Control Systems - January 9, 2023

Student name: ______________________________ Matricola: ____________

1) Compute the forced output response of the system $F(s)$ to the input $u(t)$ with

$$F(s) = \frac{1}{s+1}, \qquad \text{and} \qquad u(t) = \begin{cases} 0 & \text{for} \quad t < 1 \\ t-1 & \text{for} \quad 1 \leq t < 2 \\ -t+3 & \text{for} \quad 2 \leq t < 3 \\ 0 & \text{for} \quad t \geq 3 \end{cases}$$

2) Let the plant be represented by the transfer function $P(s) = 10\dfrac{(s+10)^2}{(s-10)^3}$

1. Establish if it is possible to stabilize the given plant with the static controller $C(s) = K$ (in a unit feedback interconnection).
2. Draw the corresponding root locus and determine the values of the gain K for which some closed loop poles cross the imaginary axis.
3. Determine, on the basis of the previous analysis, if the closed loop is asymptotically stable for $K = 2$.
4. Confirm the stability analysis (as a function of K) through the Nyquist criterion. The associated Bode plots are required.

3) Consider the three dynamical systems

$$\mathcal{S}_1 : \begin{cases} \dot{x}_1 = x_1 + u_1 \\ y_1 = x_1 \end{cases} \qquad \mathcal{S}_2 : \begin{cases} \dot{x}_2 = -2x_2 + u_2 \\ y_2 = x_2 \end{cases} \qquad \mathcal{S}_3 : \begin{cases} \dot{x}_3 = 3x_3 + u_3 \\ y_3 = -x_3 \end{cases}$$

which form an interconnected system $\mathcal{S}$ with input u and output y characterized by the following interconnecting equations: $u = u_1$, $u_2 = y_1$, $u_3 = y_1$. The output y will be defined afterwords.

1. Find the A (dynamic matrix) and B (input matrix) matrices of $\mathcal{S}$.
2. Determine the output matrix considering alternatively $y = y_2$ (will define C_2) and $y = y_3$ (will define C_3).
3. We want to stabilize the system $\mathcal{S}$ using the separation principle having available only one output, either $y = y_2$ or $y = y_3$. Discuss the possibility of solving the stabilization problem in both cases.
4. Using the given interconnection equations, draw the corresponding block scheme of $\mathcal{S}$ showing how the systems are interconnected.
5. Based on the previous block scheme, design an alternative stabilizing controller.

4) Consider the plant $P(s)$ and the controller $C(s)$ given by $P(s) = \dfrac{100}{s+100}$ and $C(s) = \dfrac{K}{s}$. Determine, if possible, a value of the controller gain such that, at steady state

- a sinusoidal disturbance, with frequency $\omega \leq 10$ rad/s, acting on the plant's input is attenuated by at least a factor 10 (ten times smaller), and
- a sinusoidal measurement noise, with frequency $\omega \geq 1000$ rad/s, is attenuated by at least a factor 100 (hundred times smaller).

Possible useful numbers for some problem: $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $\sqrt{5} \approx 2.24$, $\sqrt{7} \approx 2.64$, $\sqrt{11} \approx 3.31$.

1) Compute the forced output response of the system $F(s)$ to the input $u(t)$ with

$$F(s) = \frac{1}{s+1}, \qquad \text{and} \qquad u(t) = \begin{cases} 0 & \text{for} \quad t < 1 \\ t-1 & \text{for} \quad 1 \le t < 2 \\ -t+3 & \text{for} \quad 2 \le t < 3 \\ 0 & \text{for} \quad t \ge 3 \end{cases}$$

1 - Sol.) The input $u(t)$ is shown in Fig. 1 and can be rewritten as the sum of delayed signals

$$u(t) = (t-1)\delta_{-1}(t-1) - 2(t-2)\delta_{-1}(t-2) + (t-3)\delta_{-1}(t-3)$$

