Control Systems - January 31, 2022

1) Consider the plant $P(s)$ and the controller $C(s) = K_c$ with

$$P(s) = \frac{s+1}{s^2+1}$$

Study the closed loop stability for $K_c \in I\!R$ varying (both positive and negative) with

1. Routh stability criterion
2. Nyquist stability criterion drawing clear Bode and Nyquist plots
3. Root locus (both positive and negative) computing also the singular points.

2) Consider a pure gain controller $C(s) = K_c$ and the plant

$$P(s) = \frac{10}{s(s+100)}$$

A) Using correctly the closed loop functions approximations, determine the value of the gain K_c which guarantees:

- that the effect of sinusoidal signals with frequency greater equal than 100 rad/s acting on the feedback loop are attenuated, at steady state, by at least a factor 1/10;
- a phase margin of at leat 45°;
- and maximum output attenuation, at steady state, of the effect of sinusoidal disturbances acting at the plant's output.

B) Evaluate approximately how sinusoidal disturbances acting at the plant's input would affect the controlled output at steady state.

3) Let a system be represented by

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -3 \\ 0 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

1. Find, if possible, a stabilizing state feedback.
2. How would you choose (if possible) an output such that the reconstruction error converges to 0 as fast as e^{-2t}? Explain.

1 - Sol. The closed loop polynomial, having the loop function $L(s) = K_c P(s)$, is

$$p(s, K_c) = s^2 + 1 + K_c(s + 1) = s^2 + K_c s + (1 + K_c)$$

and therefore the N. & S. condition requires K_c strictly positive to have closed loop asymptotic stability. The Bode diagrams for $K_c = 1$ are shown in Fig. 1 while the resulting Nyquist diagrams are reported in Fig. 2 for both $K_c = 1$ and $K_c = -1$.

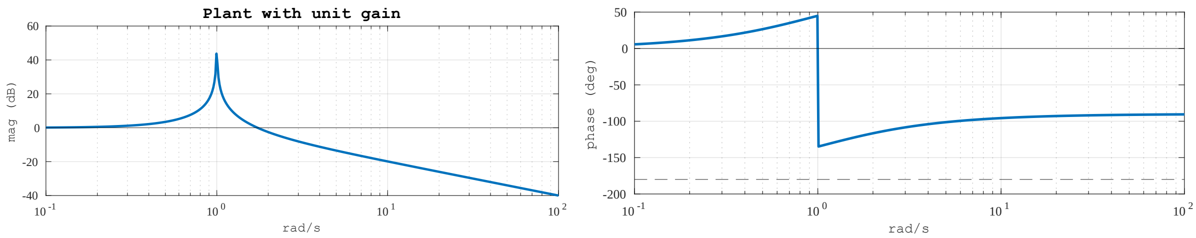

Figure 1: Bode plots Exercise 1

From the Nyquist plots we have the confirmation that for $K_c > 0$ the closed loop system is asymptotically stable since $n_{cc} = N_p = 0$. When K_c is negative, if $K_c < -1$ the number of encirclements around the point $(-1, 0)$ is $n_{cc} = -1$ while if $K_c > -1$ we have $n_{cc} = -2$; in both situations $n_{cc} \neq 0$.

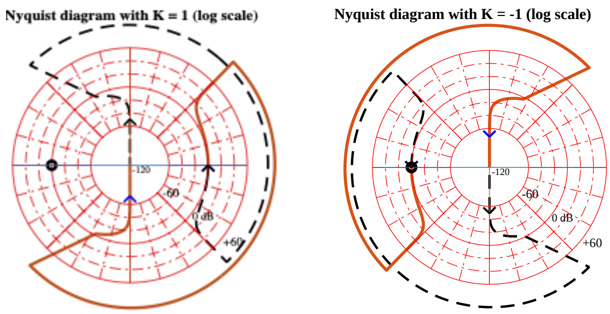

Figure 2: Nyquist diagrams plots in a log scale Exercise 1

For the root locus plot, in order to compute the singular points candidates, we can use the formula

$$\frac{1}{s+j} + \frac{1}{s-j} - \frac{1}{s+1} = \frac{2s}{s^2+1} - \frac{1}{s+1} = \frac{2s(s+1) - (s^2+1)}{*} = \frac{s^2 + 2s - 1}{*} = 0$$

which has the two real solutions

$$s_1^* = -1 - \sqrt{2}, \qquad s_2^* = -1 + \sqrt{2}$$

Finally the root locus is shown in Fig. 3 which confirms all the previous results.

