Control Systems - January 08, 2024

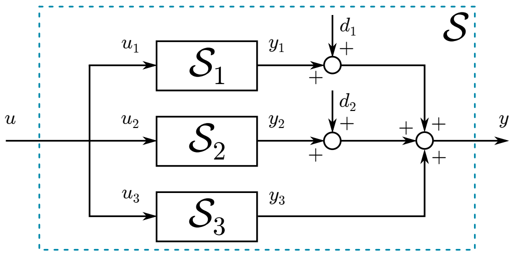
Student name: ______________________ Matricola: __________

1) Let the plant be

$$P(s) = \frac{0.1}{s(s+0.1)}$$

1. Design a controller and a control scheme which guarantee a crossover frequency $\omega_c^* = 1$ rad/s and a phase margin of at least $PM^* = 30°$.
2. Draw qualitatively the control sensitivity function magnitude and compute the control input at steady state corresponding to a reference input $r(t) = t\delta_{-1}(t)$

2) Consider the system $\mathcal{S}$ shown in figure



$$\mathcal{S}_1 : \begin{cases} \dot{x}_1 = -x_1 + u_1 \\ y_1 = x_1 \end{cases}$$

$$\mathcal{S}_2 : \begin{cases} \dot{x}_2 = -x_2 - u_2 \\ y_2 = -2x_2 \end{cases}$$

$$\mathcal{S}_3 : \begin{cases} \dot{x}_3 = -x_3 + 2u_3 \\ y_3 = -2x_3 \end{cases}$$

where d_1 and d_2 are unknown constant disturbances.

1. Find the state space representation of the system $\mathcal{S}$ when $d_1 = d_2 = 0$.
2. Find the eigenvalues together with their algebraic and geometric multiplicity.
3. Study explicitly controllability and observability of the system $\mathcal{S}$.
4. Using a feedback control scheme (which should be drawn), establish if it is necessary that the controller has a pole in $s = 0$ to make the control scheme astatic w.r.t. constant unknown disturbances d_1 and d_2.
5. Find a controller which guarantees astatism w.r.t. constant unknown disturbances d_1 and d_2.
6. Compute the value at steady state of y_1 when a constant non-zero disturbance d_1 is present and $d_2 = 0$.

3) Design a control system which stabilizes the plant $P(s)$ and ensures that the output is capable of reproducing precisely a reference ramp $r(t) = t\delta_{-1}(t)$ at steady state

$$P(s) = \frac{(s+1)(s-10)}{s(s+10)}.$$

4) Design, if possible, a stabilizing dynamic controller based on the separation principle for the system

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 \end{pmatrix}, \qquad D = 0.$$

You need to give the final expression of the dynamic controller (state space or transfer function, you decide).

1 - Sol.) Since there are no steady state requirements (except stability), the plant coincides with the "modified plant" that is $\widehat{P}(s) = P(s)$ and therefore we can proceed to the Bode plots (see Fig. 1) to check if the other requests are already met (and implicitly also closed loop stability if we use Bode's stability theorem). After writing the transfer function in its canonical form

$$P(s) = \frac{1}{s(1+10s)}$$

we see that we need an amplification of 20 dB and a phase increase of at least 30°.

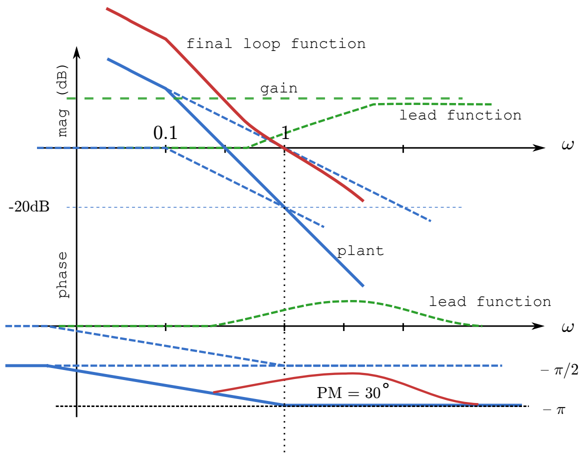


Figure 1: Problem 1: Bode plots of $P(j\omega)$

We focus on the necessary phase lead, the amplification will be adjusted with the controller gain K_c. This can be obtained, for example, with a lead function characterized by $m_a = 4$ and normalized frequency 1 (many different choices are possible), that is $\tau_a = 1$. The lead function gives also an amplification of roughly 2.5 dB. The remaining amplification is given by a gain K_c such that $K_c|_{dB} = 17.5$ dB that is $K_c = 10^{\frac{17.5}{20}}$. Approximate magnitude and phase of the lead function and the gain are also shown in Fig. 1. We are in the conditions for the applicability of Bode's stability theorem so the closed loop system is asymptotically stable.

