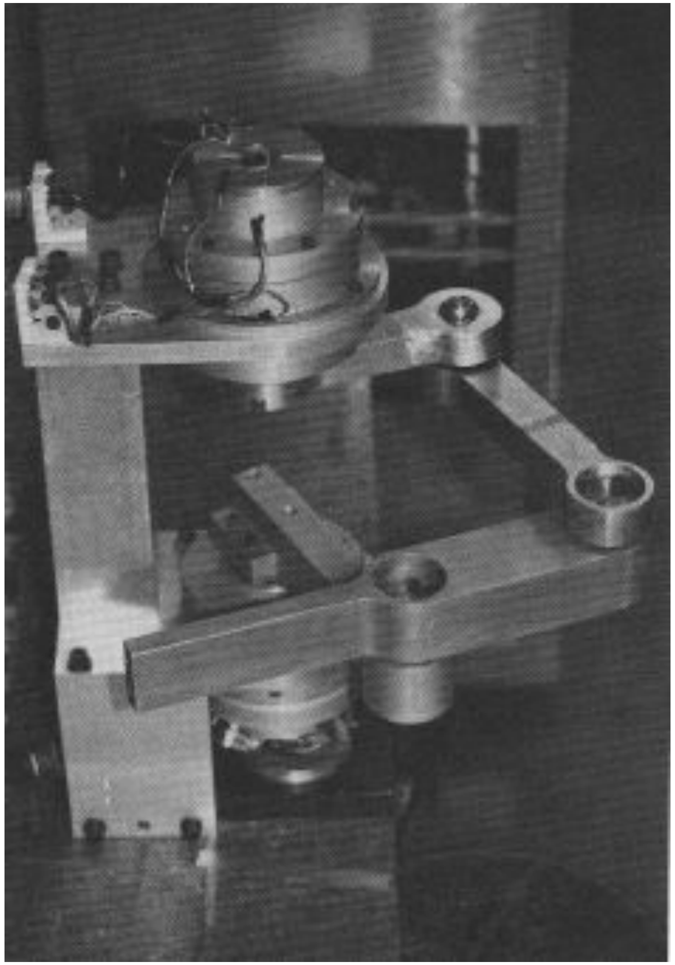
# Robots with closed kinematic chains - 1



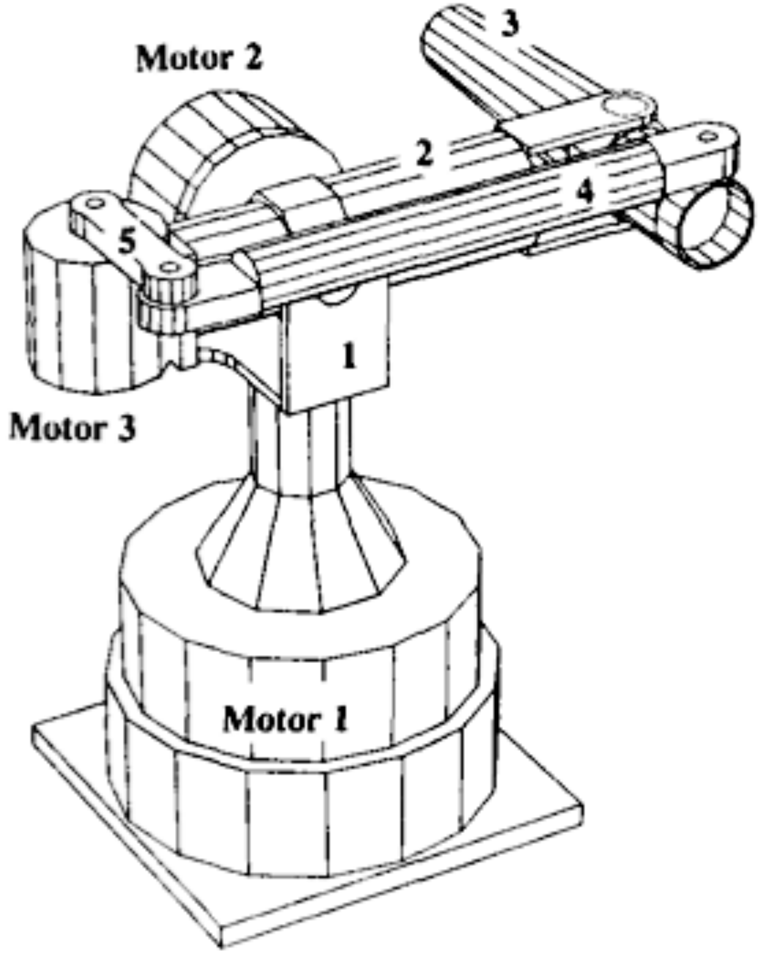
Comau Smart NJ130

MIT Direct Drive Mark II and Mark III

# Robots with closed kinematic chains - 2



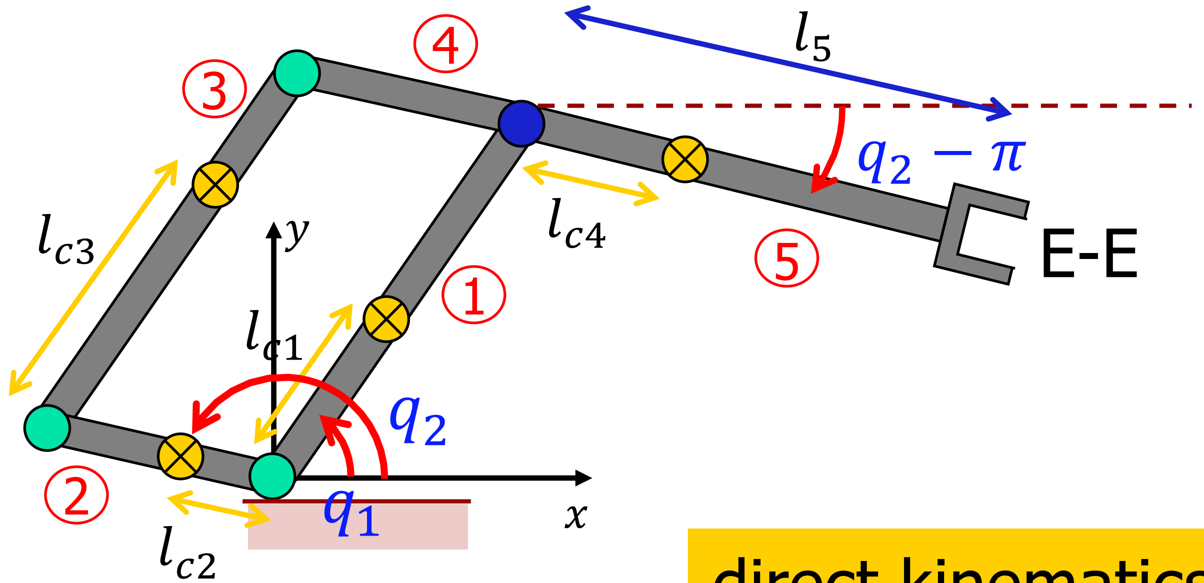
MIT Direct Drive Mark IV
(planar five-bar linkage)




UMinnesota Direct Drive Arm
(spatial five-bar linkage)