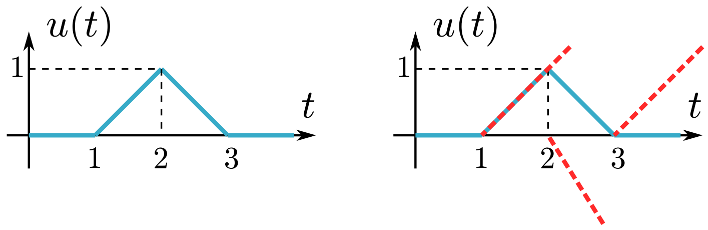

Figure 1: Exercise 1: input $u(t)$

Thanks to the translation theorem in s, we just need to compute the output response to the signal

$$u_0(t) = t\delta_{-1}(t)$$

that is the inverse Laplace transform of

$$Y(s) = F(s)u(s) = \frac{1}{s^2(s+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} = \frac{s^2(a+c) + s(a+b) + b}{s^2(s+1)}$$

which gives $a = -1$, $b = 1$ and $c = 1$. We could have used the residues formula alternatively with

$$a = \left[\frac{d}{ds}\left\{s^2 \frac{1}{s^2(s+1)}\right\}\right]_{s=0} = \left[\frac{-1}{(s+1)^2}\right]_{s=0} = -1, \qquad b = \left[\frac{1}{s+1}\right]_{s=0} = 1, \qquad c = \left[\frac{1}{s^2}\right]_{s=-1} = 1$$

Therefore the response $y_0(t)$ to $u_0(t)$ is

$$y_0(t) = (a + bt + e^{-t})\delta_{-1}(t)$$

and the response to $u(t)$ (translation theorem) is

$$y(t) = y_0(t-1) - 2y_0(t-2) + y_0(t-3)$$

Note that the step function is in $y_0(t)$, so there is no need to write $y_0(t-1)\delta_{-1}(t-1)$ and so on.

Typical errors:

- Wrong decomposition of $u(t)$ as a linear combination of functions whose Laplace transform is immediate.
- Wrong expression for the Laplace transform of $t\delta_{-1}(t)$.
- Some have considered the time function in each single interval, for example $t-1$ for $t \in [1, 2)$ and computed the Laplace transform using (erroneously) the translation theorem in s, that is $u_2(s) = e^{-s}/s^2$ (or even simply $1/s^2 - 1/s$) and then computing the output as $P(s)u_2(s)$. This wrong reasoning also leads to having the output being identically 0 after $t = 3$ since the input is 0.

2) Let the plant be represented by the transfer function $P(s) = 10\frac{(s+10)^2}{(s-10)^3}$

1. Establish if it is possible to stabilize the given plant with the static controller $C(s) = K$ (in a unit feedback interconnection).
2. Draw the corresponding root locus and determine the values of the gain K for which some closed loop poles cross the imaginary axis.

3. Determine, on the basis of the previous analysis, if the closed loop is asymptotically stable for $K = 2$.
4. Confirm the stability analysis (as a function of K) through the Nyquist criterion. The associated Bode plots are required.

2 - Sol.) Since the system is minimum phase and $n - m = 1$, a sufficiently high positive gain will for sure stabilize the closed loop system. Set $K' = 10K$, then the closed loop pole polynomial is

$$p(s, K') = s^3 + s^2(K' - 30) + s(20K' + 300) + 100(K' - 10)$$

or in K

$$p(s, K) = s^3 + s^2(10K - 30) + s(200K + 300) + 1000(K - 1)$$

Setting $K' = 10K$ is of course not necessary for studying stability of the closed loop system through the Routh criterion, however it is important when drawing the root locus, for example when computing the singular points with the simplified formula. The candidate extra singular points (clearly -10 and 10 are singular points) are the solutions of

$$\frac{3}{s-10} - \frac{2}{s+10} = \frac{s+50}{\dots} = 0$$

and being the solution $s^* = -50$ real, it is for sure a singular point. The resulting root locus (positive and negative) is shown in Fig. 2. Note the angles of the branches at the singular points.

The Routh table for $p(s, K')$ (equivalent results would have been obtained with $p(s, K)$) is

$$\begin{array}{|cc} 1 & 20K' + 300 \\ K' - 30 & 100(K' - 10) \\ \alpha & \\ K' - 10 & \end{array}$$

with

$$\alpha = \frac{20(K'^2 - 20K' - 400)}{K' - 30} \qquad \Rightarrow \qquad \alpha = 0 \quad \text{for} \quad K' = 10(1 \pm \sqrt{5})$$

