Outline

- a general feedback control scheme
- typical specifications
- the 3 sensitivity functions
- constraints in the specification definitions
- steady-state requirements w.r.t. references
- system type
- steady-state requirements w.r.t. disturbances
- effects of the introduction of integrators
- transient characterization in the frequency domain
- closed-loop to open-loop transient specifications
general **feedback control scheme**

all this is “built” on top of the system to be controlled

→ this choice will influence therefore the type of measurement noise present

→ we concentrate on the choice of the controller
general **feedback control scheme**

![Diagram of feedback control scheme](image)

- \( r(t) \) reference signal
- \( e(t) \) error
- \( m(t) \) control input
- \( d(t) \) disturbance
- \( y(t) \) controlled output
- \( n(t) \) measurement noise

**controller**

\[
\begin{align*}
C(s) & = \left\{ A_c, B_c, C_c, D_c \right\} \\
\text{equivalent} & \quad \text{(since we design the controller)}
\end{align*}
\]
+ model for the controlled system (plant)

\[ r(t) \rightarrow e(t) \rightarrow C(s) \rightarrow m(t) \rightarrow + u(t) \rightarrow P(s) \rightarrow y(t) \]

\[ d_1(t) \rightarrow + \rightarrow \text{plant} \rightarrow d_2(t) \]

\[ n(t) \rightarrow + \rightarrow \text{plant} \]

distinguish if the disturbance acts at the input of the plant \( d_1(t) \) or at the output \( d_2(t) \)

signals \( d_1(t), d_2(t) \) and \( n(t) \) may be the output of some other system too
• several inputs act simultaneously on the control system
• we may be interested in several variables
we may need to give specifications on different pairs (Input, Output)

the effect of each input on any output is determined using the superposition principle that is by considering each input at a time (for example if we want to determine the effect of the input \( r(t) \) on \( m(t) \) we set \( d_1(t) = d_2(t) = n(t) = 0 \) and compute the single input - single output transfer function from \( r(t) \) to \( m(t) \))
define the **loop function** $L(s) = C(s)P(s)$ and

$$S(s) = \frac{1}{1 + L(s)} \quad \text{sensitivity function}$$

$$T(s) = \frac{L(s)}{1 + L(s)} \quad \text{complementary sensitivity function} \quad \text{since} \quad S(s) + T(s) = 1$$

$$S_u(s) = \frac{C(s)}{1 + L(s)} \quad \text{control sensitivity function}$$

check that, using the superposition principle,

$$y(s) = T(s)r(s) + P(s)S(s)d_1(s) + S(s)d_2(s) - T(s)n(s)$$

$$e(s) = S(s)r(s) - P(s)S(s)d_1(s) - S(s)d_2(s) - S(s)n(s)$$

$$m(s) = S_u(s)r(s) - T(s)d_1(s) - S_u(s)d_2(s) - S_u(s)n(s)$$
Model uncertainties (plant)

- **Parametric uncertainties:**
  the real (perturbed) parameters of the controlled system are different from the ones (nominal) used to design the controller
  - slowly time-varying parameters
  - wear & tear (damage caused by use)
  - difficulty to determine true values
  - change of operating conditions (linearization), ...

- **Un-modeled dynamics:**
  typically high-frequency
  - dynamics deliberately neglected for design simplification,
  - difficulty in modeling
Parametric uncertainties
(MSD example)

\[ P(s) = \frac{1}{ms^2 + \mu s + k} \]

\[ m \text{ in } [0.9, 1.1] \]
\[ \mu \text{ in } [0.05, 2] \]
\[ k \text{ in } [2, 3] \]

\[ \omega_1 = 1 \text{ rad/s} \]
\[ \omega_2 = 1.5 \text{ rad/s} \]
\[ \omega_3 = 2 \text{ rad/s} \]
\[ \omega_4 = 3 \text{ rad/s} \]
**Un-modeled dynamics**

This example illustrates the dangers of designing a controller (static $K = 1$ in this case) based on dominant dynamics.

\[ F_2(s) = \frac{100}{(s + 1)(0.025s + 1)^2} \]

