Control Systems

Bode plots

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Outline

- Bode’s canonical form for the frequency response
- Magnitude and phase in the complex plane
- The decibels (dB)
- Logarithmic scale for the abscissa
- Bode’s plots for the different contributions
Frequency response

The **steady-state response** of an asymptotically stable system \( P(s) \) to a sinusoidal input \( u(t) = \sin \bar{\omega} t \) is given by

\[
y_{ss}(t) = |P(j \bar{\omega})| \sin(\bar{\omega} t + \angle P(j \bar{\omega}))
\]

\( P(j \omega) \) is the restriction of the transfer function to the imaginary axis \( P(j \omega)|_{s=j\omega} \)

**Frequency response** \( P(j \omega) \) \[
\begin{align*}
\{ & |P(j \omega)| & \text{gain curve} \\
& \angle P(j \omega) & \text{phase curve}
\end{align*}
\]

for \( \omega \in \mathbb{R}^+ = [0, +\infty) \)
Bode canonical form

pole/zero representation of the transfer function

\[ F(s) = K' \frac{1}{s^m} \frac{\prod_{k} (s - z_k) \prod_{\ell} (s^2 + 2\zeta_\ell \omega_{n\ell} s + \omega_{n\ell}^2)}{\prod_{i} (s - p_i) \prod_{z} (s^2 + 2\zeta_z \omega_{nz} s + \omega_{nz}^2)} \]

with \( m \) such that

- \( m = 0 \) if no pole or zero in \( s = 0 \)
- \( m < 0 \) if \( m \) zeros in \( s = 0 \)
- \( m > 0 \) if \( m \) poles in \( s = 0 \)

remarks

- numerator and denominator are by hypothesis coprime
- denominator is monic
- \( K' \) is not the system gain
Bode canonical form

- The terms \((s - z_k)\) and \((s - p_i)\) are relative to
  - Real zeros (in \(s = z_k\))
  - Real poles (in \(s = p_i\))

- The terms \((s^2 + 2\zeta_\ell \omega_n s + \omega_{n\ell}^2)\) and \((s^2 + 2\zeta_z \omega_{nz} s + \omega_{nz}^2)\) are relative to
  - Complex conjugate zeros (in \(s = \alpha_\ell \pm j\beta_\ell\))
  - Complex conjugate poles (in \(s = \alpha_z \pm j\beta_z\))

With
  - Natural frequency \(\omega_{n*} = \sqrt{\alpha_*^2 + \beta_*^2}\)
  - Damping coefficient \(\zeta_* = -\alpha_*/\omega_{n*} = -\alpha_*/\sqrt{\alpha_*^2 + \beta_*^2}\)
Bode canonical form

factoring out the constant terms

\[ s - z_k = -z_k(1 - 1/z_k s) = -z_k(1 + \tau_k s) \quad \text{with} \quad \tau_k = -1/z_k \]
\[ s - p_i = -p_i(1 - 1/p_i s) = -p_i(1 + \tau_i s) \quad \text{with} \quad \tau_i = -1/p_i \]

with \( \tau_i \) and \( \tau_k \) being time constants

\[
F(s) = K' \frac{1}{s^m} \prod_k (-z_k) \prod_{\ell} (\omega_{n\ell}^2) \prod_k (1 + \tau_k s) \prod_{\ell} (1 + 2\zeta_{\ell}/\omega_{n\ell} s + s^2/\omega_{n\ell}^2) \\
\prod_i (-p_i) \prod_z (\omega_{nz}^2) \prod_i (1 + \tau_i s) \prod_z (1 + 2\zeta_z/\omega_{nz} s + s^2/\omega_{nz}^2)
\]

defining \( K = K' \frac{\prod_k (-z_k) \prod_{\ell} (\omega_{n\ell}^2)}{\prod_i (-p_i) \prod_z (\omega_{nz}^2)} \) \( K = [s^m F(s)]_{s=0} \quad \text{for any} \quad m \geq 0 \)

how to compute \( K \)

