Reasoning about Knowledge and Strategies

Bastien Maubert and Aniello Murano
Distributed synthesis

**Input:** A concurrent game structure and a formula $\varphi \in \text{LTL}$

**Output:** A distributed strategy to enforce $\varphi$

$p, q$ are atomic propositions

$\circ, \bullet$ are actions

strategies $\sigma : \text{Histories} \rightarrow \text{Actions}$

indistinguishability relations $\sim_\alpha$ on states
Imperfect information

1. Strategies must be consistent with players’ information

Constraint on strategies:

If \( h \sim_a h' \), then \( \sigma_a(h) = \sigma_a(h') \).

2. Makes epistemic reasoning meaningful and useful

Example: opacity

A system is \textit{opaque} for property \( P \) if a spy never knows whether the current execution is in \( P \).

Classic definition:

\[
\forall h, \exists h' \text{ s.t. } h \sim_{\text{spy}} h' \text{ and } h' \notin P
\]

With epistemic temporal logic:

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G \neg K_{\text{spy}} P
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Imperfect information

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With epistemic temporal logic:

$$\exists \sigma(c, \sigma) \Box \neg K_{\text{spy}} P$$
Semantics of knowledge when reasoning about strategies

Yellow subtree: controller’s strategy
Blue arrows: spy’s indistinguishability relation

Two possible semantics:
- Uninformed semantics: players ignore each other’s strategy
  → $K_{\text{spy}} P$ does not hold
- Informed semantics: players know each other’s strategy
  → $K_{\text{spy}} P$ holds
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Overview

Peterson and Reif (1979), Pnueli and Rosner (1990)
Distributed synthesis for reachability objective is undecidable.

Peterson and Reif (1979), Pnueli and Rosner (1990)
Decidable for LTL objectives when information is hierarchical.

For epistemic temporal objectives,
Distributed synthesis with hierarchical information is
- Undecidable for informed semantics
  [van der Meyden and Wilke, 2005]
- Decidable for uninformed semantics
  [Puchala, 2010]
SL with imperfect information and knowledge

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<th>(Chatterjee et al. 2010, Mogavero et al. 2014)</th>
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<td>⊨ ∃σ ϕ</td>
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### SL with imperfect information and knowledge

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Distributed synthesis for opacity:

$$\exists^1 \sigma_1 (c_1, \sigma_1) \exists^2 \sigma_2 (c_2, \sigma_2) G\neg K_s P$$
SL with imperfect information and knowledge

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Existence of Nash equilibria:

$$\exists^o \sigma_1 (a_1, \sigma_1) \exists^o \sigma_2 (a_2, \sigma_2) \left( \bigwedge_i \exists^{o_i} \sigma' (a_i, \sigma') \text{ Win}_i \rightarrow \text{ Win}_i \right)$$
Main result

Hierarchical instances

An ESL formula $\Phi$ is **hierarchical** if:

- innermost strategies observe better than outermost ones
- epistemic subformulas do not talk about current strategies

Considering the uninformed semantics of knowledge:

**Theorem**

Model-checking hierarchical instances of ESL is decidable.
Corollaries:

On systems with **hierarchical information**, for **epistemic temporal** specifications,

**We can solve**

- distributed synthesis,
- module checking,
- existence of Nash equilibria,
- rational synthesis,
- ...
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Thank you!