-40

-1

full dynamics

\[ F_1(s) = \frac{100}{s + 1} \quad \text{same gain as } F_2(s) \text{ but only dominant dynamics (approximation)} \]
open-loop similar

\[ F_2(s) = \frac{100}{(s + 1)(0.025s + 1)^2} \quad F_1(s) = \frac{100}{s + 1} \]

but closed-loop different

\[ W_1(s) = \frac{100}{s + 101} \quad \text{stable} \]

\[ W_2(s) = \frac{160000}{(s + 83.9254)(s^2 - 2.9254s + 1925.5)} \]

unstable dynamics (Nyquist criterion)

differ in high frequency content
Specifications

**Stability** of the control system (closed-loop system)
- **nominal stability** (can be checked with Routh, Nyquist, root locus ...)
- **robust stability** guarantees that, even in the presence of parameter uncertainty and/or un-modeled dynamics, stability of the closed-loop system is guaranteed. We have seen two useful indicators (gain and phase margins) others are possible (based on the Nyquist stability criterion or on a surprising result known as the Kharitonov theorem).

**Performance**
- **nominal performance**
  - **static** (or at steady-state) on the desired behavior between the different input/output pairs of interest
  - **dynamic**: on the dynamic behavior during transient
- **robust performance**: we ask that the performance obtained in nominal conditions is also guaranteed, to some extent, under perturbations (parameter variations, un-modeled dynamics).
Specifications

being

\[ y(s) = T(s)r(s) + P(s)S(s)d_1(s) + S(s)d_2(s) - T(s)n(s) \]
\[ e(s) = S(s)r(s) - P(s)S(s)d_1(s) - S(s)d_2(s) - S(s)n(s) \]
\[ m(s) = S_u(s)r(s) - T(s)d_1(s) - S_u(s)d_2(s) - S_u(s)n(s) \]

ideally we would like to have

- the output accurately reproducing instantaneously the reference
  i.e. we ask the complementary sensitivity \( T(s) \) to be as close as possible to 1
- the disturbances and the noise not affecting the output
  i.e. the complementary sensitivity \( T(s) \) should be as close as possible to 0
  (or equivalently the sensitivity \( S(s) \) close to 1 being \( S(s) + T(s) = 1 \))

\[ T(s) = 1 \] and \( T(s) = 0 \) simultaneously

conflicting requirement!

requirements need to be carefully chosen (compromise)
Specifications

example

• static (at steady-state) reference/output behavior w.r.t. standard signals (sinusoidal or polynomial)
• static disturbance/output behavior for some standard signal (sinusoidal or constant)
• dynamic (transient) reference/output behavior
  - by setting limits to the step response parameters like overshoot or rise time
  - by setting some equivalent bounds on the frequency response (bandwidth, resonance peak defined soon)

+ closed-loop stability  
  most important requirement always present even if not explicitly stated

note how we relaxed some requirements on the performance w.r.t. reference and disturbance by asking the fulfillment only at steady-state that is

\[
\lim_{t \to \infty} (r(t) - y(t)) = 0 \quad \text{instead of} \quad y(t) = r(t), \quad \forall t
\]
Steady-state specifications - reference

**Hyp**: closed-loop system will be asymptotically stable

Let the **canonical signal of order** $k$ be

- **order 0 (step function)**
  \[ \delta_{-1}(t) \]

- **order 1 (ramp function)**
  \[ t \delta_{-1}(t) \]

- **order 2 (quadratic function)**
  \[ \frac{t^2}{2} \delta_{-1}(t) \]
**Def** a system is of type $k$ if its steady-state response to an input of order $k$ differs from the input by a non-zero constant or, equivalently, if the error at state-state (output minus input) is constant and different from zero.

apply this definition to a feedback control system where the input is the reference signal and the output is the controlled output and we look for conditions which guarantee that a feedback system is of type $k$. 

