Control Systems

Control Design: Loop shaping

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Outline

• specifications
• open-loop shaping principle
• lead and lag controllers
• 4 basic situations
• sketch of PID controllers
Specifications (closed-loop)

**Static**
- desired behavior w.r.t. order \( k \) inputs in terms of a maximum allowed absolute error
- desired attenuation level w.r.t. constant disturbances acting on the forward loop
- tracking of a sinusoidal reference
- attenuation of sinusoidal disturbances and measurement noise

**Dynamic**
- location of eigenvalues/poles in the complex plane
- time domain specifications on the step response (mainly on the reference to output behavior)
- frequency domain specifications through the resonance peak and the bandwidth (mainly on the reference to output behavior)

**Stability**
- location of eigenvalues/poles in the complex plane
- robustness in terms of gain and phase margin
Specifications

CLOSED-LOOP System

**Static**
equivalent to

**Dynamic**
bandwidth $B_3$ (and rise time $t_r$)
resonant peak $M_r$ (and overshoot $M_p$)

OPEN-LOOP System

presence of a sufficient number of poles in 0 and/or at the reference/disturbance angular frequency sufficiently high gain (in absolute value)

**Necessary conditions** require the following structure in the controller:

$$\frac{K_c}{s^h} \left( \frac{1}{s^2 + \omega^2} \right)$$

for sinusoidal reference or disturbance

$\omega_c$ crossover frequency
$PM$ phase margin

taken care by $R(s)$

$$C(s) = \frac{K_c}{s^h} R(s)$$

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Loop shaping

necessary part of the controller
to be still chosen in order to satisfy the dynamic specifications

controller

extended plant

plant

dynamic specifications
static specifications

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Open Loop Shaping

Basic idea: being the plant $P(s)$ fixed, choose the controller $C(s)$ to **shape** the loop function frequency response $L(j\omega)$

\[ C(s) \] such that

\[ C(s) = C_1(s)R(s) \]

$C_1(s)$ static specs

$R(s)$ dynamic performance specs

$\frac{K_c}{s^h}$

$L(j\omega)$ with some desired characteristics

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hyp: the necessary part of the controller $C(s)$ has been determined so we have the extended plant

$$\hat{P}(s) = C_1(s)P(s) = \frac{K_c}{s^h}P(s)$$

we need to determine $C(s)$ so to satisfy also the dynamics specifications and ensure stability specifications

$\omega_c^*$ desired crossover frequency at which we want to have a phase margin of at least $PM^*$

from the extended plant frequency response we need to check which action is necessary both in terms of magnitude and phase by comparing the actual value of the magnitude and phase at the future crossover frequency $\omega_c^*$

**Magnitude**
- Amplification: we need to increase the magnitude at some frequency
- Attenuation: we need to decrease the magnitude at some frequency

**Phase**
- Lead: we need to increase the phase at some frequency
- Lag: phase can be decreased at some frequency if necessary

we want $PM \geq PM^*$ so if we have extra phase we can keep it
2 possible actions on the magnitude at some frequency

2 possible actions on the phase at some frequency
Remember that the static specifications have already been met (if we ensure stability) and therefore we do not want to alter this first step.

Therefore, in general, we are not going to use a

- zero in $s = 0$ to obtain a phase lead
- pole in $s = 0$ to attenuate at some frequency
- a gain smaller than 1 in magnitude to attenuate if we have a constraint on the loop gain from the static requirements (while we may use a gain greater than 1 to amplify)

elementary functions that can provide these magnitude and phase contributions

\[
R_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s} \quad \tau_a > 0 \quad m_a > 1
\]

\[
R_i(s) = \frac{1 + \tau_i s}{1 + \tau_i s} \quad \tau_i > 0 \quad m_i > 1
\]

both have **unit gain** so no magnitude change in $\omega = 0$
**Lead compensator**

\[
R_a(s) = \frac{1 + \tau_a s}{1 + \frac{\tau_a}{m_a} s}
\]

- \( \tau_a > 0 \)
- \( m_a > 1 \)

*zero in* \( \frac{1}{\tau_a} \)

*pole in* \( \frac{m_a}{\tau_a} \)

\[m_a = 10\]
**Lag compensator**

\[
R_i(s) = \frac{1 + \frac{\tau_i}{m_i} s}{1 + \tau_i s}
\]

\[\tau_i > 0\]

\[m_i > 1\]

zero in \[\frac{m_i}{\tau_i}\]

pole in \[\frac{1}{\tau_i}\]

\[m_i = 10\]
**Choice of** $R(s)$

we assume $C_1(s)$ (static specs) has already been chosen. Therefore we need to

- by evaluating the actual values of the extended plant magnitude and phase at the desired crossover frequency $\omega_c^*$, understand what **action** needs to be undertaken:
  - amplification or **attenuation** at $\omega_c^*$
  - **phase lead** or maximum allowed **phase lag** at $\omega_c^*$

