Control Systems - January 8, 2020 - (**with solution**)

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1) Compute the zero-state output response to the input of Fig. 1 of the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 2u$$
$$y = x_1 + x_2$$

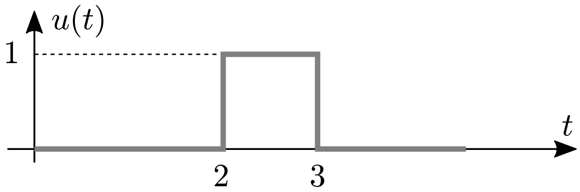

Figure 1: (Ex. 1) input $u(t)$

2) Consider the system

$$\dot{x}_1 = x_1 + 2u$$
$$\dot{x}_2 = -6x_1 - x_2 - 5u$$
$$y = 6x_1 + 2x_2$$

Study controllability and observability and write the corresponding decomposition (when necessary). Can the system be stabilized by state or output feedback?

3) Draw the Bode and Nyquist plot for the system

$$F(s) = \frac{s(s+10)}{(s^2+1)(s-1)}$$

and determine if the closed-loop system is asymptotically stable or not.

4) Let the plant be

$$P(s) = \frac{1}{s+10}$$

It is required to determine a control system such that a sinusoidal disturbance with frequency $\omega = 1$ rad/s acting on the plant's input is perfectly rejected (i.e., no effect) at steady state. Moreover the control system should guarantee a steady-state error smaller equal than 0.1 for a unit step reference. Confirm closed-loop stability through the root-locus plot.

5) Consider the plant

$$P(s) = \frac{1}{s+10}$$

Determine the control scheme and the controller such that:

- the control scheme is astatic w.r.t. a constant disturbance acting at the plant's input,
- the steady-state error w.r.t. a sinusoidal reference $r(t) = \sin(\omega t)$ is lower than 0.1 in magnitude for any $\omega \in (0, 0.1]$ rad/s,
- the effect at steady state of a sinusoidal measurement noise $n(t) = \sin(\omega t)$ is lower than 0.1 in magnitude for any $\omega \in (1, 100]$ rad/s.

You can use the approximate magnitude plots for the binomial term i.e. without the ±3 dB correction at the cut-off frequency.

Sol. 1 The system is characterized by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and therefore the transfer function is

$$F_1(s) = \frac{2(s+1)}{s^2}$$

The input can be written as

$$u(t) = \delta_{-1}(t-2) - \delta_{-1}(t-3)$$

and therefore we only need to compute the response to a step input $\delta_{-1}(t)$ and use the translation theorem. The output zero-state response to the unit step is

$$y_1(s) = F_1(s)\frac{1}{s} = \frac{2(s+1)}{s^3} = \frac{2s}{s^3} + \frac{2}{s^3} = \frac{2}{s^2} + \frac{2}{s^3}$$

(note that there is no need to compute the residues since these are evident) which, in the time domain, becomes,

$$y_1(t) = 2\left(\frac{t^2}{2} + t\right)\delta_{-1}(t)$$

The final response is therefore

$$\begin{aligned} y(t) &= y_1(t-2) - y_1(t-3) \\ &= 2\left(\frac{(t-2)^2}{2} + (t-2)\right)\delta_{-1}(t-2) - 2\left(\frac{(t-3)^2}{2} + (t-3)\right)\delta_{-1}(t-3) \end{aligned}$$

Typical errors:

- writing the matrices (A, B, C) from the system of differential equations;
- confusing the zero-state output response to an input with the (output) impulsive response;
- computing explicitly the convolution integral

  $$\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

  is possible with the right expression of the matrix exponential but it's longer (this is one of the reasons why we introduced the Laplace transform) and the integral has to be computed carefully since one may think erroneously that, being the input different from 0 only in the interval $[2, 3]$, then also the integral (and the response) is non-zero in the same time interval;
- the spectral form does not hold for this matrix exponential since the matrix is not diagonalizable.

Sol. 2 The system is represented by

$$A = \begin{pmatrix} 1 & 0 \\ -6 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \qquad C = \begin{pmatrix} 6 & 2 \end{pmatrix}$$

with (A lower triangular) eigenvalues ± 1. Being the controllability matrix non-singular

$$R = \begin{pmatrix} 2 & 2 \\ -5 & -7 \end{pmatrix} \quad \rightarrow \quad \text{rk}[R] = 2,$$

the system is controllable and therefore can be stabilized by state feedback.