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**Lanari: CS - Control basics II**

Sunday, November 30, 2014
An asymptotically stable **negative unit feedback** control system is of **type** $k$
if and only if

the **open-loop system** $L(s)$ has $k$ **poles** in $s = 0$

Basic ideas for proof:

- closed-loop system is asymptotically stable by hypothesis

- the error at steady-state is constant and non-zero if and only if there are $k$ zeros
  in $s = 0$ in the transfer function from the reference to the error, that is the
  sensitivity function $S(s)$

- we can apply the final value theorem

- the zeros of $S(s)$ coincide with the poles of the loop function $L(s)$ since if
  $L(s) = N_L(s)/D_L(s)$ then

$$
S(s) = \frac{1}{1 + L(s)} = \frac{D_L(s)}{D_L(s) + N_L(s)}
$$
Define with $K_P$ and $K_C$ the generalized gain respectively of the plant and the controller, therefore the generalized gain of the loop function $L(s)$ is $K_L = K_P K_C$

- order $k = 0$ reference

$$e_0 = S(0) = \begin{cases} \frac{1}{1+K_L} & \text{if Type 0} \\ 0 & \text{if Type } k \geq 1 \end{cases}$$

since the presence of 1 or more roots in $s = 0$ in the denominator $D_L(s)$ of the loop function makes the numerator of $S(s)$ become zero.

- order $k \geq 1$ reference

$$e_k = \lim_{s \to 0} \left[ sS(s) \frac{1}{s^{k+1}} \right] = \begin{cases} \infty & \text{if Type } < k \\ \frac{1}{K_L} & \text{if Type } = k \\ 0 & \text{if Type } > k \end{cases}$$

if the denominator $D_L(s)$ has roots in $s = 0$ with multiplicity $h$, we factor $D_L(s)$ as $s^h D'_L(s)$ such that $K_L = N_L(0)/D'_L(0)$. We obtain the different situations depending on the multiplicity $h$, that is $h < k$, $h = k$ and $h > k$.
**Summarizing table:** error w.r.t. the reference

<table>
<thead>
<tr>
<th>Input order</th>
<th>error</th>
<th>System type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\delta_{-1}(t)$</td>
<td>$0$</td>
</tr>
<tr>
<td>1</td>
<td>$t\delta_{-1}(t)$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{t^2}{2}\delta_{-1}(t)$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{t^3}{3!}\delta_{-1}(t)$</td>
<td>$+\infty$</td>
</tr>
</tbody>
</table>

*Note: The table represents the error in the system as a function of input order, where $\delta_{-1}(t)$ is the impulse response of the system.*
Therefore

we can define the specifications on the reference to output behavior in terms of system type and value of maximum allowable error or, equivalently,

- presence of the sufficient number of poles in \( s = 0 \) in the open-loop system
- absolute value of the open-loop gain \( K_L \) sufficiently large in order to guarantee the maximum allowed error

\[
|e_k| \leq e_{kmax} \iff \begin{cases} 
\frac{1}{|1+K_L|} \leq e_{kmax} & \iff |1 + K_L| \geq \frac{1}{e_{kmax}} \quad \text{if Type 0} \\
\frac{1}{|K_L|} \leq e_{kmax} & \iff |K_L| \geq \frac{1}{e_{kmax}} \quad \text{if Type } k \geq 1
\end{cases}
\]

We have translated the closed-loop specifications in equivalent open-loop ones
**Steady-state specifications - disturbance**

The disturbance is just another - undesired - input.

Let us consider the **constant disturbance input** case and use the same basic principle as for the reference.

To make an asymptotically stable control system controlled output insensible (**astatic**), at steady-state, to a constant input $d_1$ or $d_2$, we just need to ensure the presence of a pole in $s = 0$ before the entering point of the disturbance.