The final controller is

$$C(s) = \frac{K_c(1+\tau_a\, s)}{1+\tau_a/m_a\, s} = \frac{10^{\frac{17.5}{20}}(1+s)}{1+1/4\, s}$$

The control sensitivity function magnitude is shown in Fig. 2 using the usual approximation. In order to compute the steady state control input (steady state exists since the closed loop system is asymptotically stable) corresponding to $r(t) = t\delta_{-1}(t)$ (which has Laplace transform $1/s^2$), we can first write the control sensitivity function as

$$\begin{aligned} S_u(s) &= \frac{C(s)}{1+C(s)P(s)} = \frac{N_c(s)/D_c(s)}{1+N_c(s)N_p(s)/(D_c(s)D_p(s))} = \frac{N_c(s)D_p(s)}{D_c(s)D_p(s)+N_c(s)N_p(s)} \\ &= \frac{K_c(1+\tau_a\, s)s(s+0.1)}{s(s+0.1)(1+\tau_a/m_a\, s)+0.1K_c(1+\tau_a\, s)} \end{aligned}$$

Using the final value theorem (we can use it since the expression of m_{ss} has all the roots of the denominator with real parte less than 0), we have

$$m_{ss} = \lim_{s\to 0} sS_u(s)\frac{1}{s^2} = \lim_{s\to 0} s\frac{K_c(1+\tau_a\, s)s(s+0.1)}{s(s+0.1)(1+\tau_a/m_a\, s)+0.1K_c(1+\tau_a\, s)}\frac{1}{s^2} = 1$$

Other solution: cancel the stable pole and manipulate with gain then add HF pole

Typical errors:

- After choosing the lead function and the gain one should always discuss closed loop stability and many forgot.
- It is tempting to choose a unique lead function which not only gives the desired phase lead but simultaneously also the required amplification (for example with $m_a = 12$ and normalized frequency $\omega\tau = 20$) but we also know that this choice is to be avoided due to low robustness issues (see slides `Lec16_Loop_Shaping.pdf`).

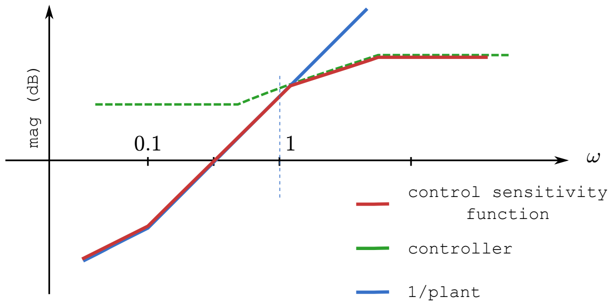


Figure 2: Problem 1: Control sensitivity function magnitude (approximate) $|S_u(j\omega)|$

- Still some evaluate the required phase and magnitude action at the current crossover frequency instead of at the desired one.
- Some read "sensitivity function" instead of "control sensitivity function" ...

2 - Sol.) The three systems are in parallel and when we set $d_1 = d_2 = 0$, the interconnection equations are $u = u_1 = u_2 = u_3$ and $y = y_1 + y_2 + y_3$. Choosing as state vector $x = (x_1,\, x_2,\, x_3\,)^T$ we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_1 + u \\ -x_2 - u \\ -x_3 + 2u \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} u = Ax + Bu$$

with output

$$y = \begin{pmatrix} 1 & -2 & -2 \end{pmatrix} x = C\,x$$

Being the dynamic matrix diagonal, the eigenvalues are evident and we have three coincident eigenvalues in $\lambda_1 = -1$ (and therefore algebraic multiplicity is equal to 3, $m_a(\lambda_1 = -1) = 3$) and a characteristic polynomial $p_A(\lambda) = (\lambda + 1)^3$. Since the matrix is already diagonal the geometric multiplicity clearly equals the algebraic one, $m_g(\lambda_1 = -1) = 3$. As a check, we could compute the dimension of the $\mathrm{Ker}(A - \lambda_1 I)$ which obviously turns out to be the null matrix and therefore its Kernel is generated by the whole state space and has dimension 3.