The observability matrix is singular

$$O = \begin{pmatrix} 6 & 2 \\ -6 & -2 \end{pmatrix} \quad \rightarrow \quad \text{rk}[O] = 1$$

and the decomposition w.r.t. observability (required by the exercise) is obtained choosing

$$\text{Ker}[O] = \text{gen}\left\{\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\} \quad \rightarrow \quad T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \quad \rightarrow \quad T = \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

and therefore

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 6 & 0 \end{pmatrix}$$

Being the unobservable subsystem unstable, the system (which can be stabilized by state feedback) cannot be stabilized by output feedback. Clearly the transfer function will have the only pole in -1 and there is an unstable hidden dynamics characterized by the eigenvalue $+1$.

Remarks:

- since the eigenvalues were so evident one could also check controllability and observability through the PBH test, but since the decomposition was required anyway, computing the controllability and observability is faster;

Typical errors:

- writing the wrong matrices (A, B, C) from the system of differential equations;
- numerical errors when computing matrix inverse or ranks;
- not being able to compute the Image or the Kernel of a matrix;
- getting confused by the original lower triangular structure of the system which may induce erroneously some conclusions on the presence of a stable uncontrollable subsystem (block A_{22}).
- confusing, when the system is observable (or even detectable), the possibility of constructing an asymptotic observer with output stabilizability.

Sol. 3 The Bode are reported in Fig. 2 and are obtained by drawing the contribution of each term of the Bode canonical form. The corresponding Nyquist plot is shown in Fig. 3. Since there is an open-loop pole with positive real part [1] (i.e., $n_{\text{ol}}^+ = 1$) and the Nyquist plot makes either $N_{cc} = 0$ or $N_{cc} = -2$ encirclements of the point $(-1, 0)$ (depending on where the point $(-1, 0)$ is), the closed-loop system is not asymptotically stable. It is unstable since the Nyquist plot does not go through the point $(-1, 0)$ and $n_{\text{ol}}^+ \neq N_{cc}$. In particular, the closed loop system will have $n_{\text{cl}}^+ = n_{\text{ol}}^+ - N_{cc} = 1$ or $n_{\text{cl}}^+ = n_{\text{ol}}^+ - N_{cc} = 3$ poles with positive real part.

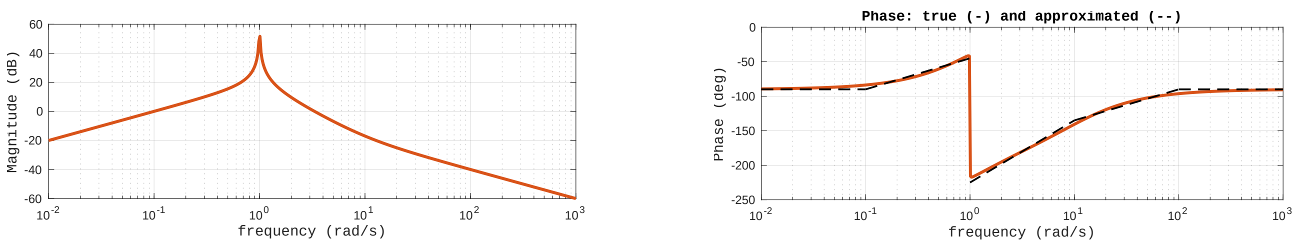

Figure 2: (Ex. 3) Bode plots of $F(j\omega)$

Typical errors:

- not factoring correctly the transfer function in its Bode canonical form;
- confusing the zero in $s = 0$ with a pole in $s = 0$;
- not considering the trinomial factor with $\xi = 0$;
- not drawing properly the closures at infinity;
- letting, in $\omega = 0$, the plot start with infinite magnitude instead of 0 magnitude (i.e., from the origin).