# Robot with parallelogram structure

## (planar) kinematics and dynamics




center of mass: arbitrary $l_{ci}$

parallelogram:

$l_1 = l_3$

$l_2 = l_4$

**direct kinematics**

$$p_{EE} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + \begin{pmatrix} l_5 \cos(q_2 - \pi) \\ l_5 \sin(q_2 - \pi) \end{pmatrix} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} - \begin{pmatrix} l_5 c_2 \\ l_5 s_2 \end{pmatrix}$$

**position of center of masses**

$$p_{c1} = \begin{pmatrix} l_{c1} c_1 \\ l_{c1} s_1 \end{pmatrix} \quad p_{c2} = \begin{pmatrix} l_{c2} c_2 \\ l_{c2} s_2 \end{pmatrix} \quad p_{c3} = \begin{pmatrix} l_2 c_2 \\ l_2 s_2 \end{pmatrix} + \begin{pmatrix} l_{c3} c_1 \\ l_{c3} s_1 \end{pmatrix} \quad p_{c4} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} - \begin{pmatrix} l_{c4} c_2 \\ l_{c4} s_2 \end{pmatrix}$$

# Kinetic energy

**linear/angular velocities**

$$v_{c1} = \begin{pmatrix} -l_{c1}s_1 \\ l_{c1}c_1 \end{pmatrix}\dot{q}_1 \quad v_{c3} = \begin{pmatrix} -l_{c3}s_1 \\ l_{c3}c_1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} -l_2 s_2 \\ l_2 c_2 \end{pmatrix}\dot{q}_2 \quad \omega_1 = \omega_3 = \dot{q}_1$$

$$v_{c2} = \begin{pmatrix} -l_{c2}s_2 \\ l_{c2}c_2 \end{pmatrix}\dot{q}_2 \quad v_{c4} = \begin{pmatrix} -l_1 s_1 \\ l_1 c_1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} l_{c4}s_2 \\ -l_{c4}c_2 \end{pmatrix}\dot{q}_2 \quad \omega_2 = \omega_4 = \dot{q}_2$$

Note: a (planar) 2D notation is used here!

**$T_i$**

$$T_1 = \frac{1}{2}m_1 l_{c1}^2\dot{q}_1^2 + \frac{1}{2}I_{c1,zz}\dot{q}_1^2 \qquad T_2 = \frac{1}{2}m_2 l_{c2}^2\dot{q}_2^2 + \frac{1}{2}I_{c2,zz}\dot{q}_2^2$$

$$T_3 = \frac{1}{2}m_3(l_2^2\dot{q}_2^2 + l_{c3}^2\dot{q}_1^2 + 2l_2 l_{c3}c_{2-1}\dot{q}_1\dot{q}_2) + \frac{1}{2}I_{c3,zz}\dot{q}_1^2$$

$$T_4 = \frac{1}{2}m_4(l_1^2\dot{q}_1^2 + l_{c4}^2\dot{q}_2^2 - 2l_1 l_{c4}c_{2-1}\dot{q}_1\dot{q}_2) + \frac{1}{2}I_{c4,zz}\dot{q}_2^2$$

# Robot inertia matrix

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$M(q) = \begin{pmatrix} I_{c1,zz} + m_1 l_{c1}^2 + I_{c3,zz} + m_3 l_{c3}^2 + m_4 l_1^2 & \text{symm} \\ (m_3 l_2 l_{c3} - m_4 l_1 l_{c4}) c_{2-1} & I_{c2,zz} + m_2 l_{c2}^2 + I_{c4,zz} + m_4 l_{c4}^2 + m_3 l_2^2 \end{pmatrix}$$

structural condition
in mechanical design

$$m_3 l_2 l_{c3} = m_4 l_1 l_{c4} \quad (*)$$

$M(q)$ diagonal and constant ⇒ centrifugal and Coriolis terms $\equiv 0$

mechanically DECOUPLED and LINEAR dynamic model (up to the gravity term $g(q)$) ⟺

$$\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

big advantage for the design of a motion control law!

# Potential energy and gravity terms

from the $y$-components of vectors $p_{ci}$

$U_i$

$$U_1 = m_1 g_0 l_{c1} s_1 \qquad U_2 = m_2 g_0 l_{c2} s_2$$

$$U_3 = m_3 g_0 (l_2 s_2 + l_{c3} s_1) \qquad U_4 = m_4 g_0 (l_1 s_1 - l_{c4} s_2)$$

$$U = \sum_{i=1}^{4} U_i$$

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T = \begin{pmatrix} g_0(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1) c_1 \\ g_0(m_2 l_{c2} + m_3 l_2 - m_4 l_{c4}) c_2 \end{pmatrix} = \begin{pmatrix} g_1(q_1) \\ g_2(q_2) \end{pmatrix}$$

gravity components are always "decoupled"

in addition, when (*) holds ⇒

$$m_{11}\ddot{q}_1 + g_1(q_1) = u_1$$
$$m_{22}\ddot{q}_2 + g_2(q_2) = u_2$$

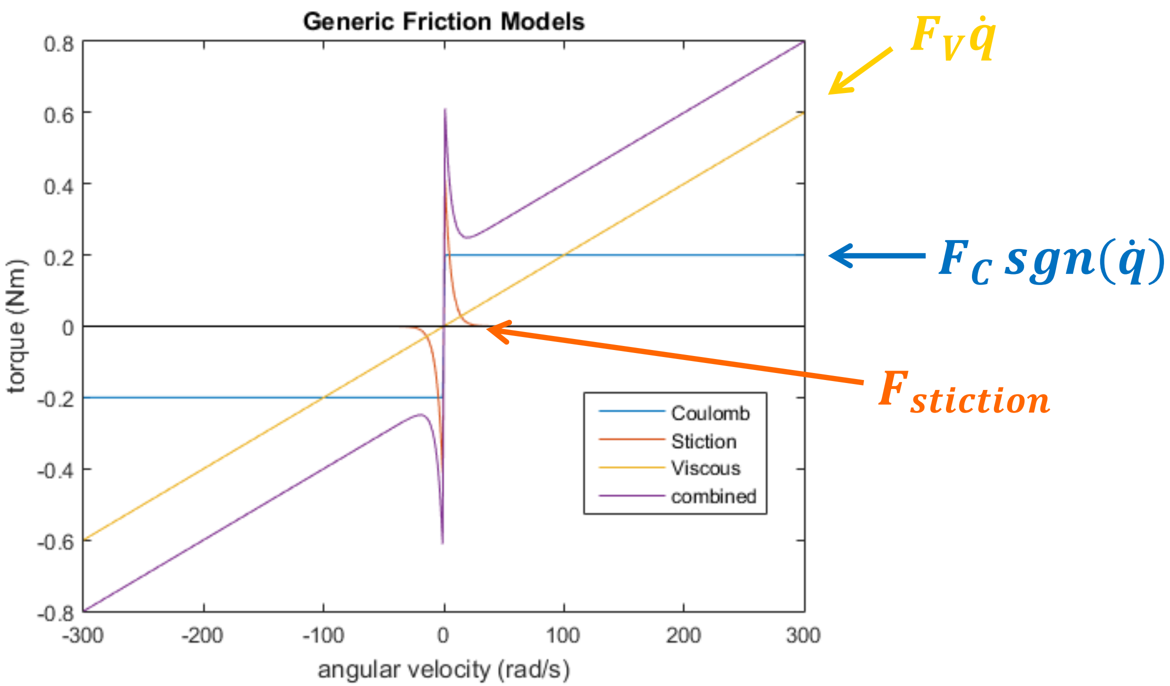
$u_i$ are (non-conservative) torques performing work on $q_i$

further structural conditions in the mechanical design lead to $g(q) \equiv 0$!!