Although we already know from the theory that this interconnection leads to uncontrollable and unobservable dynamics (of the common eigenvalues) we have seen only the case of two systems in parallel and therefore here we do not know if the unobservable and uncontrollable subsystem has dimension 1 or 2. We can therefore compute the following controllability and observability matrices

$$P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{pmatrix}, \quad \mathrm{rank}[P] = 1, \qquad O = \begin{pmatrix} 1 & -2 & -2 \\ -1 & 2 & 2 \\ 1 & -2 & -2 \end{pmatrix}, \quad \mathrm{rank}[O] = 1$$

which both have rank equal to 1 (the matrix is singular, all minors of order 2 are zero and there are matrix elements – minors of order1 – non-null). Therefore the uncontrollable subsystem and the unobservable one have both dimension 2 (the dimension of the $\mathrm{Ker}[O]$ is 2 thanks to the rank-nullity theorem). This result is expected since the index of the eigenvalue $\lambda_1 = -1$ is 1 and therefore already in $(sI - A)^{-1}$ the common denominator is going to be $(s + 1)$; of the three coincident eigenvalues only one will potentially become a pole of the transfer function (see the slides `Lec07_Laplace_Analysis.pdf`).

In order to understand how the disturbances affect the output, we can transform the block scheme representing system $\mathcal{S}$ noticing that the output is given by

$$y = (y_1 + d_1) + (y_2 + d_2) + y_3 = (y_1 + y_2 + y_3) + (d_1 + d_2)$$

and therefore the effect of the disturbances on the output is equivalent to a single disturbance $d_1 + d_2$ acting directly at the output of the three systems in parallel as illustrated in Fig. 3. We have therefore that the system $\mathcal{S}$ of Fig. 3 is represented by the transfer function

$$F(s) = F_1(s) + F_2(s) + F_3(s) = \frac{1}{s+1} + \frac{2}{s+1} + \frac{-4}{s+1} = \frac{-1}{s+1}$$

where $F_i(s)$ is the transfer function of the single system $\mathcal{S}_i$, $i = 1, ..., 3$.

Now we are in a situation already considered by the theory and therefore, provided the closed loop system is asymptotically stable, a pole in $s = 0$ in the controller guarantees astatism w.r.t. constant disturbances. Since

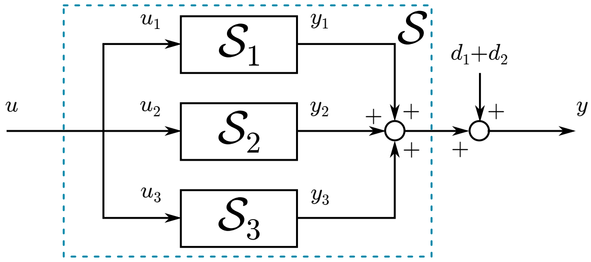


Figure 3: System $\mathcal{S}$

the sensitivity function $S(s)$ relates $d_1 + d_2$ (an output disturbance) to y, highlighting the pole at the origin in the controller $C(s) = C'(s)/s$ we have

$$S(s) = \frac{1}{1 + C(s)(F_1(s) + F_2(s) + F_3(s))} = \frac{s}{s + C'(s)(F_1(s) + F_2(s) + F_3(s))} \implies S(0) = 0$$

and therefore, since we know that for an asymptotically stable system the steady state response to a constant input $d_1 + d_2$ is given by the transfer function gain times the input, we have $S(0)(d_1 + d_2) = 0$.