- choose the elementary function(s) $R_a(s)$ and/or $R_i(s)$ (since multiple actions can be combined) needed
  - choose $m_a$ (or $m_i$) and the normalized frequency $\omega_T$
  - deciding to obtain the desired action at $\omega_c^*$ choose $\tau_a$ (or $\tau_i$) so that

\[
\omega_c^* \tau_a = \omega_T \quad \text{(or } \omega_c^* \tau_i = \omega_T)\]

- frequency chosen at which we want to obtain the desired action
- we solve in $\tau_a$

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Universal diagrams

\[ R_{a}(s) = \frac{1 + \tau_{a} s}{1 + \frac{\tau_{a}}{m_{a}} s} \]

\( \tau_{a} > 0 \)

\( m_{a} > 1 \)

for different values of \( m_{a} \)

for \( R_{i}(s) \) just change sign to the ordinates
**Case I**

**specs** \( \omega_c^* = \omega_c \quad PM \geq PM^* \)

**actions needed:**

- magnitude: as small amplification as possible in order not to move the current crossover frequency which coincides with the desired one
- phase: increase the phase, in this case of exactly \( PM^* \)
example  we need a phase lead of $25^\circ$ with the smallest amplification possible

we have chosen $m_a = 16$

$\omega_T = 0.5$

therefore to obtain this lead of $25^\circ$ together with the amplification of 1 dB at $\omega_c^* = 0.3$ rad/s we choose $\tau_a$ as $\tau_a = 0.5/0.3$

0.5 is the smallest (for these plots) normalized frequency at which we obtain the desired phase lead (for $m_a = 16$)

smallest amplification obtainable (around 1 dB)
Case II

**specs** \( \omega_c^* \)  \( PM \geq PM^* \)

**actions needed:**
- magnitude: attenuation of \( |\hat{P}(j\omega_c^*)|dB \)
- phase: since
  \[
  \angle \hat{P}(j\omega_c^*) + \pi > PM^*
  \]
  we can tolerate at most a lag of
  \[
  \angle \hat{P}(j\omega_c^*) + \pi - PM^*
  \]

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example we need an attenuation of 17 dB and can tolerate a maximum lag of 7°

we have chosen

\( m_i = 8 \)
\( \omega \tau = 60 \)

therefore to obtain this attenuation of 1 dB together with a lag smaller than 7° at \( \omega_c^* = 0.1 \text{ rad/s} \) we choose \( \tau_i \) as

\( \tau_i = 60/0.1 \)

maximum allowed lag

required attenuation

several choices of \( m_i \) are possible (at different normalized frequencies), we need to find one which is compatible with the magnitude requirement

\( 60 \) is the smallest (for these plots) normalized frequency at which we obtain the desired attenuation (for \( m_i = 8 \))
on the choice of the normalized frequency (case II)

basic consideration (see sensitivity functions): we usually want open-loop high gain at low frequency, therefore anything that gives an excess of attenuation at low frequency should be avoided if possible

case II with two alternative choices of the $\omega_T$

$$R_{i1}(s) \quad m_i = 8 \quad \omega_T = 70$$

$$R_{i2}(s) \quad m_i = 8 \quad \omega_T = 200 \quad \text{starts attenuation before strictly needed}$$

both compensators solve the specifications
Case III

specs $\omega_c^*$ $PM \geq PM^*$

actions needed:
- magnitude: amplification of $-|\hat{P}(j\omega_c^*)|dB$
- phase: since

$$\angle \hat{P}(j\omega_c^*) + \pi < PM^*$$

we need to obtain a phase lead of

$$PM^* - \left(\angle \hat{P}(j\omega_c^*) + \pi\right)$$
**example** we need an amplification of 20 dB and a phase lead of at least 25°

two choices

- find a lead compensator that will give simultaneously the required amplification and phase lead
  - usually requires a choice of the normalized frequency on the right-hand side of the phase “bell” shape which gives poor robustness w.r.t. increases in the crossover frequency since the phase of both the extended plant and the compensator are decreasing at the chosen frequency
  - may be not easy to find

- find a lead compensator that gives the required lead and gives some amplification, integrate the required amplification with an additional gain $K_{c2}$ greater than 1.