# Adding dynamic terms ...

1) dissipative phenomena due to friction at the joints/transmissions
   - viscous, Coulomb, stiction, Stribeck, LuGre (dynamic)...
   - local effects at the joints
   - difficult to model in general, except for:

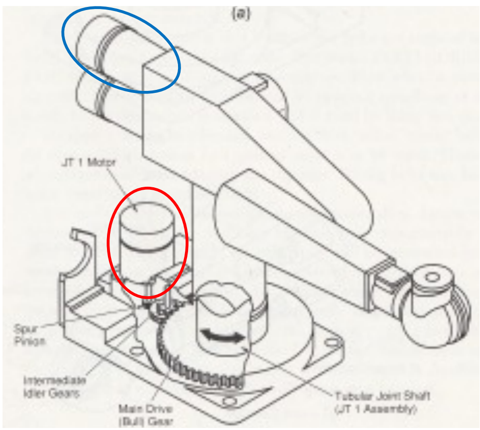
$$u_{V,i} = -F_{V,i}\,\dot{q}_i \qquad u_{C,i} = -F_{C,i}\,\mathrm{sgn}(\dot{q}_i)$$

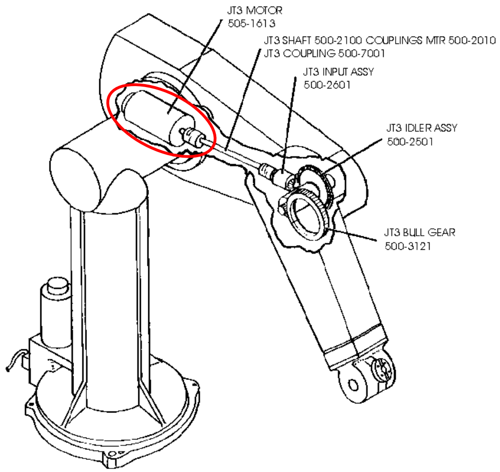


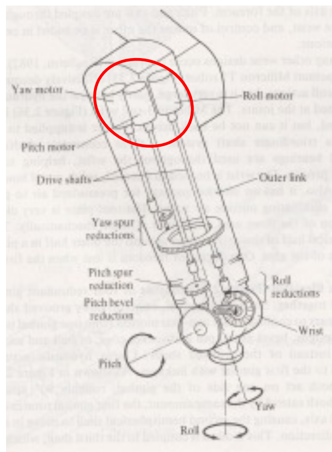
# Adding dynamic terms ...

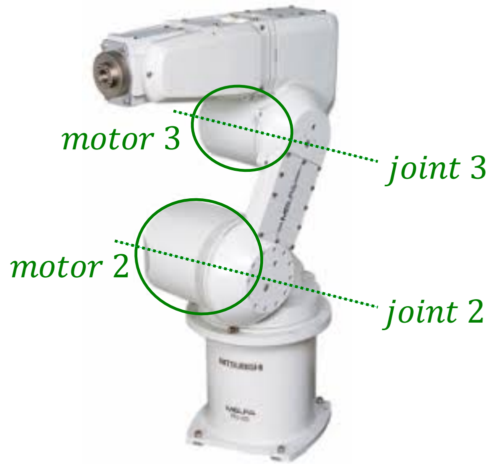
2) inclusion of electrical actuators (as additional rigid bodies)
- motor $i$ mounted on link $i-1$ (or before), with very few exceptions
- often with its spinning axis aligned with joint axis $i$
- (balanced) mass of motor included in total mass of carrying link
- (rotor) inertia has to be added to robot kinetic energy
- transmissions with reduction gears (often, large reduction ratios)
- in some cases, multiple motors cooperate in moving multiple links: use a transmission coupling matrix $\Gamma$ (with off-diagonal elements)