Alternatively we can proceed algebraically and compute the two transfer functions from d_1 and d_2 to y. We can first write looking at the original block scheme in a feedback control system (with $u = -C(s)y$ since we have no reference)

$$y = (y_1 + d_1) + (y_2 + d_2) + y_3 = d_1 + d_2 + (F_1(s) + F_2(s) + F_3(s))u = d_1 + d_2 + (F_1(s) + F_2(s) + F_3(s))C(s)y$$

where all the signals are intended as Laplace transforms. This leads to

$$(1 + (F_1(s) + F_2(s) + F_3(s))C(s))y = d_1 + d_2$$

In order to compute the $d_1 \to y$ transfer function, we set $d_2 = 0$ (thanks to the superposition principle) and obtain

$$W_{d_1 y}(s) = \frac{1}{1 + C(s)(F_1(s) + F_2(s) + F_3(s))} = S(s)$$

and similarly for the $d_2 \to y$ transfer function, setting $d_1 = 0$ we have

$$W_{d_2 y}(s) = \frac{1}{1 + C(s)(F_1(s) + F_2(s) + F_3(s))} = S(s)$$

Note, however, that if we set the controller just equal to $C(s) = 1/s$, the closed loop is not asymptotically stable since

$$S(s) = \frac{1}{1 + C(s)(F_1(s) + F_2(s) + F_3(s))} = \frac{s}{s - \frac{1}{s+1}} = \frac{s}{s^2 + s - 1}$$

Being the temporary loop function with the necessary part of the controller

$$\frac{1}{s}\frac{-1}{s+1}$$

we have $n - m = 2$, no zeros and a negative center of asymptotes. Since we have to rewrite the function in the form

$$K\frac{\prod(s - z_i)}{\prod(s - p_j)} = -K\frac{1}{s(s+1)} = K_c\frac{1}{s(s+1)}$$

we can now state that a large in magnitude and negative gain $-K$ will certainly stabilize the closed loop system. In other words we have

$$C(s) = \frac{-K}{s} \quad \text{with} \quad K > 0 \qquad \implies \qquad L(s) = C(s)P(s) = \frac{K}{s(s+1)}$$

and the closed loop pole polynomial $p_{CL}(s, K) = s^2 + s + K$ has solutions in the left half-plane if and only if $K > 0$ (so in this case we do not even need a large magnitude gain).

Now that we have an asymptotically stable closed loop system (see Fig. 4) that also guarantees astatism, we can compute the value at steady state of y_1 when $d_2 = 0$. So we need to compute the transfer function $d_1 \to y_1$. We can proceed algebraically with the following steps:

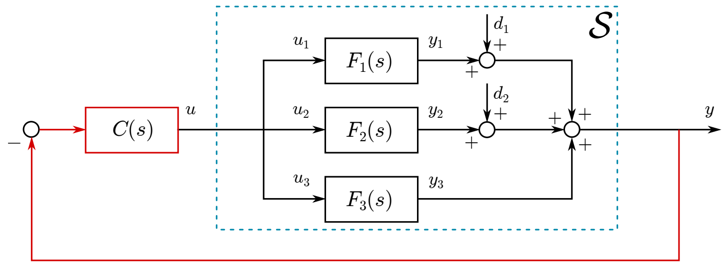


Figure 4: Control scheme

- find u from

$$u = -C(s)\,y = -C(s)(y_1 + d_1 + y_2 + y_3) = -C(s)(y_1 + d_1 + F_2(s)u + F_3(s)u)$$

  that is

$$u = \frac{-C(s)(y_1 + d_1)}{1 + (F_2(s) + F_3(s))C(s)}$$

- and finally obtain y_1 from

$$y_1 = F_1(s)u = \frac{-C(s)F_1(s)(y_1 + d_1)}{1 + (F_2(s) + F_3(s))C(s)}$$

  that is

$$y_1 = \frac{-F_1(s)C(s)}{1 + (F_1(s) + F_2(s) + F_3(s))C(s)} d_1$$

Since the closed loop system is asymptotically stable, the steady state value of y_1 to the constant disturbance d_1 is

$$y_{1,ss} = \left. \frac{-F_1(s)C(s)}{1 + (F_1(s) + F_2(s) + F_3(s))C(s)} \right|_{s=0} d_1 = \left. \frac{\frac{K}{s+1}}{s + \frac{K}{s+1}} \right|_{s=0} d_1 = 1 \times d_1 = d_1$$

An interesting aspect which is clear if we move the disturbances outside the parallel interconnection (as already shown previously), is to notice that the value $y_{1,ss}$ at steady state does not cancel d_1 (as would normally happen for a standard output disturbance when the control system is astatic) even when $d_2 = 0$. This is due to the fact that the disturbance d_1 also affects y_2 and y_3 due to the feedback. Figure 5 shows a simulation with the three outputs y_1, y_2 and y_3 when a disturbance $d_1 = -5\delta_{-1}(t)$ is applied and $d_2 = 0$.