From the root locus it is clear, as predicted by the theory, that a sufficiently high value of K' (and therefore of K) will stabilize the closed loop system. This will correspond to having all the first elements of the first column in the Routh table positive (since the first one is positive), i.e., $K' > \max\{10, 30, 10(1 + \sqrt{5})\} = 10(1 + \sqrt{5})$. We can also deduce directly that the value of K' that nullifies the known term in $p(s, K')$, i.e., $K' = 10$ gives the critical value K'_{c2} corresponding to the crossing of the origin (one of the three closed loop poles will be 0 for $K' = 10$). So one of the three closed loop poles, the one corresponding to the branch that goes from 10 to -10 along the real axis, becomes negative as soon as K' is greater than 10. Moreover, since all three poles have negative real part for $K' > 10(1 + \sqrt{5})$, the other critical value corresponding to two closed poles crossing the imaginary axis is $K'_{c3} = 10(1 + \sqrt{5}) \approx 32.4$. The last crossing corresponding to a negative value of the gain (negative locus, red plot) is the only negative value that nullifies a first element in the first column, that is $K'_{c1} = 10(1 - \sqrt{5}) \approx -12.4$.

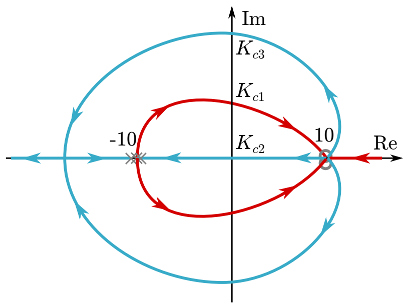

Figure 2: Exercise 2: root locus

For confirmation, it is also possible to count the number of sign changes in the first column from Table 1; note that an intermediate row (in red) has been introduced to help finding the sign of α and should not be considered in counting the number of sign changes.

Analyzing the number of sign changes which denote the number of poles in the right half plane (RHP), we can conclude that

- when K' increases and crosses $K'_{c1} = 10(1 - \sqrt{5})$, 2 poles move in the RHP (# sign changes from 1 to 3)

Table 1: Sign changes in first column of the Routh table as a function of K'

		$10(1-\sqrt{5})$		10		30		$10(1+\sqrt{5})$	
1	+		+		+		+		+
$K'-30$	-		-		-		+		+
$(K'^2-20K'-400)$	+		-		-		-		+
α	-		+		+		-		+
$K'-10$	-		-		+		+		+
# sign changes	1		3		2		2		0

- when K' increases and crosses $K'_{c2} = 10$, 1 pole moves from the RHP to the left half plane (LHP) (# sign changes from 3 to 2)
- when K' increases and crosses $K' = 30$, there are no changes in the number of poles in the RHP and therefore there is no crossing of the imaginary axis for this value of K
- when K' increases and crosses $K'_{c3} = 10(1+\sqrt{5})$, 2 poles moves from the RHP to the LHP (LHP) (# sign changes from 2 to 0).

Therefore the three values of K' corresponding to the crossing of the imaginary axis are now evident. Clearly the closed loop system is asymptotically stable for $K' > K'_{c3}$.

Since $K' = 20$ (corresponding to $K = 2$) is in the interval $(10, 30)$, from the previous table we see there are two poles in the RHP, so the closed loop system is unstable. One could also rewrite the closed loop polynomial for that value of K'

$$p(s, K' = 20) = s^3 - 10s^2 + 500s + 1000$$

and notice that the necessary condition, for all the roots having negative real part, is not satisfied. The Nyquist plot of Fig. 3 drawn for $K = 1$ (from easy Bode plots we see that the magnitude decreases from 1 to 0 while the phase increases from $-\pi$ to $3\pi/2$) shows that there will be three counterclockwise encirclements of the point $(-1, 0)$ for high values of K (larger than K_{c3}) so $N_{cc} = n_L^+$ and the Nyquist criterion is verified.

Typical errors:

- It is true that, since the system is minimum phase, it is possible to stabilize it; however here the controller is just a pure gain, so it it necessary to also recall than $n - m = 1$.
- In the Bode plots, not putting the transfer function in the Bode canonical form and therefore not seeing the negative gain.
- Calculus problems in finding the roots of a second order equation (!), α in the Routh table. Also when you write, for example, $(-2 \pm \sqrt{20})/(-2)$, please simplify it.