**Unimation PUMA family**



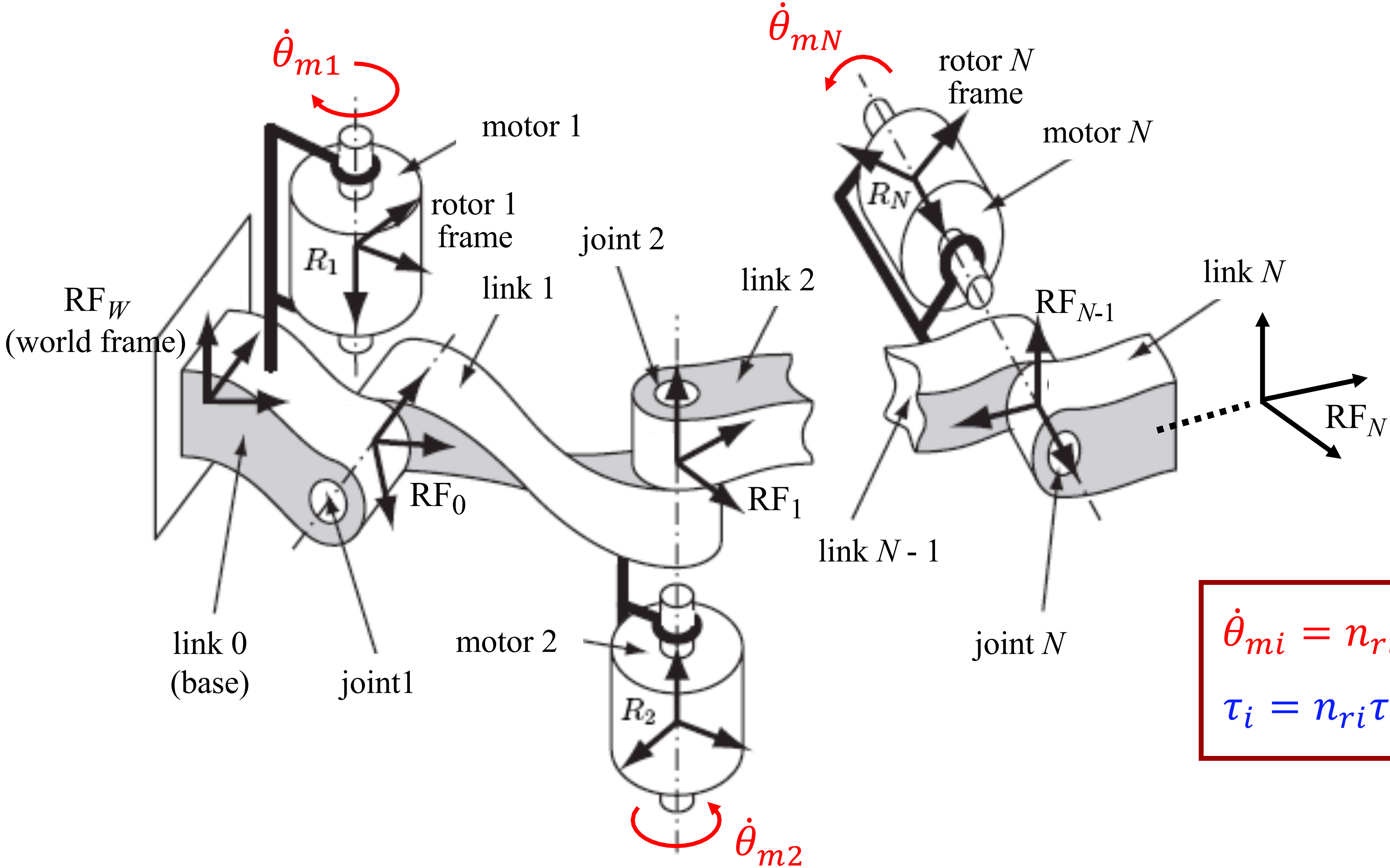




**Mitsubishi RV-3S**

# Placement of motors along the chain




$$\dot{\theta}_{mi} = n_{ri}\dot{\theta}_i$$

$$\tau_i = n_{ri}\tau_{mi}$$

# Resulting dynamic model

- simplifying assumption: in the rotational part of the kinetic energy, only the "spinning" rotor velocity is considered

$$T_{mi} = \frac{1}{2} I_{mi} \dot{\theta}_{mi}^2 = \frac{1}{2} I_{mi} n_{ri}^2 \dot{q}_i^2 = \frac{1}{2} B_{mi} \dot{q}_i^2 \qquad T_m = \sum_{i=1}^{N} T_{mi} = \frac{1}{2} \dot{q}^T B_m \dot{q}$$

diagonal, $> 0$

- including all added terms, the robot dynamics becomes

$$(M(q) + B_m)\ddot{q} + c(q, \dot{q}) + g(q) + F_V \dot{q} + F_C \,\mathrm{sgn}(\dot{q}) = \tau$$

moved to the left ...

constant → does NOT contribute to $c$

$F_V > 0, F_C > 0$ diagonal

motor torques (**after** reduction gears)

- scaling by the reduction gears, looking from the motor side

$$\left(I_m + \mathrm{diag}\left\{\frac{m_{ii}(q)}{n_{ri}^2}\right\}\right)\ddot{\theta}_m + \mathrm{diag}\left\{\frac{1}{n_{ri}}\right\}\left(\sum_{j=1}^{N} \bar{M}_j(q)\ddot{q}_j + f(q, \dot{q})\right) = \tau_m$$

diagonal

except the terms $m_{jj}$

motor torques (**before** reduction gears)

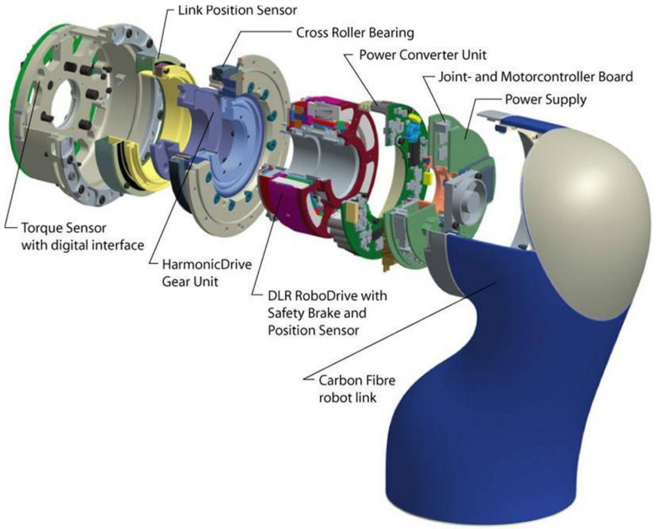
# Including joint elasticity

- in industrial robots, use of motion transmissions based on
  - belts
  - harmonic drives
  - long shafts

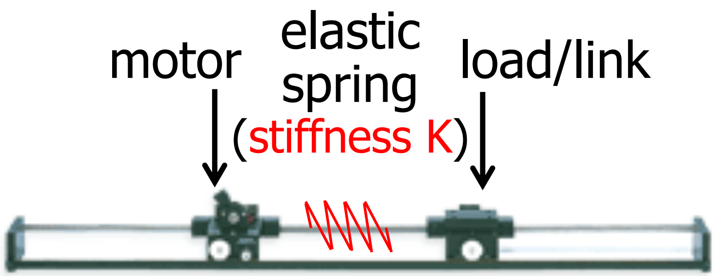
  introduces flexibility between actuating motors (input) and driven links (output)
- in research robots compliance in transmissions is introduced on purpose for safety (human collaboration) and/or energy efficiency
  - actuator relocation by means of (compliant) cables and pulleys
  - harmonic drives and lightweight (but rigid) link design
  - redundant (macro-mini or parallel) actuation, with elastic couplings
- in both cases, flexibility is modeled as concentrated at the joints
- in most cases, assuming small joint deformation (elastic domain)

# Robots with joint elasticity

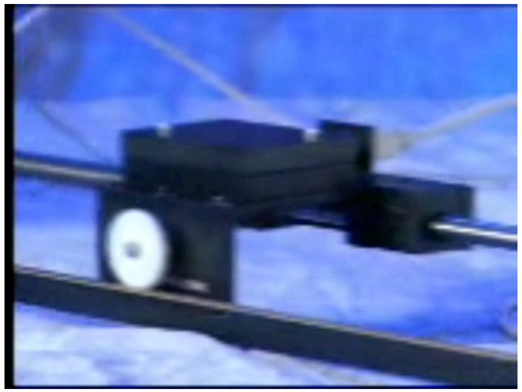
Dexter
with cable transmissions



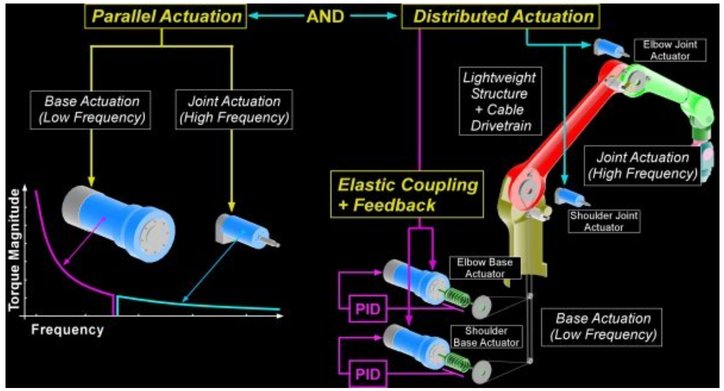

DLR LWR-III
with harmonic drives


Quanser Flexible Joint
(1-dof linear, educational)



video




Stanford DECMMA
with micro-macro actuation

# Dynamic model of robots with elastic joints

- introduce $2N$ generalized coordinates
  - $q = N$ link positions
  - $\theta = N$ motor positions (after reduction, $\theta_i = \theta_{mi}/n_{ri}$)
- add motor kinetic energy $T_m$ to that of the links $T_q = \frac{1}{2}\dot{q}^T M(q)\dot{q}$

$$T_{mi} = \frac{1}{2} I_{mi}\dot{\theta}_{mi}^2 = \frac{1}{2} I_{mi} n_{ri}^2 \dot{\theta}_i^2 = \frac{1}{2} B_{mi}\dot{\theta}_i^2 \qquad T_m = \sum_{i=1}^{N} T_{mi} = \frac{1}{2}\dot{\theta}^T B_m \dot{\theta}$$

diagonal, $> 0$

- add elastic potential energy $U_e$ to that due to gravity $U_g(q)$
  - $K =$ matrix of joint stiffness (diagonal, $> 0$)

$$U_{ei} = \frac{1}{2} K_i \left( q_i - \left( \frac{\theta_{mi}}{n_{ri}} \right) \right)^2 = \frac{1}{2} K_i (q_i - \theta_i)^2 \quad U_e = \sum_{i=1}^{N} U_{ei} = \frac{1}{2}(q-\theta)^T K (q-\theta)$$