Sol. 4 In a unit feedback scheme, in order to reject a sinusoidal disturbance at frequency $\bar{\omega}$ acting at the plant's input the controller needs to necessarily have a pair of imaginary poles in $\pm j\bar{\omega}$, i.e.,

$$\frac{1}{s^2 + \bar{\omega}^2}$$

This arises as for the constant disturbance:

[1] Here I changed notation for the number of open loop poles, n_{ol}^+ instead of n_F^+

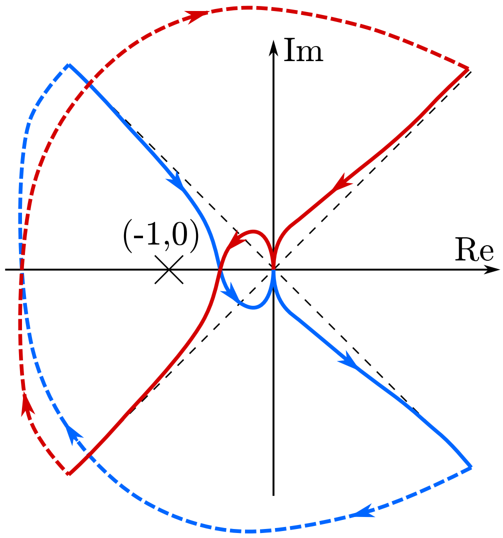

Figure 3: (Ex. 3) Nyquist plot of $F(j\omega))$

- computing the transfer function from the disturbance to the output

$$W_d(s) = P(s)S(s) = \frac{P(s)}{1 + C(s)P(s)} = \frac{N_p(s)D_c(s)}{D_c(s)D_p(s) + N_c(s)N_p(s)}$$

- we want the steady state response corresponding to $d(t) = d \sin \bar{\omega} t$ to be zero that is

$$\lim_{t \to \infty} y_d(t) = 0$$

- the final value theorem is not applicable as such

$$\lim_{t \to \infty} y_d(t) = \lim_{s \to 0} sY_d(s) = \lim_{s \to 0} sW_d(s)d(s) = \lim_{s \to 0} s \frac{N_p(s)D_c(s)}{D_c(s)D_p(s) + N_c(s)N_p(s)} \frac{\bar{\omega}}{(s^2 + \bar{\omega}^2)}$$

  since not all the roots of the denominator have negative real part (even if the closed loop is asymptotically stable) due to the term $(s^2 + \bar{\omega}^2)$ which therefore has to be cancelled. Since the plant does not have the factor $(s^2 + \bar{\omega}^2)$ at the numerator $N_p(s)$ so to cancel $(s^2 + \bar{\omega}^2)$ at the denominator of the previous expression, it must be present in $D_c(s)$.

There is moreover the specification on the steady-state error which requires a Type 0 system with an appropriate value of the loop gain K_L. The requirement in absolute value is

$$\frac{1}{|1 + K_L|} \leq 0.1 \qquad \Leftrightarrow \qquad |1 + K_L| \geq 10$$

and with the temporary controller being

$$C'(s) = \frac{K_c}{s^2 + \bar{\omega}^2}$$

being $K_p = 1/10$, the loop gain is $K_L = K_p K_c / \bar{\omega}^2$, so the requirement is

$$|1 + K_L| \geq 10 \qquad \Leftrightarrow \qquad \left| 1 + \frac{K_c}{10\,\bar{\omega}^2} \right| = \left| 1 + \frac{K_c}{10} \right| \geq 10$$

since for this exercise $\bar{\omega} = 1$. Finally, to guarantee the existence of the steady state, we also need to ensure asymptotic stability of the closed-loop system.

As a first option, being the temporary loop function

$$L'(s) = \frac{K_c}{(s^2 + 1)} \frac{1}{(s + 10)}$$

minimum phase and with $n - m = 3$, we could verify if a simple gain could be sufficient (obviously not with high positive values). The closed-loop polynomial would be

$$p(s, K_c) = (s^2 + 1)(s + 10) + K_c = s^3 + 10s^2 + s + 10 + K_c$$

which has Routh table

$$\begin{array}{|cc} 1 & 1 \\ 10 & 10+K_c \\ -K_c & \\ 10+K_c & \end{array}$$

so the closed-loop system is asymptotically stable for $-10 < K_c < 0$. If we want to draw the root locus, there are two compatible negative root loci as shown in Fig. 4 (the positive root locus has evident poles positive real part)