Note that nothing is said for the noise $n$

Let’s check
constant unit disturbances

\[ d_1(s) = d_2(s) = n(s) = \frac{1}{s} \]

being

\[ y(s) = T(s)r(s) + P(s)S(s)d_1(s) + S(s)d_2(s) - T(s)n(s) \]

we have (setting the reference to zero)

\[ y_{ss} = [P(s)S(s)]_{s=0} + S(0) - T(0) \]

therefore we need to compute the value of the terms

\[ d_1 \rightarrow y_{ss} \quad [P(s)S(s)]_{s=0} \]
\[ d_2 \rightarrow y_{ss} \quad S(0) \]
\[ n \rightarrow y_{ss} \quad -T(0) \]

define

\[ C(s) = \frac{N_C(s)}{D_C(s)} \quad P(s) = \frac{N_P(s)}{D_P(s)} \quad L(s) = \frac{N_L(s)}{D_L(s)} = \frac{N_C(s)N_P(s)}{D_C(s)D_P(s)} \]
to have no steady-state contribution to the output $y_{ss}$ from a constant disturbance $d_2$ we need to have $S(0) = 0$ that is, being

$$S(s) = \frac{1}{1 + L(s)} = \frac{D_L(s)}{D_L(s) + N_L(s)} = \frac{D_C(s)D_P(s)}{D_L(s) + N_L(s)} = \frac{D_C(s)D_P(s)}{D_P(s)D_P(s) + N_C(s)N_P(s)}$$

the zeros of the sensitivity function $S(s)$ coincide with the poles of the loop function $L(s)$ so we will have $S(0) = 0$ (i.e. $s = 0$ is a zero of $S(s)$) if and only if we have at least one pole at the origin in the open-loop system (and for this disturbance, this is equivalent to asking at least a pole in $s = 0$ before the entry point of the disturbance).

so either a pole in $s = 0$ is already present in the plant or we need to introduce it in the controller (necessary part of the controller to cancel out the effect of the constant disturbance $d_2$ at steady-state on the output).
if no pole in $s = 0$ is present in loop we have a steady-state effect of a constant unit disturbance $d_2$ given by

$$y_{ss} = S(0) = \frac{1}{1 + K_L} = \frac{1}{1 + K_C K_P}$$

so a high-gain controller will reduce the effect of the given disturbance provided the system remains asymptotically stable
\( d_1 \rightarrow y_{ss} \)

- for the steady-state contribution to the output \( y_{ss} \) of a constant disturbance \( d_1 \) note that

\[
P(s)S(s) = \frac{N_P(s)D_C(s)}{D_P(s)D_P(s) + N_C(s)N_P(s)}
\]

therefore in order to get a zero contribution at steady-state we can either

- have a pole in \( s = 0 \) in \( C(s) \) (ahead of the entry point of the disturbance) or
- have a zero in \( s = 0 \) in \( P(s) \) but this leads also to a zero steady-state contribution of a constant reference to the output, i.e. zero gain \( T(0) = 0 \) while we would like this gain to be as close to 1 as possible (avoid when possible)

otherwise we have a finite non-zero contribution given by

\[
\begin{align*}
\frac{K_P}{1 + K_PK_C} & \quad \text{if } P(s) \text{ has no poles in 0} \\
\frac{1}{K_C} & \quad \text{if } P(s) \text{ has poles in 0}
\end{align*}
\]

proof as exercise
• for the steady-state contribution to the output $y_{ss}$ of a constant noise $n$ note that

$$T(s) = \frac{N_P(s)N_C(s)}{D_P(s)D_P(s) + N_C(s)N_P(s)}$$

therefore in order to get a zero contribution at steady-state we can either
- have a zero in $s = 0$ in $P(s)$ and or $C(s)$ but this leads also to a zero steady-state contribution of a constant reference to the output, i.e. zero gain $T(0) = 0$ while we would like this gain to be as close to 1 as possible (avoid when possible)

otherwise we have a finite non-zero contribution given by

$$\frac{K_PK_C}{1 + K_PK_C} \quad \text{if } L(s) \text{ has no poles in } 0$$

$$1 \quad \text{if } L(s) \text{ has poles in } 0$$

High-gain makes things worse w.r.t. noise disturbance
Effect of integrators on stability

The previous analysis has shown that, provided the control system remains stable, adding integrators in the forward path has beneficial effects on the steady-state behavior of the closed-loop system.

However, integrators in the open-loop system have a destabilizing effect on the closed-loop as shown in the following Nyquist plot or equivalently by noting the lag effect on the phase ($-\pi/2$ for each pole in $0$). In the design process we will introduce the minimum number of integrators necessary.