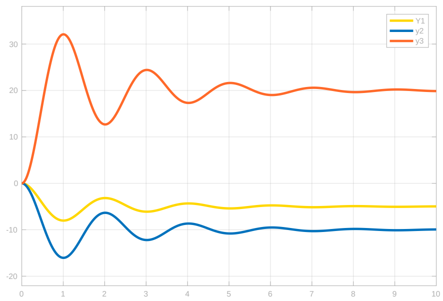
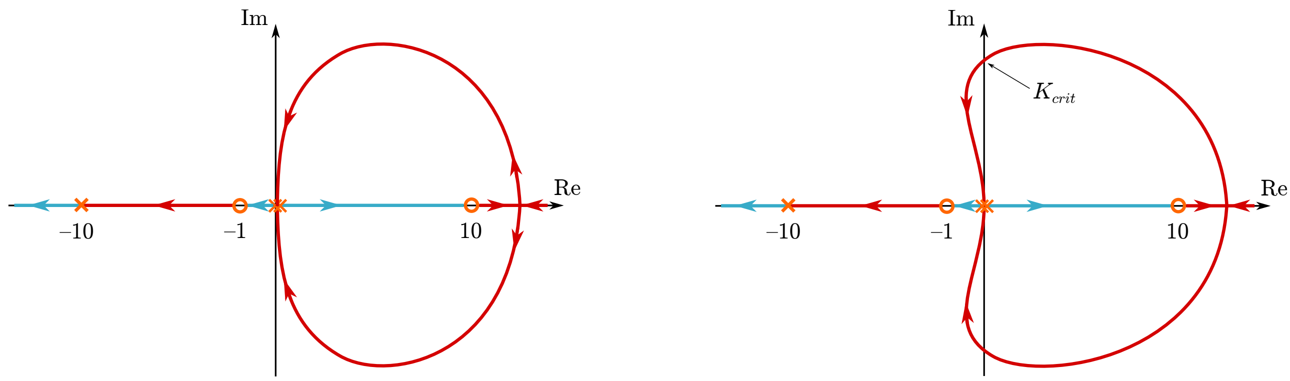


Figure 5: The three outputs $y_1(t)$, $y_2(t)$ and $y_3(t)$ when $d_1 = -5\delta_{-1}(t)$ and $d_2 = 0$

The plot shows how all three outputs $y_i(t)$ contribute to make the output y independent from the value of the disturbance (astatism), not just $y_{1,ss}$ and also that the disturbed output $y_1 + d_1$ does not tend to zero at steady state (in the simulation it tends to -10). All outputs contribute to counterbalance the disturbance(s), it's not only y_1 since also y_2 and y_3 are affected by the disturbance d_1 (even when $d_2 = 0$).

Typical errors:

- Some wrote that the geometric multiplicity is $m_g(\lambda_1 = -1) = 1$ (wrong); the dynamic matrix is already diagonal and this can happen if and only if $m_a(\lambda_i) = m_g(\lambda_i)$ for all eigenvalues. (Study theory).
- You have a diagonal matrix with 3 coincident eigenvalues and the rank of the controllability is 1 which means that the controllable subsystem has dimension 1 and the uncontrollable 2. Is there a need to make a decomposition? You will find that both the controllable and the uncontrollable subsystem are characterized by the eigenvalue $\lambda_1 = -1$.



(a) Root locus 1: no stabilizing K (b) Root locus 2: there is an interval of stabilizing K

Figure 6: Exercise 2 - Two alternative root loci.