\[
F(s) = K \frac{1}{s^m} \prod_k (1 + \tau_k s) \prod_{\ell} (1 + 2\zeta_{\ell}/\omega_{n\ell} s + s^2/\omega_{n\ell}^2) \\
\prod_i (1 + \tau_i s) \prod_z (1 + 2\zeta_z/\omega_{nz} s + s^2/\omega_{nz}^2)
\]
Gains

\[ F(s) = K \frac{1}{s^m} \frac{\prod_k (1 + \tau_k s) \prod_{\ell} (1 + 2\zeta_{\ell}/\omega_{n\ell} s + s^2/\omega_{n\ell}^2)}{\prod_i (1 + \tau_i s) \prod_z (1 + 2\zeta_z/\omega_{nz} s + s^2/\omega_{nz}^2)} = 1 \text{ for } s = 0 \]

generalized gain

\[ K = [s^m F(s)]_{s=0} \text{ for any } m \geq 0 \]

Note that

- for a system with no poles (i.e. \( m \) negative or zero) we have defined as dc-gain (or static gain)

\[ K_s = F(s) \bigg|_{s=0} = F(0) \]

if \( m < 0 \) (zeros in \( s = 0 \)) we have \( F(0) = 0 \)

- static and generalized gain coincide only when \( m = 0 \)

\[ K = K_s \iff m = 0 \]

- for an asymptotically stable system, the step response tends to the static gain \( K_s = F(0) \)
Bode canonical form

Examples

\[ F(s) = \frac{s - 1}{2s^2 + 6s + 4} = \frac{s - 1}{2(s + 1)(s + 2)} = -\frac{1}{2} \frac{1 - s}{(1 + s)(1 + s/2)} \]

\[ K = K_s = -\frac{1}{2} \]

\[ F(s) = \frac{s(s - 1)}{2(s + 1)^2(s + 2)} = -\frac{1}{2} \frac{s(1 - s)}{(1 + s)^2(1 + s/2)} \]

\[ K = -\frac{1}{2} \quad \# K_s = 0 \]

\[ F(s) = \frac{s - 1}{2s(s + 1)(s + 2)} = -\frac{1}{2} \frac{1 - s}{s(1 + s)(1 + s/2)} \]

\[ K = -\frac{1}{2} \quad \# K_s \]
Bode canonical form

frequency response

\[ F(j\omega) = K \frac{1}{(j\omega)^m} \prod_{k} \frac{(1 + j\omega \tau_k) \prod_{\ell}(1 + 2\zeta_{\ell}j\omega/\omega_{n\ell} + (j\omega)^2/\omega_{n\ell}^2)}{\prod_{i}(1 + j\omega \tau_i) \prod_{z}(1 + 2\zeta_{z}j\omega/\omega_{n_z} + (j\omega)^2/\omega_{n_z}^2)} \]

has 4 elementary factors

1. constant \( K \) (generalized gain)
2. monomial \( j\omega \) (zero or pole in \( s = 0 \))
3. binomial \( 1 + j\omega \tau \) (non-zero real zero or pole)
4. trinomial \( 1 + 2\zeta(j\omega)/\omega_n + (j\omega)^2/\omega_n^2 \) (complex conjugate pairs of zeros or poles)

so first check which kind of root you have and then factor it out
Bode diagrams

for any real value of the angular frequency $\omega$ the frequency response $F(j\omega)$ is a complex number

$|F(j\omega)|$  magnitude of the frequency response as a function of the angular frequency $\omega$

$\angle F(j\omega)$  angle or phase of the frequency response as a function of the angular frequency $\omega$

$F(j\omega) = \text{Re}[F(j\omega)] + j\text{Im}[F(j\omega)]$

$F(j\omega) = |F(j\omega)|e^{j\angle F(j\omega)}$

$|F(j\omega)| = \sqrt{\text{Re}[F(j\omega)]^2 + \text{Im}[F(j\omega)]^2}$  

$\angle F(j\omega) = \text{atan2}(\text{Im}[F(j\omega)], \text{Re}[F(j\omega)])$
Phase

The phase of a product is the sum of the phases:

\[ \text{Phase}[F \cdot G] = \text{Phase}[F] + \text{Phase}[G] \]