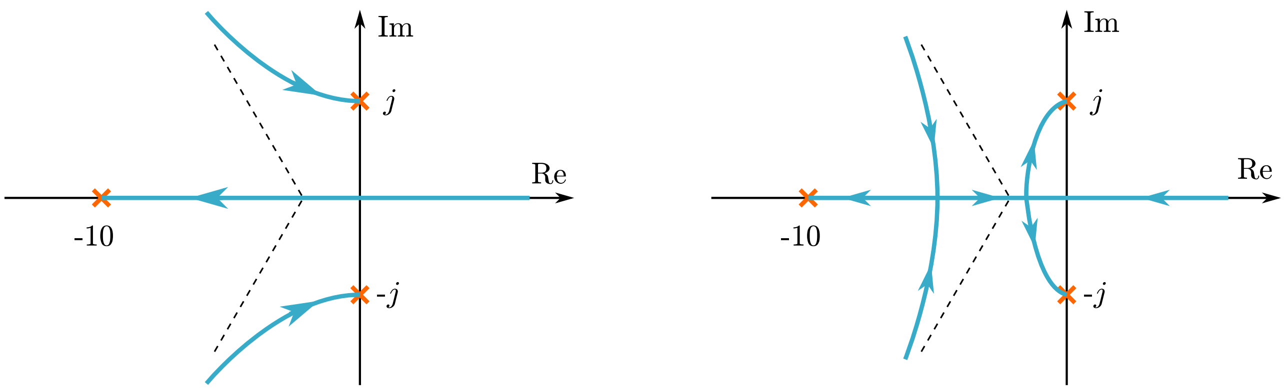

Figure 4: (Ex. 4): two alternative negative root loci of $\frac{K_c}{(s+10)(s^2+10)}$

To determine the candidate singular points we can find the roots of

$$\frac{1}{s+j} + \frac{1}{s-j} + \frac{1}{s+10} = 0 \qquad \Leftrightarrow \qquad 3s^2 + 20s + 1 = 0$$

which has the two real roots $(-6.6163, -0.0504)$ and therefore these are two singular points. The correct negative root locus in Fig.4 is then the one on the right (but it is not important for the solution). Comparing the stability range $(-10, 0)$ for K_c with the steady state requirement

$$\left|1 + \frac{K_c}{10}\right| \geq 10$$

we understand that a simple K_c is not sufficient.

As a successive step, we could add a zero to reduce to the $n - m = 2$ case or even two zeros if we desire $n - m = 1$. In the first case we need to choose the zero such that the center of asymptotes is negative, which is possible without adding any pole/zero pair, if the zero is to the right of -10 but still negative; for example with

$$C_1(s) = \frac{K_c(s+5)}{(s^2+1)} \qquad \rightarrow \qquad L_1(s) = \frac{K_c(s+5)}{(s^2+1)(s+10)}$$

one obtains the root locus of Fig. 6 (Left). The pole polynomial is

$$p(s, K_c) = (s^2+1)(s+10) + K_c(s+5) = s^3 + 10s^2 + (K_c+1)s + 5K_c + 10$$

with the corresponding Routh table

$$\begin{array}{|cc} 1 & K_c+1 \\ 10 & 5K_c+10 \\ 5K_c & \\ 5K_c+10 & \end{array}$$

The closed-loop system is asymptotically stable for any value of $K_c > 0$ and therefore we can also meet the requirement on the reference error.

Second option. Alternatively, choosing to reduce to $n - m = 1$ for example using

$$C_2(s) = \frac{K_c(s+1)(s+2)}{(s^2+1)} \qquad \rightarrow \qquad L_2(s) = \frac{K_c(s+1)(s+2)}{(s^2+1)(s+10)}$$

gives

$$p(s, K_c) = (s^2+1)(s+10) + K_c(s+2)(s+1) = s^3 + (10+K_c)s^2 + 3K_c s + 2K_c$$

with the Routh table

$$
\begin{array}{l|cc}
 & 1 & 3K_c \\
 & 10+K_c & 2K_c \\
 & (3K_c+28)K_c & \\
 & 2K_c &
\end{array}
$$

The closed-loop system is asymptotically stable for any value of $K_c > 0$ and therefore we can also meet the requirement on the reference error.