- Some spent time to compute the transfer function (when $d_1 = d_2 = 0$) from the state space representation while it is more evident and easier from the parallel of the 3 transfer functions (being one dimensional systems these were evident).
- It should be clear that the input to system is represented by u so it is not possible to control separately (as has been wrongly suggested in some solutions) each subsystem by placing a controller $C_i(s)$ on each branch.
- When computing the steady state value $y_{1,ss}$ an interesting error is: since we made the output astatic, the steady state value of the output when only the constant disturbance is applied is 0 (true). Therefore when we compute $y_1(s)$ and write the algebraic relationship

$$y_1(s) = F_1(s)u_1(s) = F_1(s)u(s) = -F_1(s)C(s)y(s)$$

one could think that being at steady state $y(t) = 0$ then also $y_{1,ss}(t)$ would be zero. However one forgets the presence of the pole is $s = 0$ (i.e., an integrator) and although at steady state the input of the controller is 0, the output $u(t)$ is not (see the last slides of `Performance.pdf`).

3 - Sol.) Being the reference of order 1, in order to have a 0 steady state tracking error the control system needs to be at least of type 2 and therefore the controller needs a pole in $s = 0$. The resulting modified plant

$$\widehat{P}(s) = \frac{(s+1)(s-10)}{s^2(s+10)}$$

is such that $n - m = 1$ but since the system is non-minimum phase due to the presence of the positive zero in $s = 10$, it is not possible to stabilize the plant with high positive gain (which does not mean we cannot stabilize the system with a pure gain). However, plotting the root locus (that is verifying if a simple gain is sufficient to stabilize the control system) we could have two possible situations shown in Fig. 6. From the Routh criterion applied to the closed loop polynomial

$$p(s, K) = s^2(s + 10) + K(s + 1)(s - 10) = s^3 + (10 + K)s^2 - 9Ks - 10K$$

we see that the necessary condition requires $-10 < K < 0$. The Routh table is

$$\begin{array}{|cc} 1 & -9K \\ 10+K & -10K \\ -K(9K+80) & \\ -10K & \end{array}$$

we clearly see that the closed loop is asymptotically stable for $K \in (-80/9, 0)$. This corresponds to the root locus of Fig. 6b. In particular, if we analyze the sign changes of the terms in the first column, we have that the two unstable poles (the third one is always real and negative) move to left half plane when K crosses the value $-80/9$.

The resulting Nyquist plot (for completeness) is shown in Fig. 7.

Other solutions: one could obtain similar results through a loop shaping approach choosing a negative gain and guaranteeing, after the introduction of the necessary pole in $s = 0$ that the conditions for the applicability of Bode's stability conditions were met.

Typical errors:

	−10	-80/9	0	
1	+	+	+	+
$10+K$	-	+	+	+
$-K(9K+80)$	-	-	+	-
$-10K$	+	+	+	-
# sign changes	2	2	0	1

Table 1: Sign changes in first column of the Routh table as a function of K

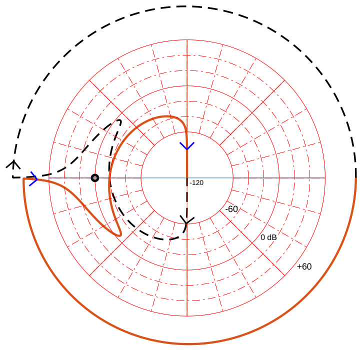


Figure 7: Nyquist plot in a log scale

- One of the worse error is to choose a controller that cancels the non-minimum phase zero and thus creates hidden unstable dynamics which cannot be modified by any output feedback controller.

4 - Sol.) Let us first study controllability and observability

$$\text{rank}[P] = \text{rank}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 1, \qquad \text{rank}[O] = \text{rank}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 1$$

so the system has an uncontrollable eigenvalue and an unobservable eigenvalue (they may not coincide in general). The eigenvalues are easily found to be $\lambda_1 = 1$ and $\lambda_2 = -2$ since the characteristic polynomial is

$$p_A(\lambda) = \det\begin{pmatrix} \lambda+1 & -2 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + \lambda - 2 = (\lambda-1)(\lambda+2)$$

So in order for the system to be stabilizable by state feedback (necessary condition for output stabilizability) and detectable, the eigenvalue $\lambda_1 = 1$ needs to be controllable and observable which can be checked through the PBH tests

$$\text{rank}\begin{pmatrix} A-\lambda_1 & B \end{pmatrix} = \text{rank}\begin{pmatrix} -2 & 2 & 2 \\ 1 & -1 & 2 \end{pmatrix} = 2, \qquad \text{rank}\begin{pmatrix} A-\lambda_1 \\ C \end{pmatrix} = \text{rank}\begin{pmatrix} -2 & 2 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} = 2,$$