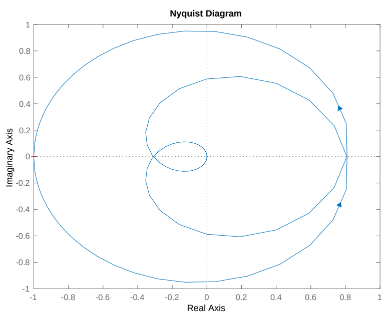

Figure 3: Exercise 2: Nyquist plot

3) Consider the three dynamical systems

$$\mathcal{S}_1 : \begin{cases} \dot{x}_1 = x_1 + u_1 \\ y_1 = x_1 \end{cases} \qquad \mathcal{S}_2 : \begin{cases} \dot{x}_2 = -2x_2 + u_2 \\ y_2 = x_2 \end{cases} \qquad \mathcal{S}_3 : \begin{cases} \dot{x}_3 = 3x_3 + u_3 \\ y_3 = -x_3 \end{cases}$$

which form an interconnected system $\mathcal{S}$ with input u and output y characterized by the following interconnecting equations: $u = u_1$, $u_2 = y_1$, $u_3 = y_1$. The output y will be defined afterwords.

1. Find the A (dynamic matrix) and B (input matrix) matrices of $\mathcal{S}$.
2. Determine the output matrix considering alternatively $y = y_2$ (will define C_2) and $y = y_3$ (will define C_3).
3. We want to stabilize the system $\mathcal{S}$ using the separation principle having available only one output, either $y = y_2$ or $y = y_3$. Discuss the possibility of solving the stabilization problem in both cases.
4. Using the given interconnection equations, draw the corresponding block scheme of $\mathcal{S}$ showing how the systems are interconnected.
5. Based on the previous block scheme, design an alternative stabilizing controller.

3 - Sol.) Choosing as state the vector $x = (x_1, x_2, x_3)$ and using the interconnecting equations, we obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 + u \\ x_1 - 2x_2 \\ x_1 + 3x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u = Ax + Bu$$

Choosing y_2 or y_3 as output we obtain respectively

$$y = y_2 = x_2 \quad \Rightarrow \quad C_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \qquad \text{or} \qquad y = y_3 = -x_3 \quad \Rightarrow \quad C_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$$

We need to check whether the system is stabilizable via state feedback and detectable. Computing the controllability matrix

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 4 \end{pmatrix}, \qquad \text{nonsingular}$$

we see that the system is controllable while both observability matrices are singular

$$O_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 4 & 0 \end{pmatrix}, \quad \text{Ker}[O_2] = \text{gen} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad O_3 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -3 \\ -4 & 0 & -9 \end{pmatrix}, \quad \text{Ker}[O_3] = \text{gen} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and we need to check the asymptotic stability of the unobservable subsystem.

Choosing $y = y_2$ as output, we can choose T^{-1} as the identity matrix and therefore matrix A is already decomposed according to the observability property and the unobservable 1-dimensional subsystem is characterized by the element $(3, 3)$ of A, that is by the unstable eigenvalue $\lambda = 3$. Therefore choosing $y = y_2$ does not guarantee detectability (and we cannot build an asymptotic observer) and we cannot stabilize the system with an output feedback (we have an unstable hidden unobservable dynamics).

Choosing as output $y = y_3$, the change of coordinates can be chosen as

$$T_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = T_3$$

and the dynamic matrix in the new coordinates is

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

which shows that the unobservable subsystem is characterized by the negative eigenvalue $\lambda_2 = -2$ and therefore the system is detectable and we can build a 3-dimensional asymptotic observer. In conclusion, in order to stabilize the system with output feedback, we need to choose the output $y = y_3$.

From the block scheme corresponding to the given interconnecting equations shown in Fig. 4 it turns out that, being the subsystem $\mathcal{S}_2$ asymptotically stable, it can be kept out from the feedback loop and therefore the previous choice $y = y_3$ for the output feedback stabilization problem is evident. This is also telling us that we can "neglect" the subsystem $\mathcal{S}_2$ in the output stabilization problem (considering only the two systems in the blue dashed box as the plant) and therefore we can focus on the transfer function resulting from the series interconnection between $\mathcal{S}_1$ and $\mathcal{S}_3$, that is

$$F(s) = \frac{1}{s-1} \frac{-1}{s-3} = \frac{-1}{(s-1)(s-3)}$$

Note that the first question was asking the state space representation of the whole 3-dimensional system, not only of the series interconnection $\mathcal{S}_1\mathcal{S}_2$ or $\mathcal{S}_1\mathcal{S}_3$.