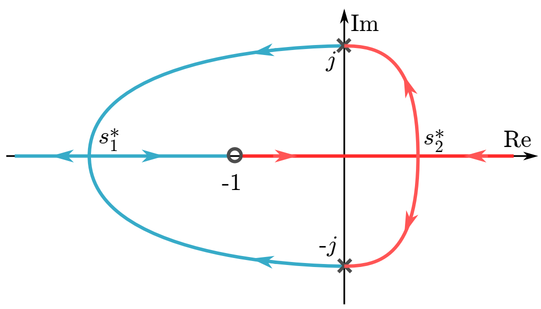

Figure 3: Root locus Exercise 1

2 - Sol. We have a mix of closed loop (attenuation of sinusoidal signals at steady state on the closed loop) and open loop (phase margin) requirements. Let us first draw the Bode plots for $P(s)$ which is written in the Bode canonical form as

$$P(s) = \frac{1}{10} \frac{1}{s(1 + s/100)}$$

The corresponding approximate Bode plots are shown in Fig. 4.

We need to find the value of the gain K_c which simultaneously allows the closed loop system to satisfy the requirements.

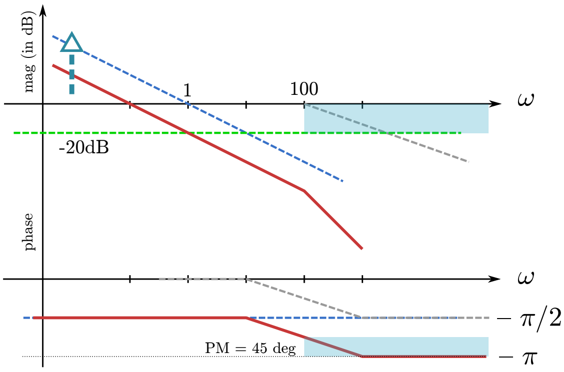

Figure 4: Approximate Bode plots Exercise 2. Red is the sum of all contributions.

- Since we want to guarantee at least 45° of phase margin and since the phase remains the same for any positive value of K_c, the crossover frequency must at most be $\omega_c^{\max} = 100$ rad/s.
- Using the approximations for the sensitivity and complementary sensitivity functions, we can transform the specifications on the closed loop into specifications on the magnitude of the loop function $|K_c P(j\omega)|$. For the disturbances acting at the plant's output which affect the output through the sensitivity function $S(s)$, we should have
$$\min |S(j\omega)| \qquad \Rightarrow \qquad \max |K_c P(j\omega)|$$
and since the gain K_c translates the magnitude uniformly, this means that we should have the maximum possible crossover frequency (compatible with the other requirements).
- The requirement on the sinusoidal signals acting in the feedback loop (measurement noise) is translated in
$$|T(j\omega)| \leq 0.1 \quad \text{for} \quad \omega \geq 100 \text{ rad/s} \qquad \Rightarrow \qquad |T(j\omega)|_{\text{dB}} \leq -20 \text{ dB} \quad \text{for} \quad \omega \geq 100 \text{ rad/s}$$
This means that the crossover frequency of the loop function will be smaller than 100 rad/s and therefore using the approximation
$$|T(j\omega)|_{\text{dB}} \approx |K_c P(j\omega)|_{\text{dB}} \leq -20 \text{ dB} \quad \text{for} \quad \omega \geq 100 \text{ rad/s}.$$

Putting all these requirements together we see that K_c should be chosen as

- $K_c\big|_{\text{dB}} = 40$ dB or $K_c = 100$ if we use the segment approximation for the magnitude of the binomial term
- $K_c\big|_{\text{dB}} = 43$ dB or $K_c = 100\sqrt{2}$ if we use the true value for the magnitude of the binomial term