![Nyquist plot diagram](image)

*The shown Nyquist plot are not complete since it’s only for $\omega$ in $(0^+, +\infty)$.*
Other steady-state requirements

- asymptotic tracking of a sinusoidal function. Let the reference be (for positive $t$)

$$
r(t) = \sin \bar{\omega} t
$$

with Laplace transform

$$
r(s) = \frac{\bar{\omega}}{s^2 + \bar{\omega}^2}
$$

to asymptotically track this reference the controlled output needs to tend asymptotically to the reference or, equivalently, the difference (error signal) $r(t) - y(t)$ needs to tend to zero as $t$ tends to infinity. Recalling that the transfer function from the reference to the error is

$$
e(s) = \frac{r(s) - y(s)}{r(s)} = S(s)
$$

and that, for an asymptotically stable system, the steady-state response to a sinusoidal is

$$
e_{ss}(t) = |S(j\bar{\omega})| \sin(\bar{\omega}t + \angle S(j\bar{\omega}))
$$

it is clear that, in order to achieve zero asymptotic error we need, at the specific input frequency, to be able to ensure that

$$
|S(j\bar{\omega})| = 0 \quad \iff \quad S(s)\bigg|_{s=j\bar{\omega}} = 0
$$
that is the sensitivity function must have a pure imaginary zero (and its conjugate) at the frequency of the input signal $\bar{\omega}$

from the previous analysis we also know that the zeros of the sensitivity function coincide with the poles of the open-loop function (in a unit feedback scheme), therefore the necessary and sufficient condition becomes

$$
\text{in order to guarantee asymptotic tracking of a sinusoid of frequency } \bar{\omega} \text{ in an asymptotically stable feedback system, the open-loop system needs to have a pair of conjugate poles in } s = \pm j\bar{\omega}
$$

Being $L(s) = C(s)P(s)$ and assuming that the plant has no poles in $s = \pm j\bar{\omega}$ the controller needs to be of the form

$$
C'(s) = \frac{N_C(s)}{(s^2 + \bar{\omega}^2)D_C'(s)}
$$
• asymptotic rejection of a sinusoidal disturbance (similarly)

\[ d_1(t) = \sin \bar{\omega}t \quad d_1 \rightarrow y_{ss} \quad |P(j\bar{\omega})S(j\bar{\omega})| = 0 \quad [P(s)S(s)]_{s=j\bar{\omega}} = 0 \]

\[ d_2(t) = \sin \bar{\omega}t \quad d_2 \rightarrow y_{ss} \quad |S(j\bar{\omega})| = 0 \quad S(s)\big|_{s=j\bar{\omega}} = 0 \]

Assuming that the plant has no poles in \( s = \pm j\bar{\omega} \) the controller needs to be of the form

\[ C(s) = \frac{N_C(s)}{(s^2 + \bar{\omega}^2)D'_C(s)} \]
**Transient specifications**

We already know how to characterize the transient and therefore define requirements on the closed-loop dynamic behavior in terms of

- poles (and zeros) location in the complex plane (time constants, damping coefficients, natural frequencies)
- particular quantities of the step response like rise-time, overshoot and settling time

We can also define two quantities in the frequency domain related to the transient behavior

- bandwidth $B_3$
- resonant peak $M_r$

which will be related to the rise time and the overshoot establishing interesting connections between time and frequency domain characterization of the transient
for the typical magnitude plots encountered so far, we define the bandwidth $B_3$ as the first frequency such that for all frequencies greater than the bandwidth the magnitude is attenuated by a factor greater than $1/\sqrt{2}$ from its value in $\omega = 0$

$$B_3 : \quad |W(jB_3)| = \frac{|W(j0)|}{\sqrt{2}}$$

and being

$$20 \log_{10} \left( \frac{1}{\sqrt{2}} \right) \approx -3 \, dB$$

$$B_3 : \quad |W(jB_3)|_{dB} = |W(j0)|_{dB} - 3$$
simplest example

\[ W(s) = \frac{K}{1 + \tau s} \quad \text{asymptotically stable system (therefore } \tau > 0) \]

normalized w.r.t.