- apply Euler-Lagrange equations w.r.t. $(q, \theta)$

$2N$ 2nd-order differential equations

$$\begin{cases} M(q)\ddot{q} + c(q,\dot{q}) + g(q) + K(q-\theta) = 0 \\ B_m \ddot{\theta} + K(\theta - q) = \tau \end{cases}$$

← no external torques performing work on $q$

# Use of the dynamic model

## inverse dynamics

- given a desired trajectory $q_d(t)$
  - twice differentiable ($\exists\ \ddot{q}_d(t)$)
  - possibly obtained from a task/Cartesian trajectory $r_d(t)$, by (differential) kinematic inversion

  the input torque needed to execute this motion (in free space) is

$$\tau_d = (M(q_d) + B_m)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) + F_V \dot{q}_d + F_C\,\mathrm{sgn}(\dot{q}_d)$$

- useful also for control (e.g., nominal feedforward)
- however, this way of performing the algebraic computation ($\forall t$) is not efficient when using the above Lagrangian approach
  - symbolic terms grow much longer, quite rapidly for larger $N$
  - in real time, numerical computation is based on Newton-Euler method

# State equations

## direct dynamics

**Lagrangian dynamic model**

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

$N$ differential $2^{nd}$ order equations

defining the vector of state variables as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \in \mathbb{R}^{2N}$

**state equations**

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1,x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u$$

$$= f(x) + G(x)u$$

$f(x)$: $2N \times 1$; $G(x)$: $2N \times N$

$2N$ differential $1^{st}$ order equations

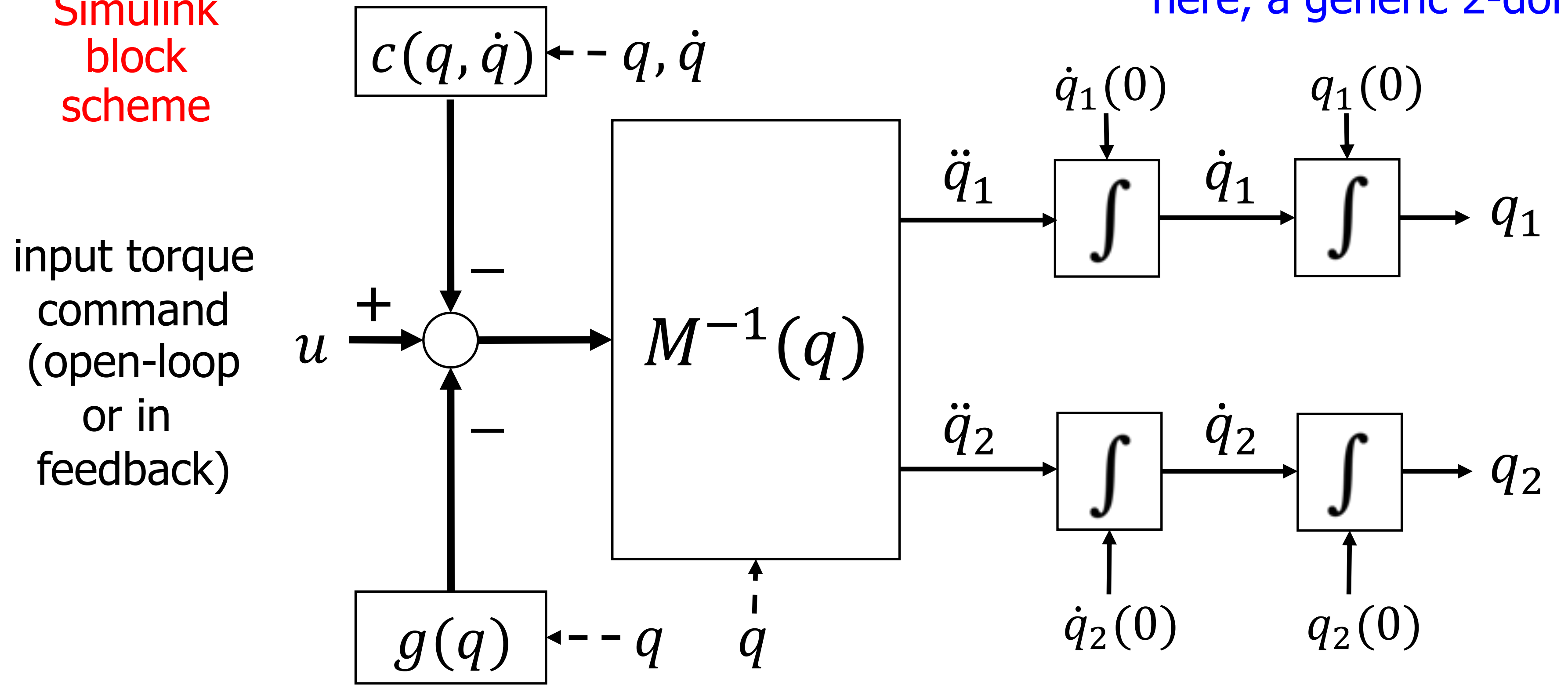
another choice... $\tilde{x} = \begin{pmatrix} q \\ M(q)\dot{q} \end{pmatrix}$ ⟵ generalized momentum

$\dot{\tilde{x}} = \ldots$ (do it as exercise)

# Dynamic simulation

Simulink block scheme

here, a generic 2-dof robot




including "inv(M)"

- initialization (dynamic coefficients and initial state)
- calls to (user-defined) Matlab functions for the evaluation of model terms
- choice of a numerical integration method (and of its parameters)

# Approximate linearization

- we can derive a linear dynamic model of the robot, which is valid locally around a given operative condition
  - useful for analysis, design, and gain tuning of linear (or, the linear part of) control laws
  - approximation by Taylor series expansion, up to the first order
  - linearization around a (constant) equilibrium state or along a (nominal, time-varying) equilibrium trajectory
  - usually, we work with (nonlinear) state equations; for mechanical systems, it is more convenient to directly use the 2nd order model
    - same result, but easier derivation

equilibrium state $(q, \dot{q}) = (q_e, 0)$ $[\, \ddot{q} = 0 \,]$ ➡ $g(q_e) = u_e$

equilibrium trajectory $(q, \dot{q}) = (q_d(t), \dot{q}_d(t))$ $[\, \ddot{q} = \ddot{q}_d(t) \,]$