The phase of a ratio is the difference of the phases:

\[ \text{Phase} \left[ \frac{F}{G} \right] = \text{Phase}[F] - \text{Phase}[G] \]

Very useful since we can find the contribution to the phase of each term and then just do an algebraic sum.
Phase

\[ \text{atan2}(\beta, \alpha) = \begin{cases} 
\arctan\left(\frac{\beta}{\alpha}\right) & \text{if } \alpha > 0 \quad \text{(I & IV quadrant)} \\
\arctan\left(\frac{\beta}{\alpha}\right) + \pi & \text{if } \beta \geq 0 \text{ and } \alpha < 0 \quad \text{(II quadrant)} \\
\arctan\left(\frac{\beta}{\alpha}\right) - \pi & \text{if } \beta < 0 \text{ and } \alpha < 0 \quad \text{(III quadrant)} \\
\frac{\pi}{2} \text{sign}(\beta) & \text{if } \alpha = 0 \text{ and } \beta \neq 0 \\
\text{undefined} & \text{if } \alpha = 0 \text{ and } \beta = 0
\end{cases} \]

principal argument takes on values in \((-\pi, \pi]\) and is implemented by the function with two arguments atan2.

\[ P = \alpha + j\beta \]

\[ \text{atan2}(\beta, \alpha) = \begin{cases} 
\arctan\left(\frac{\beta}{\alpha}\right) & \text{if } \alpha > 0 \quad \text{(I & IV quadrant)} \\
\arctan\left(\frac{\beta}{\alpha}\right) + \pi & \text{if } \beta \geq 0 \text{ and } \alpha < 0 \quad \text{(II quadrant)} \\
\arctan\left(\frac{\beta}{\alpha}\right) - \pi & \text{if } \beta < 0 \text{ and } \alpha < 0 \quad \text{(III quadrant)} \\
\frac{\pi}{2} \text{sign}(\beta) & \text{if } \alpha = 0 \text{ and } \beta \neq 0 \\
\text{undefined} & \text{if } \alpha = 0 \text{ and } \beta = 0
\end{cases} \]
Magnitude

in order to have the same property we need to go through some logarithmic function

\[ |F(j\omega)|_{\text{dB}} = 20 \log_{10} |F(j\omega)| \]

decibels (dB)

\[ |F \cdot G|_{\text{dB}} = |F|_{\text{dB}} + |G|_{\text{dB}} \]

same nice properties as phase

\[ \left| \frac{1}{F} \right|_{\text{dB}} = -|F|_{\text{dB}} \]

\[ |1|_{\text{dB}} = 0 \text{ dB} \]
\[ |0.1|_{\text{dB}} = -20 \text{ dB} \]
\[ |\sqrt{2}|_{\text{dB}} \approx 3 \text{ dB} \]
\[ |10|_{\text{dB}} = 20 \text{ dB} \]
\[ |100|_{\text{dB}} = 40 \text{ dB} \]
\[ |F|_{\text{dB}} \rightarrow +\infty \text{ if } |F| \rightarrow \infty \]
\[ |F|_{\text{dB}} \rightarrow -\infty \text{ if } |F| \searrow 0 \]

Lanari: CS - Bode plots

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Logarithmic scale

we use a logarithmic ($\log_{10}$) scale for the abscissa (angular frequency $\omega$)

A decade corresponds to multiplication by 10

$log_{10}(\omega)$ becomes a straight line if $\omega$ is in a logarithmic scale

very useful when we add different contributions
Logarithmic scale

advantages

• quantities can vary in large range (both $\omega$ and magnitude)
• easy to build the magnitude plot in dB of a frequency response given in its Bode canonical form from the magnitudes of the single terms
• easy to represent series of systems

same data

different scales

for abscissa and ordinates

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**Bode diagrams**

\[ |F(j\omega)|_{dB} \]  

magnitude in dB of the frequency response as a function of the angular frequency \( \omega \) with logarithmic scale for \( \omega \)

\[ \angle F(j\omega) \]  

angle or phase of the frequency response as a function of the angular frequency \( \omega \) with logarithmic scale for \( \omega \)