Second option (bis). Since we are reducing to $n - m = 1$, the center of asymptotes has no importance and therefore we can choose one of the zeros so to cancel the stable pole in $s = -10$. For example

$$C_{2bis}(s) = \frac{K_c(s+10)(s+2)}{(s^2+1)} \qquad \rightarrow \qquad L_{2bis}(s) = \frac{K_c(s+2)}{(s^2+1)}$$

with

$$p(s, K_c) = (s^2+1) + K_c(s+2) = s^2 + K_c s + 1 + 2K_c$$

and therefore (necessary condition is also sufficient) the closed-loop system is asymptotically stable for any $K_c > 0$ so that the steady-state requirement on the gain can be verified. The corresponding root locus is reported in Fig. 5. The candidate singular points are the solutions of

$$\frac{1}{s+j} + \frac{1}{s-j} - \frac{1}{s+2} = \frac{s^2+4s-1}{\dots} = 0$$

which has solutions $(-4.2361, 0.2361)$.

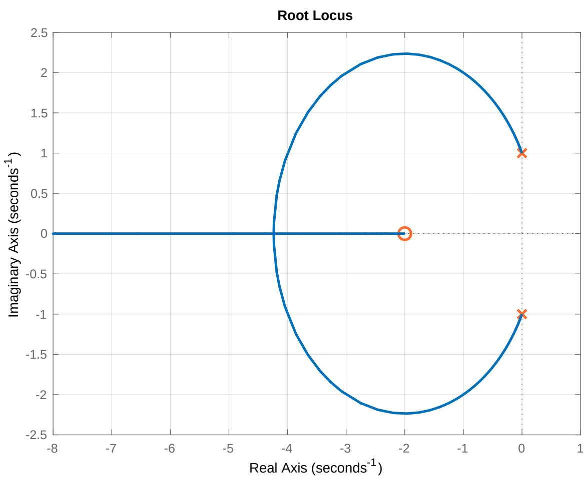

Figure 5: (Ex. 4): root locus of $\frac{K_c(s+2)}{(s^2+1)}$

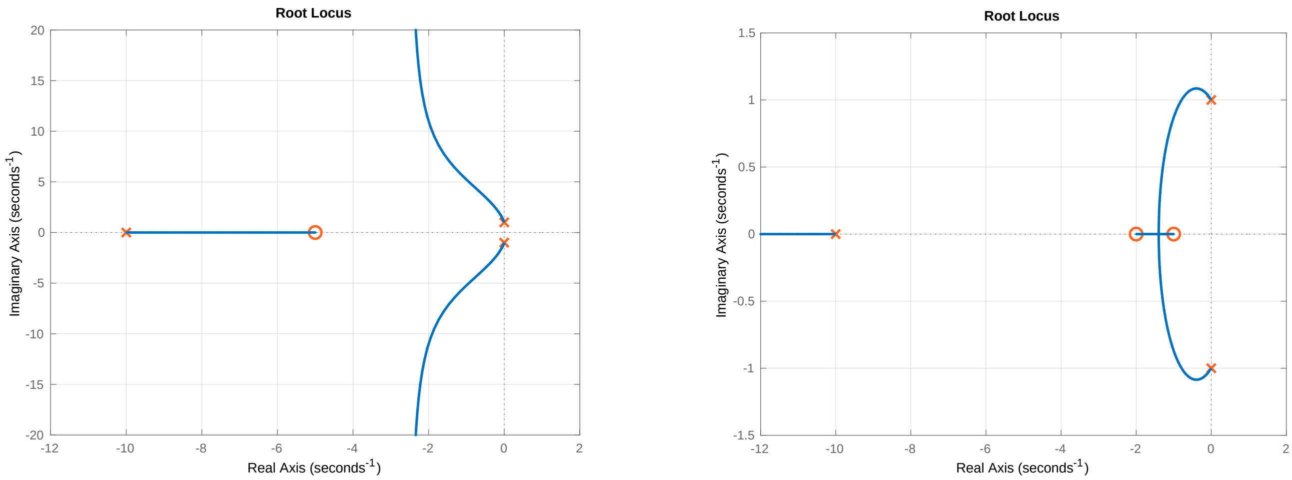

Figure 6: (Ex. 4): root locus of $\frac{K(s+5)}{(s+10)(s^2+10)}$ (Left) and $\frac{K(s+1)(s+2)}{(s+10)(s^2+10)}$ (Right)

Typical errors:

- not considering properly the specification on the sinusoidal disturbance (some, many, too many, just treated it as a constant disturbance);
- first considering the type 0 requirement and determining the minimum value of the controller gain and then changing the controller gain by introducing $(s^2 + \bar{\omega}^2)$ at the denominator;
- forgetting to study and check stability of the resulting closed loop system;
- "perfectly rejected" at steady state means 0 steady state error.