➡ $M(q_d)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) = u_d$

# Linearization at an equilibrium state

- variations around an equilibrium state

$$q = q_e + \Delta q \quad \dot{q} = \dot{q}_e + \dot{\Delta q} = \dot{\Delta q} \quad \ddot{q} = \ddot{q}_e + \ddot{\Delta q} = \ddot{\Delta q} \quad u = u_e + \Delta u$$

- keeping into account the quadratic dependence of c terms on velocity (thus, neglected around the zero velocity)

$$M(q_e)\ddot{\Delta q} + \cancel{g(q_e)} + \underbrace{\left.\frac{\partial g}{\partial q}\right|_{q=q_e}}_{G(q_e)} \Delta q + \cancel{o\left(\|\Delta q\|, \|\dot{\Delta q}\|\right)} = \cancel{u_e} + \Delta u$$

infinitesimal terms
of second or higher order

- in state-space format, with $\Delta x = \begin{pmatrix} \Delta q \\ \dot{\Delta q} \end{pmatrix}$

$$\dot{\Delta x} = \begin{pmatrix} 0 & I \\ -M^{-1}(q_e)G(q_e) & 0 \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ M^{-1}(q_e) \end{pmatrix} \Delta u = A\,\Delta x + B\,\Delta u$$

# Linearization along a trajectory

- variations around an equilibrium trajectory

$$q = q_d + \Delta q \qquad \dot{q} = \dot{q}_d + \dot{\Delta q} \qquad \ddot{q} = \ddot{q}_d + \ddot{\Delta q} \qquad u = u_d + \Delta u$$

- developing to 1st order the terms in the dynamic model ...

$$M(q_d + \Delta q)\left(\ddot{q}_d + \ddot{\Delta q}\right) + c(q_d + \Delta q, \dot{q}_d + \dot{\Delta q}) + g(q_d + \Delta q) = u_d + \Delta u$$

$$M(q_d + \Delta q) \cong M(q_d) + \sum_{i=1}^{N} \left.\frac{\partial M_i}{\partial q}\right|_{q=q_d} e_i^T \Delta q$$

$e_i$: $i$-th row of the identity matrix

$$g(q_d + \Delta q) \cong g(q_d) + G(q_d)\Delta q$$

$$c\left(q_d + \Delta q, \dot{q}_d + \dot{\Delta q}\right) \cong c(q_d, \dot{q}_d) + \overbrace{\left.\frac{\partial c}{\partial q}\right|_{\substack{q=q_d \\ \dot{q}=\dot{q}_d}}}^{C_1(q_d, \dot{q}_d)} \Delta q + \underbrace{\left.\frac{\partial c}{\partial \dot{q}}\right|_{\substack{q=q_d \\ \dot{q}=\dot{q}_d}}}_{C_2(q_d, \dot{q}_d)} \dot{\Delta q}$$

# Linearization along a trajectory (cont)

- after simplifications …

$$M(q_d)\Delta\ddot{q} + C_2(q_d, \dot{q}_d)\Delta\dot{q} + D(q_d, \dot{q}_d, \ddot{q}_d)\Delta q = \Delta u$$

with

$$D(q_d, \dot{q}_d, \ddot{q}_d) = G(q_d) + C_1(q_d, \dot{q}_d) + \sum_{i=1}^{N} \left.\frac{\partial M_i}{\partial q}\right|_{q=q_d} \ddot{q}_d e_i^T$$

- in state-space format

$$\Delta\dot{x} = \begin{pmatrix} 0 & I \\ -M^{-1}(q_d)D(q_d, \dot{q}_d, \ddot{q}_d) & -M^{-1}(q_d)C_2(q_d, \dot{q}_d) \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ M^{-1}(q_d) \end{pmatrix} \Delta u = A(t)\,\Delta x + B(t)\,\Delta u$$

a linear, but time-varying system!!

# Coordinate transformation

$q \in \mathbb{R}^N$ $$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = M(q)\ddot{q} + n(q,\dot{q}) = u_q$$ (1)

if we wish/need to use a new set of generalized coordinates $p$

$p \in \mathbb{R}^N$

$$p = f(q) \quad \Rightarrow \quad q = f^{-1}(p)$$

$$\dot{p} = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q} \quad \Rightarrow \quad \dot{q} = J^{-1}(q)\dot{p} \qquad u_q = J^T(q)u_p$$

$$\ddot{p} = J(q)\ddot{q} + \dot{J}(q)\dot{q} \quad \Rightarrow \quad \ddot{q} = J^{-1}(q)\left(\ddot{p} - \dot{J}(q)J^{-1}(q)\dot{p}\right)$$

$\Rightarrow$ (1)

$$M(q)J^{-1}(q)\ddot{p} - M(q)J^{-1}(q)\dot{J}(q)J^{-1}(q)\dot{p} + n(q,\dot{q}) = J^T(q)u_p$$

$J^{-T}(q)\,\cdot$ pre-multiplying the whole equation...

# Robot dynamic model
## after coordinate transformation

$$J^{-T}(q)M(q)J^{-1}(q)\ddot{p} + J^{-T}(q)\big(n(q,\dot{q}) - M(q)J^{-1}(q)\dot{J}(q)J^{-1}(q)\dot{p}\big) = u_p$$

$q \rightarrow p$ for actual computation, these inner substitutions are not necessary $(q,\dot{q}) \rightarrow (p,\dot{p})$

non-conservative generalized forces performing work on $p$

$$M_p(p)\ddot{p} + c_p(p,\dot{p}) + g_p(p) = u_p$$

$$M_p = J^{-T}MJ^{-1}$$ symmetric, positive definite (out of singularities)

$$g_p = J^{-T}g$$

$$c_p = J^{-T}\big(c - MJ^{-1}\dot{J}\,J^{-1}\dot{p}\big) = J^{-T}c - M_p\dot{J}\,J^{-1}\dot{p}$$ quadratic dependence on $\dot{p}$

$$c_p(p,\dot{p}) = S_p(p,\dot{p})\,\dot{p}$$ $$\dot{M}_p - 2S_p$$ skew-symmetric

when $p$ = E-E pose, this is the robot dynamic model in Cartesian coordinates

*Q: What if the robot is redundant with respect to the Cartesian task?*

# Dynamic scaling of trajectories

## uniform time scaling of motion

- given a smooth original trajectory $q_d(t)$ of motion for $t \in [0, T]$
  - suppose to rescale time as $t \to r(t)$ (a strictly *increasing* function of $t$)
  - in the new time scale, the scaled trajectory $q_s(r)$ satisfies

$$q_d(t) = q_s(r(t)) \;\Rightarrow\; \dot{q}_d(t) = \frac{dq_d}{dt} = \frac{dq_s}{dr}\frac{dr}{dt} = q_s' \, \dot{r}$$

**same** path executed
(at different instants of time)

$$\Downarrow$$

$$\ddot{q}_d(t) = \frac{d\dot{q}_d}{dt} = \left(\frac{dq_s'}{dr}\frac{dr}{dt}\right)\dot{r} + q_s'\ddot{r} = q_s''\dot{r}^2 + q_s'\ddot{r}$$