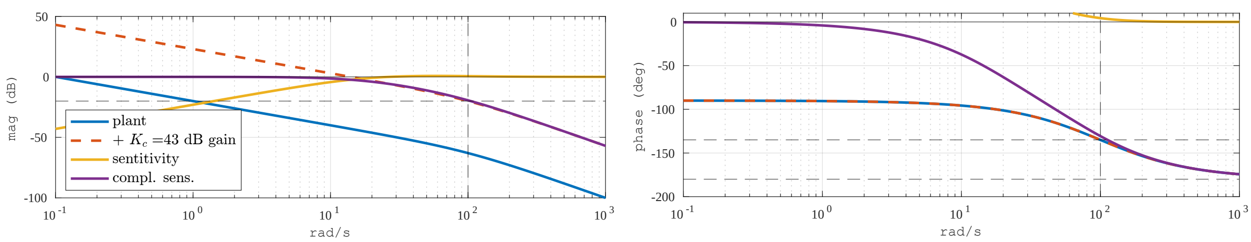

Figure 5: Bode plots Exercise 2.A

The transfer function between a disturbance entering at the input of the plant and the controlled output is $S(s)P(s)$ and therefore we should evaluate the magnitude of the corresponding frequency response. In dB this translates in looking at
$$|S(j\omega)P(j\omega)|_{\text{dB}} = |S(j\omega)|_{\text{dB}} + |P(j\omega)|_{\text{dB}}$$
Having already these plots (or at least the approximation of the sensitivity magnitude plot), we clearly see that the disturbance is attenuated by at least 40 dB (or 43 dB if we refer to the exact plots) as shown in Fig. 6.

Asymptotic stability of the closed loop is guaranteed by Bode's stability theorem.

3 - Sol. Due to the upper block triangular structure of the dynamic matrix A, the eigenvalues are
$$\lambda_1 = -1, \qquad \{\lambda_2, \lambda_3\} = \text{eig}\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$$

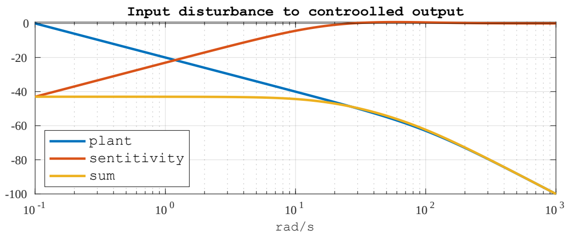

Figure 6: Bode magnitude Exercise 2.B

that is

$$\det\begin{pmatrix} \lambda-2 & 3 \\ -1 & \lambda+2 \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1) \qquad \Rightarrow \qquad \{\lambda_2, \lambda_3\} = \{-1, 1\}$$

We therefore need to check if the unstable eigenvalue λ_3 is controllable, and therefore check the rank of the matrix (PBH test)

$$\text{rank}\begin{pmatrix} \lambda_3 I - A & B \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 0 \end{pmatrix} = 3$$

Since it is full rank, the only unstable eigenvalue is controllable and therefore we can stabilize the system with state feedback. We could have directly studied the controllability of the system computing the controllability matrix

$$R = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Rightarrow \quad \det(R) = 0$$

The image of the controllability matrix is given by

$$\text{Im}(R) = \text{gen}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} = \text{gen}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and therefore a possible change of coordinates T is $z = Tx$ with

$$T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \tilde{A} = TAT^{-1} = \begin{pmatrix} 2 & -3 & -3 \\ 1 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{B} = TB = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

As expected the unstable eigenvalue is controllable, while we discover that there is an asymptotically stable uncontrollable subsystem characterized by the eigenvalue -1. We can apply the Ackermann formula to the pair $(\tilde{A}_{11}, \tilde{B}_1)$ with

$$\tilde{A}_{11} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is by construction controllable, and then go back to the original coordinates as usual (see slides).

In general we would choose the output such that the whole system is observable so that we can assign the desired decay rate of the reconstruction error. However, note that we have a repeated eigenvalue $\lambda_1 = \lambda_2 = -1$ and therefore we need to check its geometric multiplicity (see slides "Laplace Analysis") which turns out to be $m_g(\lambda_1) = 2$ since

$$\text{Ker}(A - \lambda_1 I) = \text{Ker}\begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -3 \\ 0 & 1 & -1 \end{pmatrix} = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and therefore its index is 1. In other words the matrix $(sI - A)^{-1}$ will have at its denominator the term $(s+1)$ and not $(s+1)^2$ independently from the choices of B and C. This can also be seen noticing that there exists no $C = [c_1, c_2, c_3]$ matrix which gives rank 3 to the matrix

$$\text{observability PBH test :} \qquad \text{rank}\begin{pmatrix} A - \lambda_1 I \\ C \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -3 \\ 0 & 1 & -1 \\ c_1 & c_2 & c_3 \end{pmatrix} = 2$$

since $(A - \lambda_1 I)$ has rank 1.

Moreover, note that:

- if we choose the output such that the unstable eigenvalue $\lambda_3 = 1$ is unobservable, clearly the observer does not exist;
- if we choose the output such that there exists an unobservable asymptotically stable subsystem this will necessarily include the eigenvalue -1 so the observer exists but the rate of convergence cannot be faster than e^{-t} and therefore we cannot meet the specifications.