Sol. 5 Add a pole in $s = 0$ for astatism. The temporary loop function is

$$L'(s) = P(s)\frac{1}{s}$$

For the sinusoidal reference requirement, using the approximations for $|S(j\omega)|$, the specification translates in

$$|S(j\omega)|_{db} \leq -20 \text{ dB} \quad \Rightarrow \quad |L(j\omega)|_{db} \geq 20 \text{ dB} \quad \text{for } \omega \in (0, 0.1]$$

while for the sinusoidal measurement noise, using the approximation for $|T(j\omega)|$, one has

$$|T(j\omega)|_{db} \leq -20 \text{ dB} \quad \Rightarrow \quad |L(j\omega)|_{db} \leq -20 \text{ dB} \quad \text{for } \omega \in (1, 100]$$

The constraints are shown in Fig. 7.

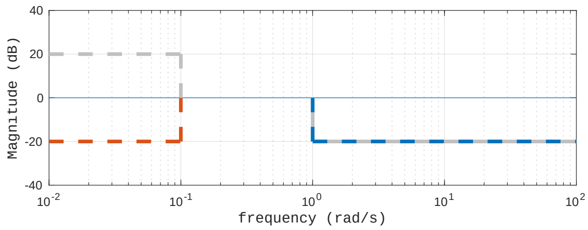

Figure 7: (Ex. 5) Constraints on the magnitude of: the sensitivity magnitude (red), complementary sensitivity (blue) and corresponding loop function (grey)

Comparing the temporary loop function magnitude with the requirements there are two possible alternatives.

[Alt. 1] One could first add an additional gain $K_c = 10$ which gives the required increase of 20 dB at $\omega = 0.1$ rad/s. The temporary loop function becomes

$$L''(s) = \frac{10}{s}P(s)$$

and the magnitude, in order to satisfy the requirement on the measurement noise, needs to be attenuated by at least 20 dB from $\omega = 1$ rad/s on. This can be achieved by introducing a binomial term with cut-off frequency 0.1 rad/s. The resulting Bode plots are reported in Fig. 8 from which the positive phase margin guarantees closed-loop stability (Bode stability theorem can be applied) and therefore the existence of the steady state. The final controller is

$$C(s) = \frac{10}{s(1 + 10s)}$$

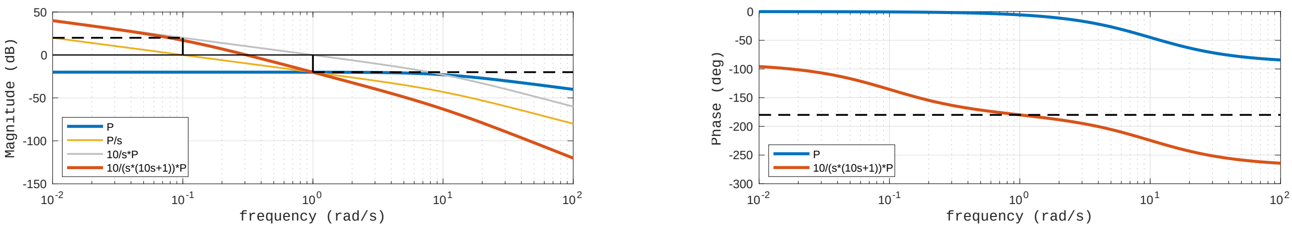

Figure 8: (Ex. 5) – Bode plots of the different steps in **[Alt. 1]**: magnitude (Left) and phase (Right).