  - uniform scaling of the trajectory occurs when $r(t) = kt$

$$\dot{q}_d(t) = k q_s'(kt) \qquad \ddot{q}_d(t) = k^2 q_s''(kt)$$

Q: what is the new input torque needed to execute the scaled trajectory?
(suppose dissipative terms can be neglected)

# Dynamic scaling of trajectories

## inverse dynamics under uniform time scaling

- the new torque could be recomputed through the inverse dynamics, for every $r = kt \in [0, T'] = [0, kT]$ along the scaled trajectory, as

$$\tau_s(kt) = M(q_s)q_s'' + c(q_s, q_s') + g(q_s)$$

- however, being the dynamic model linear in the acceleration and quadratic in the velocity, it is

$$\tau_d(t) = M(q_d)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) = M(q_s)k^2 q_s'' + c(q_s, kq_s') + g(q_s)$$
$$= k^2\big(M(q_s)q_s'' + c(q_s, q_s')\big) + g(q_s) = k^2\big(\tau_s(kt) - g(q_s)\big) + g(q_s)$$

- thus, saving separately the total torque $\tau_d(t)$ and gravity torque $g_d(t)$ in the inverse dynamics computation along the original trajectory, the new input torque is obtained directly as

$$\tau_s(kt) = \frac{1}{k^2}\big(\tau_d(t) - g(q_d(t))\big) + g(q_d(t))$$

$k > 1$: slow down $\Rightarrow$ reduce torque

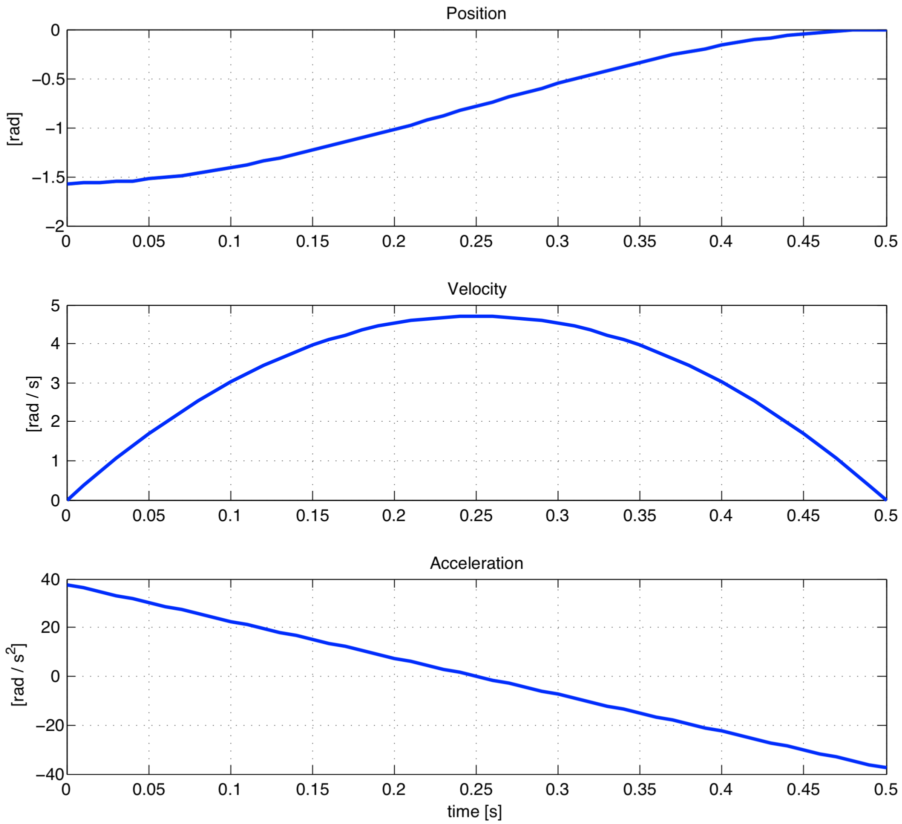
$k < 1$: speed up $\Rightarrow$ increase torque

gravity term (only position-dependent): does NOT scale!

# Dynamic scaling of trajectories

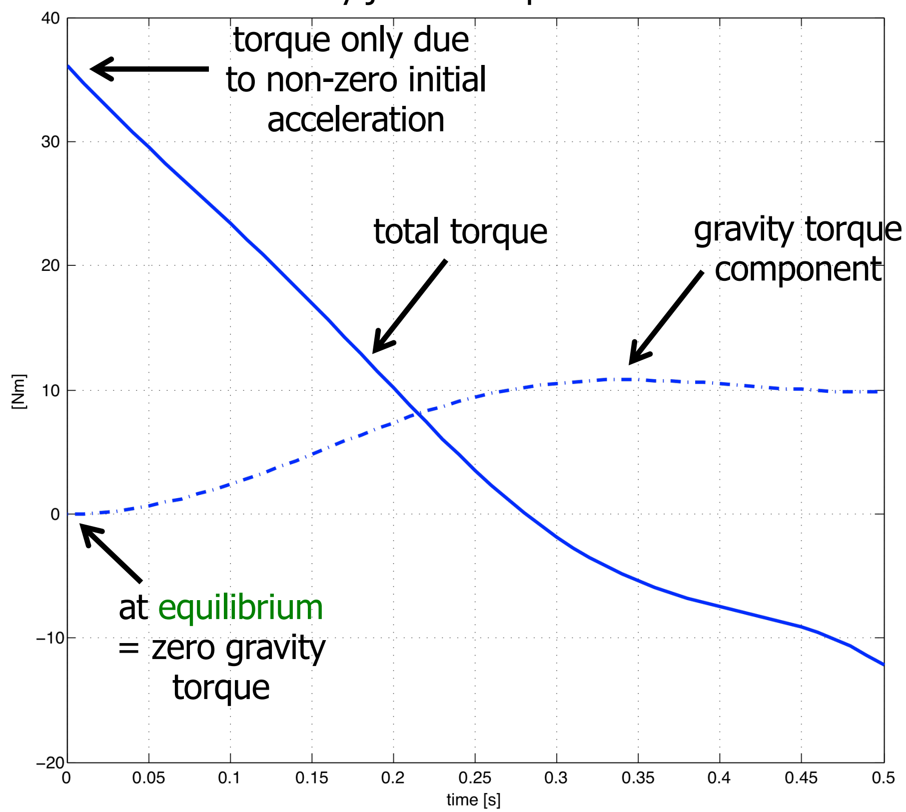
## numerical example

- rest-to-rest motion with cubic polynomials for planar 2R robot under gravity (from downward equilibrium to horizontal link 1 & upward vertical link 2)
- original trajectory lasts $T = 0.5$ s (but maybe violates the torque limit at joint 1)