we need to find the magnitude (in dB) and phase for the 4 elementary factors

1. constant \( K \) (generalized gain)
2. monomial \( j\omega \) (zero or pole in \( s = 0 \))
3. binomial \( 1 + j\omega \tau \) (non-zero real zero or pole)
4. trinomial \( 1 + 2\zeta(j\omega)/\omega_n + (j\omega)^2/\omega_n^2 \) (complex conjugate pairs of zeros or poles)

for \( \omega \in \mathbb{R}^+ = [0, +\infty) \)
**Constant**

\[ K \]

\[ K_3 = -\sqrt{10} \]

\[ K_1 = \sqrt{10} \]

\[ K_2 = 0.5 \]

\[ |\sqrt{10}|_{dB} = 10 \text{ dB} \]

\[ |-\sqrt{10}|_{dB} = 10 \text{ dB} \]

\[ |0.5|_{dB} \approx -6 \text{ dB} \]

\[ \angle \sqrt{10} = 0^\circ \]

\[ \angle -\sqrt{10} = -180^\circ = -\pi \]

\[ \angle 0.5 = 0^\circ \]

\[ \angle K \]

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**Monomial - Numerator**

\[ j\omega \]

\[ |j\omega|_{dB} = 20 \log_{10} \omega \]

\[ |j\omega|_{dB} = 20x \]

*log scale*  
magnitude

*Phase*  

\[ 90^\circ \]

\[ \omega |_{dB} = 20 \log_{10} \omega \]

\[ 20 \text{ dB/dec} \]

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from properties of log and phase

**Magnitude**

-20 dB/dec

**Phase**

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**Binomial - Numerator**

\[ 1 + j\omega \tau \]

**Magnitude**

\[ |1 + j\omega \tau|_{dB} = 20 \log_{10} \sqrt{1 + \omega^2 \tau^2} \]

approximation wrt the **cutoff frequency** \(1/|\tau|\) (corner frequency)

\[ \sqrt{1 + \omega^2 \tau^2} \approx \begin{cases} 
1 & \text{if } \omega \ll 1/|\tau| \\
\sqrt{\omega^2 \tau^2} & \text{if } \omega \gg 1/|\tau| 
\end{cases} \]

and therefore

\[ |1 + j\omega \tau|_{dB} \approx \begin{cases} 
0 \text{ dB} & \text{if } \omega \ll 1/|\tau| \\
20 \log_{10} \omega + 20 \log_{10} |\tau| & \text{if } \omega \gg 1/|\tau| 
\end{cases} \]

at the cutoff frequency \(\omega^* = 1/|\tau|\)

\[ |1 + j\tau/|\tau||_{dB} = 20 \log_{10} \sqrt{2} \approx 3 \text{ dB} \]

two half-lines approximation: 0 dB until the cutoff frequency, + 20 dB/decade after
phase depends on the sign of $\tau$

see how the phase changes as $\omega$ increases
Binomial - Numerator

\[ 1 + j\omega \tau \]

phase depends on the sign of \( \tau \)

case \( \tau > 0 \)

\[ \angle (1 + j\omega \tau) \approx \begin{cases} 
0 & \text{if } \omega \ll 1/|\tau| \\
\pi/2 & \text{if } \omega \gg 1/|\tau| \text{ and } \tau > 0
\end{cases} \]

case \( \tau < 0 \)

\[ \angle (1 + j\omega \tau) \approx \begin{cases} 
0 & \text{if } \omega \ll 1/|\tau| \\
-\pi/2 & \text{if } \omega \gg 1/|\tau| \text{ and } \tau < 0
\end{cases} \]

the two asymptotes are connected by a segment starting a decade before \((0.1/|\tau|)\) the cutoff frequency and ending a decade after \((10/|\tau|)\). The approximation is a broken line.

\[ \omega^* = 1/|\tau| \]

at the cutoff frequency

\[ \angle (1 + j\tau/|\tau|) = \begin{cases} 
\pi/4 & \text{if } \tau > 0 \\
-\pi/4 & \text{if } \tau < 0
\end{cases} \]
Binomial - numerator

\[ 1 + j\omega \tau \]

magnitude

\( \tau > 0 \)

phase

\( \tau < 0 \)

phase
**Binomial - denominator**

\[ \frac{1}{1 + j\omega \tau} \]