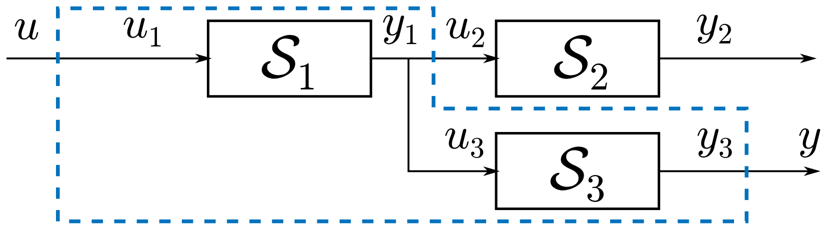

Figure 4: Exercise 3: block scheme

Being $F(s)$ minimum phase with $n - m = 2$ and positive center of asymptotes $s_0 = 2$, we know that a properly chosen zero/pole pair and a sufficiently high gain will stabilize the closed loop system. Note, however, that choosing $C(s) = K$, the loop function becomes

$$L(s) = \frac{-K}{(s-1)(s-3)} = \frac{K'}{(s-1)(s-3)}$$

and therefore we need to look for sufficiently high positive values of K' (corresponding to negative values of K). For example, choosing

$$C(s) = \frac{-K'(s+2)}{s+10}, \qquad \text{with} \quad K' > K'_c$$

will solve the output stabilization problem. Simple computation would give K'_c.

Typical errors:

- The given interconnection equations are not those of a parallel between $\mathcal{S}_2$ and $\mathcal{S}_3$ since there is no equation defining how the output of the interconnected system should be. This is defined in the second part of the problem.
- As already mentioned, you had to find the state space representation of the whole 3-dimensional system and study its controllability and detectability. Only after, noticing the particular structure of the interconnected system, we can "neglect" $\mathcal{S}_2$.
- You should not confuse observability and detectability. For building an asymptotic observer you need detectability, not necessarily the observability of the full system.

4) Consider the plant $P(s)$ and the controller $C(s)$ given by $P(s) = \dfrac{100}{s+100}$ and $C(s) = \dfrac{K}{s}$. Determine, if possible, a value of the controller gain such that, at steady state

- a sinusoidal disturbance, with frequency $\omega \leq 10$ rad/s, acting on the plant's input is attenuated by at least a factor 10 (ten times smaller), and
- a sinusoidal measurement noise, with frequency $\omega \geq 1000$ rad/s, is attenuated by at least a factor 100 (hundred times smaller).

4 - Sol.) First note that the closed loop system is asymptotically stable for any positive value of K. Then recall that the transfer function from the disturbance acting at the input of the plant to the output is given by $W_{d_i y}(s) = P(s)S(s)$ while the complementary sensitivity relates the measurement noise to the output as $-T(s)$. Using the known approximations for $|S(j\omega)|$ and $|T(j\omega)|$, we note that $K = 100$ solves both requirements. In particular one notes that for the sinusoidal input disturbance frequency range, the magnitude of the plant is equal to 1 and therefore

$$|S(j\omega)P(j\omega)| = |S(j\omega)| \qquad \text{for} \qquad \omega \leq 10 \text{ rad/s}$$

A similar reasoning can be done for the approximation of $W_{d_i y}$ using the approx of S.

Typical errors:

- Not properly considering the transfer function from the disturbance at the input of P to the output. Some have considered just the sensitivity function $S(s)$ instead of $S(s)P(s)$.
- Not studying the stability of the closed loop system. For $K = 100$ the closed loop could have been unstable and therefore no steady state would exist.
- Many., when drawing the Bode plots, write on the axis $|F(s)|_{\text{dB}}$ instead of $|F(j\omega)|_{\text{dB}}$ (similarly for the phase).

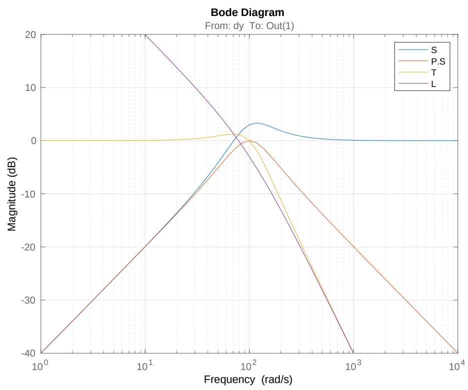

Figure 5: Exercise 4: sensitivies for $K = 100$