Therefore the asymptotically stable eigenvalue $\lambda_2 = -2$ will be both uncontrollable and unobservable. We can perform the Kalman decomposition w.r.t. controllability

$$\text{Im}[P] = \text{gen}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \Rightarrow \quad T^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and therefore

$$\tilde{A} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} 3 & 2 \end{pmatrix}$$

Assigning $\lambda_1^* = -1$ leads to

$$\tilde{F} = \begin{pmatrix} -1 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad F = \tilde{F}T = \begin{pmatrix} -1 & 0 \end{pmatrix}$$

Let's check

$$A + BF = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad p_{A+BF}(\lambda) = (\lambda+1)(\lambda+2)$$

For the design of the observer, since we know the system is detectable (and that the unobservable eigenvalue is $\lambda_2 = -2$), we can either proceed algebraically or do the Kalman decomposition w.r.t. observability. In the first case, let's define

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \quad \rightarrow \quad A - KC = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 - k_1 & 2 - 2k_2 \\ 1 - k_2 & -2k_2 \end{pmatrix}$$

which has characteristic polynomial

$$p_{A-KC}(\lambda) = \det \begin{pmatrix} \lambda + 1 + k_1 & -2 + 2k_2 \\ -1 - +k_2 & \lambda + 2k_2 \end{pmatrix}$$

We know one of the two eigenvalues is $\lambda_2 = -2$ and the other one is to be chosen, for example $\lambda^* = -10$ so that we should have

$$p_{A-KC}(\lambda) = \det \begin{pmatrix} \lambda + 1 + k_1 & -2 + 2k_2 \\ -1 - +k_2 & \lambda + 2k_2 \end{pmatrix} = (\lambda + 2)(\lambda + 10)$$

which we should solve in k_1 and k_2 (long computation ...).

Doing instead the decomposition w.r.t. observability we have

$$\text{Ker}[O] = \text{gen} \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \quad \Rightarrow \quad T^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = T$$

and therefore

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Assigning for example $\lambda^* = -10$ leads to

$$\tilde{K} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad K = T\tilde{K} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$$

Let's check

$$A - KC = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 11 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} -12 & -20 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad p_{A+BF}(\lambda) = (\lambda + 10)(\lambda + 2)$$

The observer is therefore

$$\dot{\xi} = (A - KC)\xi + Bu + Ky = \begin{pmatrix} -12 & -20 \\ 1 & 0 \end{pmatrix} \xi + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u + \begin{pmatrix} 11 \\ 0 \end{pmatrix} y$$

and the controller using $u = F\xi$ has dynamic matrix

$$A + BF - KC = \begin{pmatrix} -14 & -20 \\ -1 & 0 \end{pmatrix}, \quad \Rightarrow \quad p_{A+BF-KC}(\lambda) = \lambda^2 + 14\lambda - 20$$

which has eigenvalues

$$\lambda_1 = -7 + \sqrt{69} = 1.3066, \quad \lambda_2 = -7 - \sqrt{69} = -15.3066$$

and transfer function

$$C(s) = F(sI - (A + BF - KC))^{-1}K = \cdots = \frac{-11s}{s^2 + 14s - 20}$$

As a check, the loop function is $L(s) = C(s)P(s)$ but since we are doing a positive feedback any transfer function of the closed loop will have $1 - L(s)$ at the denominator and therefore the pole polynomial will be $D_L(s) - N_L(s)$ that is

$$D_L(s) - N_L(s) = \cdots = (s + 10)(s + 2)(s + 1)$$

Since there were no cancellations between the controller and the plant but the plant has the eigenvalue $\lambda_2 = -2$ which is not a pole, we end up with three poles (note however that we are not considering any input or output).

Typical errors:

- Some tried to assign the eigenvalue to the one-dimensional subsytem (fo example the controllable one) using the Ackermann formula. This is certainly possible. However some chose to assign 2 eigenvalues (even choosing one of the two coincident with the eigenvalue of the uncontrollable subsystem would be wrong) and chose a second order polynomial for $p^*(\lambda)$.