[Alt. 2] A possible alternative solution can be obtained by noting in Fig. 7 that the final loop magnitude should be above 20 dB up to 0.1 rad/s, have a slope of at least -40 dB/dec between 0.1 and 1 rad/s and then remain below -20 dB after 1 rad/s. $L'(s)$ already guarantees the right magnitude up to 0.1 rad/s. We therefore need to guarantee a steeper slope from this frequency and this can be achieved by introducing a second pole in $s = 0$

(adding poles in $s = 0$ introduces a phase shift of $-\pi/2$ which, in general, is not beneficial for the closed loop stability). The new temporary loop function

$$L''(s) = P(s)\frac{1}{s^2}$$

satisfies all the requirements on the magnitude, however the resulting closed-loop system is not asymptotically stable. Adding a negative zero, being the system minimum phase and with $n - m = 2$, would guarantee the existence of a sufficiently high value of the loop gain which stabilizes the closed loop provided the center of asymptotes is negative. The additional zero should however alter the magnitude in a compatible way with the requirements on $|L(j\omega)|$. Choosing for example a zero in -1 leads to the alternative controller

$$C'(s) = \frac{K(s+1)}{s^2} \qquad \Rightarrow \qquad L(s) = \frac{K(s+1)}{s^2(s+10)}$$

Stability can be verified through the Routh criterion. The Routh table

$$\begin{array}{|cc} 1 & K \\ 10 & K \\ 9K & \\ K & \end{array}$$

shows that the closed-loop system is asymptotically stable for any $K > 0$. There are two alternatives for the root locus as shown in Fig. 9; the determination of the correct one is not essential for the exercise since both indicate asymptotic stability for positive values of K. However, since the computation is easy, we should find the candidate singular points as solutions of the following equation

$$\frac{2}{s} + \frac{1}{s+1} - \frac{1}{s+1} = \frac{2s^2 + 13s + 20}{\dots} = 0$$

The singular points are therefore $(-4, -2.5)$.

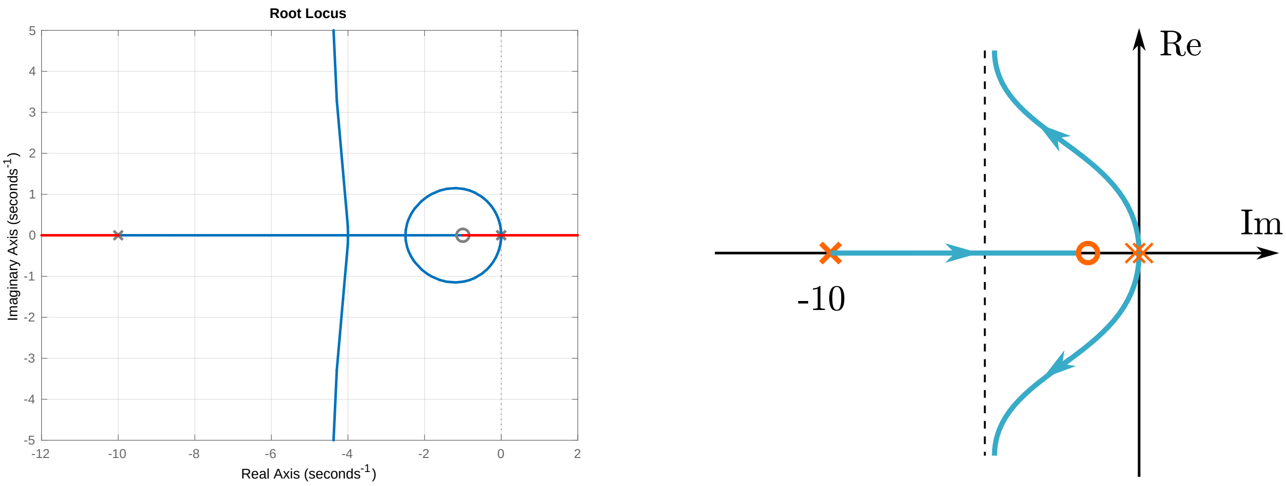

Figure 9: (Ex. 5) – Root locus **[Alt. 2]**: actual positive and negative (Left) and other possible positive root locus without computing the singular points (Right).

Typical errors:

- when considering the sinusoidal reference requirement, considering the complementary sensitivity function $T(s)$ (which represents the $r \to y$ behavior) instead of the sensitivity function $S(s)$ which also drescribes the $r \to e$ behavior;
- introducing $(s^2 + \bar{\omega}^2)$ at the controller denominator (the requirement is to attenuate, not to make it 0). Moreover the reference or measurement noise has frequencies which belong to intervals, there are not fixed single values;
- forgetting to study and check stability of the resulting closed loop system.