  This is usually possible since the static requirement (if any) asks for a gain sufficiently high.
example  we need an amplification of 20 dB and a phase lead of at least 25°

\[ K_{c2}R_a(s) \]

we can choose
\[ m_a = 3 \]
\[ \omega \tau = 0.9 \]
therefore to obtain
this lead of 25° together
with the amplification of
2.5 dB at \( \omega_c^* = 0.2 \) rad/s
we choose \( \tau_a \) as
\[ \tau_a = 0.9/0.2 \]

several choices of \( m_a \) are possible (at different normalized frequencies),
we just keep the choice of the normalized frequency
on the left of the “bell-shape”

\[ K_{c2} / [K_{c2}]_{\text{dB}} = 20 - 2.5 \]

we can obtain an amplification of 2.5 dB)
about the robustness issue

case III revisited with two alternative solutions (the gains have been chosen appropriately)

\[ K_c R_a(j\omega) \]

\[ m_a = 6 \]
\[ \omega \tau = 0.8 \]

left-hand side of the bell

\[ K_c R_a(j\omega) \]

\[ m_a = 6 \]
\[ \omega \tau = 7.5 \]

right-hand side of the bell

solution 1 is more robust w.r.t. uncertainties in the \( \omega_c \)

\[ R_a(j\omega) \]
\[ L(j\omega) \]
\[ P(j\omega) \]
Case IV

specs \( \omega_c^* \) \( PM \geq PM^* \)

actions needed:
- magnitude: attenuation of \( |\hat{P}(j\omega_c^*)|dB \)
- phase: since

\[
\angle \hat{P}(j\omega_c^*) + \pi < PM^*
\]

we need to obtain a phase lead of at least

\[
PM^* - (\angle \hat{P}(j\omega_c^*) + \pi)
\]
**example** we need an attenuation of 8 dB and a phase lead of at least 25°

we need to use both lead and a lag compensators but in the proper order

- we choose the **lead** compensator first in such a way to obtain a phase increase of the required 25° plus an extra (for example of 8°) in order to compensate the lag that will be introduced by the lag compensator  

This lead function will also introduce, at the chosen frequency, an amplification of exactly

\[|\hat{R}_a(j\omega_c^*)|_{dB}\]

- the **lag** compensator will be chosen so to introduce an attenuation of

\[8\, dB + |\hat{R}_a(j\omega_c^*)|_{dB}\]

and a lag smaller than the extra 8° previously introduced

\[\begin{align*}
m_a &= 8 & \omega \tau &= 0.8 & m_i &= 3.2 & \omega \tau &= 20 \\
34° & \quad \rightarrow & 2.5 \, dB & \quad \rightarrow & -10.5 \, dB & \quad < 8°
\end{align*}\]

(these numbers are just for illustration purposes and have not been verified)
**PID controllers**

3 basic heuristic actions

**Proportional**: the control action is set to be directly proportional to the system error (present)

**Integral**: the control action is set to be proportional to the system error integral (past)

**Derivative**: the control action is set to be proportional to the system error derivative (future)

output of the controller

\[ m(t) = K_P e(t) + K_I \int_0^t e(\tau)d\tau + K_D \dot{e}(t) \]

\[ C_{PID}(s) = \frac{m(s)}{e(s)} = K_P + \frac{K_I}{s} + K_D s \]

ideal PID controller transfer function

\[ = K_P \left(1 + \frac{1}{T_I s} + T_D s\right) \]

- widely spread

- fixed structure with 3 tunable parameters \( K_P K_D K_I \)

- can be tuned automatically even with scarce knowledge of the (simple) plant

- basic tuning refers only to static specifications

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Derivative action is physical not realizable (improper transfer function) we need to approximate

\[ s = - \frac{K_P}{K_D} N = - \frac{N}{T_D} \]

\[ C_D^a(s) = \frac{T_D s}{1 + \frac{T_D}{N} s} \]

add a high frequency pole in

approximated derivative action for various values of \( N \)

**Loop shaping**
Basic PID feedback control scheme

typical configurations

- proportional
- proportional + derivative (approximate)
- proportional + integrative
**PD configuration**

equivalent to a Lead compensator

\[ C_{PD}(s) = K_P \left( 1 + \frac{sT_D}{1 + s\frac{T_D}{N}} \right) = K_P \frac{1 + \left( \frac{N+1}{T_D N} \right) s}{1 + \frac{1}{N+1} \left( \frac{T_D N+1}{N} \right) s} \]

\[ \tau_a \]

\[ \frac{1}{m_a} \]

**PI configuration**

\[ C_{PI}(s) = K_P \left( 1 + \frac{1}{sT_I} \right) = \frac{K_P (1 + T_I s)}{T_I s} \]

zero in \( s = -1/T_I \) + pole in \( s = 0 \)

compensates the phase lag introduced by the pole in \( s = 0 \)

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