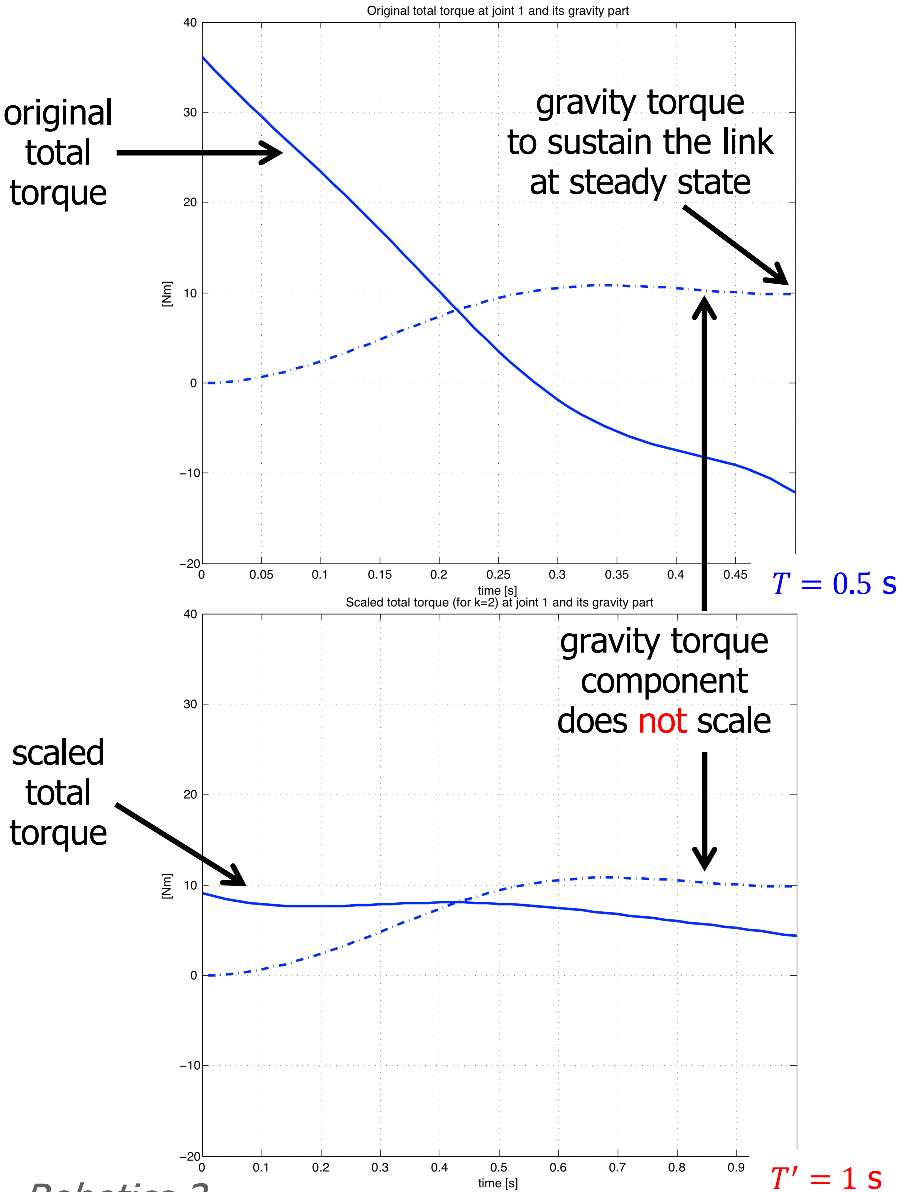

for both joints

only joint 1 torque is shown

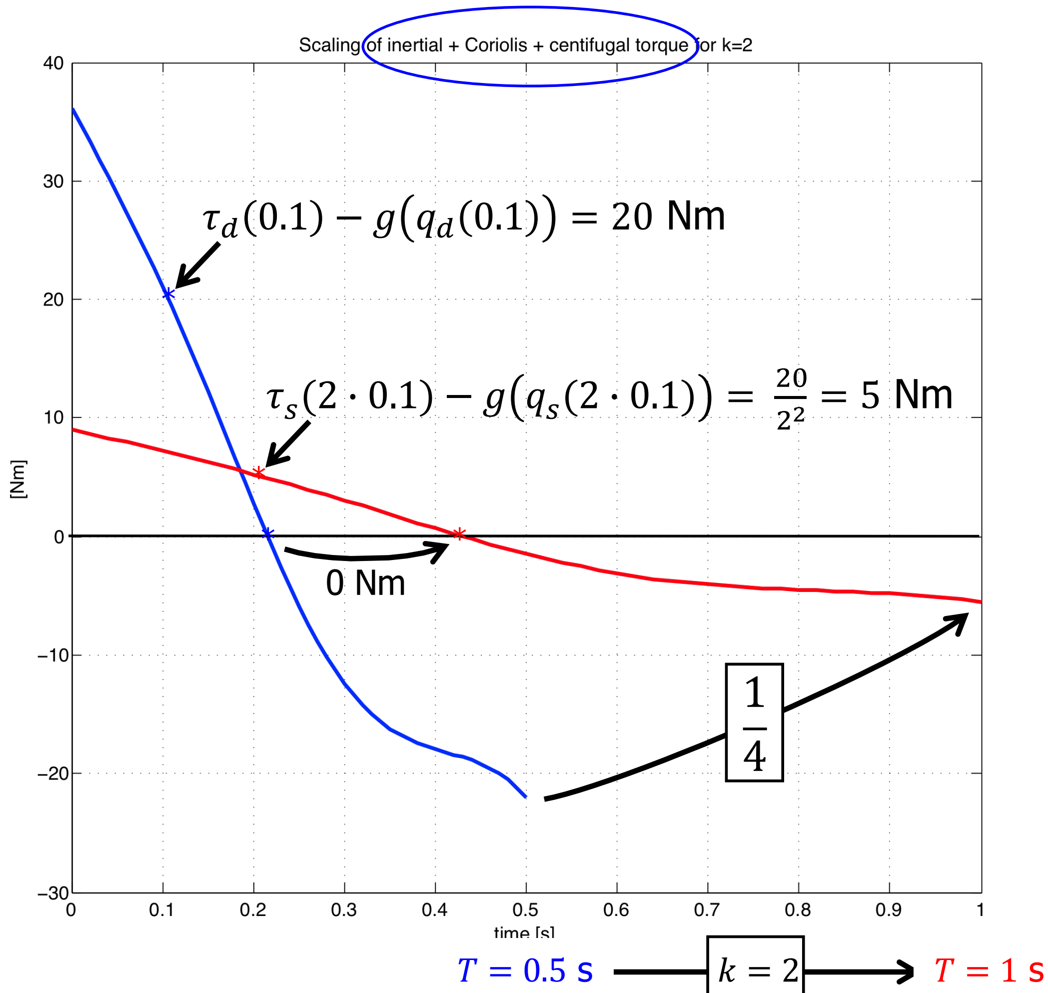


# Dynamic scaling of trajectories

## numerical example




- scaling with $k = 2$ (slower) $\rightarrow T' = 1$ s



# Optimal point-to-point robot motion
## considering the dynamic model

- given the initial and final robot configurations (at rest) and actuator torque bounds, find
  - the minimum-time $T_{min}$ motion
  - the (global/integral) minimum-energy $E_{min}$ motion

  and the associated command torques needed to execute them
- a complex nonlinear optimization problem solved numerically

video


$T_{min}$= 1.32 s, E = 306

video


T = 1.60 s, $E_{min}$ = 6.14