\[ |K|_{dB} \text{ magnitude plot} \]

being

\[
|W(j\omega)|_{dB} - |W(j0)|_{dB} = |W(j\omega)|_{dB} - |K|_{dB} \\
= |K|_{dB} + |1/(1 + j\omega\tau)|_{dB} - |K|_{dB} \\
= |1/(1 + j\omega\tau)|_{dB}
\]

and

\[ |1 + j\tau/\tau|_{dB} = 20 \log_{10} \sqrt{2} \approx 3 \text{ dB} \]

the bandwidth coincides with the cutoff frequency

\[ B_3 = \frac{1}{\tau} \]
Resonant peak

we define the resonant peak $M_r$ as the maximum value of the frequency response magnitude referred to its value in $\omega = 0$

$$M_r = \frac{\max |W(j\omega)|}{|W(j0)|}$$

or in dB

$$M_r|_{dB} = \max |W(j\omega)|_{dB} - |W(j0)|_{dB}$$

a high resonant peak indicates that the system behaves similarly to a second order system with low damping coefficient
on a plot with normalized magnitude (not in dB)
Relationships

typically (with some exceptions)

\[ B_3 t_r \approx \text{constant} \]

higher bandwidth (higher frequency components of the input signal are not attenuated and therefore are allowed to go through) leads to smaller rise time (faster system response)

\[ \frac{1 + M_p}{M_r} \approx \text{constant} \]

higher resonant peak (as if we had a second order system with lower damping coefficient) leads to higher overshoot (the oscillation damps out slower)

very useful relationships in order to understand the connections between time and frequency domain response characteristics
Transient specifications

we may want to ensure a maximum rise time $t_{r,\text{max}}$

\[
tr \leq t_{r,\text{max}} \iff B_3 \geq B_{3,\text{min}}
\]

this may be achieved by ensuring a sufficiently high bandwidth (greater than some value $B_{3,\text{min}}$)

we may want to ensure a maximum overshoot $M_{p,\text{max}}$

\[
M_p \leq M_{p,\text{max}} \iff M_r \leq M_{r,\text{max}}
\]

this may be achieved by ensuring a sufficiently low resonant peak (smaller than some value $M_{r,\text{max}}$)
Transient specifications

we want to relate some transient specifications on the closed-loop system (control system) to some characteristics of the open-loop system.

Control system

- bandwidth $B_3$ (and rise time $t_r$)
- resonant peak $M_r$ (and overshoot $M_p$)

Open-loop system

- crossover frequency $\omega_c$
- phase margin $PM$
we show these typical (with some exceptions) relationships through an example

$$F(s) = \frac{10K}{s(s + 10)(s + 1)}$$  
open-loop system

comparison for increasing values of $K$

$PM$ and $M_r$ relationship

open-loop phase margin $PM$ decreases & closed-loop resonant peak $M_r$ increases

recall that if the Nyquist plot goes through the critical point then the closed-loop system has pure imaginary poles (zero damping and thus infinite resonant peak)
same open-loop system comparison for increasing values of $K$

$\omega_c$ and $B_3$ relationship

as the open-loop crossover frequency $\omega_c$ increases the closed-loop bandwidth $B_3$ increases
Transient specifications

\[
\begin{align*}
  r(t) & \xrightarrow{\text{specs}} y(t) \xrightarrow{\text{specs}} S_{OL} \\
  S_{CL} & \xrightarrow{\text{control system}} e(t)
\end{align*}
\]

\[
\begin{align*}
  t_r & \leq t_{r,\text{max}} \iff B_3 \geq B_{3,\text{min}} \\
  \text{bandwidth } B_3 \text{ (and rise time } t_r) & \quad \omega_c \geq \omega_{c,\text{min}} \\
  M_p & \leq M_{p,\text{max}} \iff M_r \leq M_{r,\text{max}} \\
  \text{resonant peak } M_r \text{ (and overshoot } M_p) & \quad PM \geq PM_{\text{min}}
\end{align*}
\]

\[
\begin{align*}
  \text{closed-loop system} & \quad \text{open-loop system}
\end{align*}
\]