**magnitude**

\[ \tau > 0 \]

**phase**

\[ \tau < 0 \]

_Lanari: CS - Bode plots_
Trinomial

magnitude

\[ |1 + 2 \frac{\zeta}{\omega_n} (j \omega) + \frac{(j \omega)^2}{\omega_n^2}| = |1 - \frac{\omega^2}{\omega_n^2} + j2\zeta \frac{\omega}{\omega_n}| \]

\[ = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(4\zeta^2 \frac{\omega^2}{\omega_n^2}\right)} \]

approximation wrt \( \omega_n \)

\[ |\text{TRINOMIAL}| \approx \begin{cases} 
1 & \text{if } \omega \ll \omega_n \\
\sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2} = \frac{\omega^2}{\omega_n^2} & \text{if } \omega \gg \omega_n 
\end{cases} \]

\[ |\text{TRINOMIAL}|_{dB} \approx \begin{cases} 
0 \text{ dB} & \text{if } \omega \ll \omega_n \\
40 \log_{10} \omega - 20 \log_{10} \omega_n^2 & \text{if } \omega \gg \omega_n 
\end{cases} \]
Trinomial

in $\omega = \omega_n$ the magnitude $|\text{TRINOMIAL}|$ is equal to $2 |\zeta|$

| $|\zeta|$ | 0   | 0.5 | $1/\sqrt{2} \approx 0.707$ | 1   |
|-------|-----|-----|--------------------------|-----|
| $|\text{TRIN}|_{dB}$ in $\omega_n$ | $-\infty$ | 0 dB | 3 dB | 6 dB |

large variation of the magnitude in $\omega = \omega_n$ depending upon the value of the damping coefficient $\zeta$

no approximation around the natural frequency $\omega_n$
Trinomial

How does a generic complex root varies in the plane as a function of $\omega$

\[ \angle \left( 1 + 2 \frac{\zeta}{\omega_n} (j\omega) + \frac{(j\omega)^2}{\omega_n^2} \right) = \begin{cases} 
0 & \text{if } \omega \ll \omega_n \\
\pi & \text{if } \omega \gg \omega_n \text{ and } \zeta \geq 0 \\
-\pi & \text{if } \omega \gg \omega_n \text{ and } \zeta < 0
\end{cases} \]

transition between $-\pi$ and $\pi$ is symmetric wrt $\omega_n$ and becomes more abrupt as $|\zeta|$ becomes smaller. When $\zeta = 0$ the phase has a discontinuity in $\omega_n$. 

*Phase*

*Monday, November 3, 2014*
Trinomial - numerator

magnitude

phase

$\zeta \geq 0$

phase

$\zeta < 0$
Trinomial - denominator

magnitude

phase

$\zeta \geq 0$

phase

$\zeta < 0$
When $|\zeta| = 1$ the trinomial reduces to a product of identical binomials (real roots)

\[
\text{roots} = \begin{cases} 
-\omega_n & \text{if } \zeta = 1 \\
\omega_n & \text{if } \zeta = -1
\end{cases}
\]

\[
\left(1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}\right)_{\zeta=\pm1} = \left(1 \pm \frac{s}{\omega_n}\right)^2
\]

and therefore the magnitude and phase coincides with that of a double binomial with corner frequency

\[
\frac{1}{|\tau|} = \omega_n
\]

that is in $\omega = \omega_n$ when $|\zeta| = 1$

\[
2 \times (3 \text{ dB}) = 6 \text{ dB} \quad \text{(numerator)}
\]
\[
2 \times (-3 \text{ dB}) = -6 \text{ dB} \quad \text{(denominator)}
\]

example: MSD system with critical value for the damping ($\mu^2 = 4 km$)
if $|\zeta| < 1/\sqrt{2} \approx 0.707$ the magnitude of a trinomial factor at the denominator has a peak

$$|F(j\omega_r)| = \frac{1}{2|\zeta|\sqrt{1 - \zeta^2}}$$

at the resonance frequency

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

(similarly for the anti-resonance peak)