Control Systems - January 29, 2024

A Student name: ____________________ Matricola: __________

1) A lift to move material is described by the following differential equation

$$m\ddot{p} + d\dot{p} = f - m_\ell\, g$$

where p is the height, m is the total weight of the lift platform m_p plus the weight of the load m_ℓ (i.e., $m = m_p + m_\ell$), d a friction coefficient f a control force provided by a motor and g the gravity acceleration. The weight of the empty lift is counterbalanced mechanically so it does not enter into the gravity term. Moreover the load weight m_ℓ is unknown but can be up to $m_\ell^{\max}$, that is $m_\ell \in [0, m_\ell^{\max}]$ kg

Draw a control scheme and design a controller which guarantees that the load is moved from the ground ($p = 0$) to a given constant height p^d. Assume $g = 10$ m/s^2, $d = 40$ kg/s, the empty lift (platform) weight is $m_p = 10$ kg while the maximum load weight is $m_\ell^{\max} = 5$ kg (in other words, the total mass m belongs to the interval $[m_p, m_p + m_\ell^{\max}] = [10, 15]$ kg).

2) Consider the system with state space representation

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad D = 0$$

1. Study controllability and observability properties.
2. Establish whether the system is stabilizable with state feedback and output feedback.
3. Compute the output impulse response.
4. Compute the transfer function.
5. Find a controller which assigns the bandwidth $B_3 = 10$ rad/s to the closed loop system (reference to output).

3) Consider the system represented by the transfer function

$$P(s) = \frac{1}{s+1}$$

1. Find a control scheme and a controller which guarantees that the output exactly follows the reference $r(t) = 3\sin(t)$ at steady state.
2. With the controller found at the previous point, assume its gain K_c can be changed and is still variable. Draw the positive and negative root locus.
3. With the chosen controller at the first point, draw the Nyquist plot.

4) For the system characterized by the state space representation

$$A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0$$

write the spectral form of the matrix exponential and draw in the (x_1, x_2) plane the typical trajectories arising starting from different initial conditions.

1 - Sol.) Taking the Laplace transform of the differential equation (and starting from 0 initial conditions) we have

$$(m\,s + d)s\,p(s) = f(s) - \mathcal{L}[m_\ell\, g]$$

which shows that gravity is acting as an input disturbance to the plant described by the transfer function

$$P(s) = \frac{p(s)}{f(s) - \mathcal{L}[m_\ell\, g]} = \frac{1}{(ms+d)s}$$

and we can then draw the control scheme of Fig. 1.

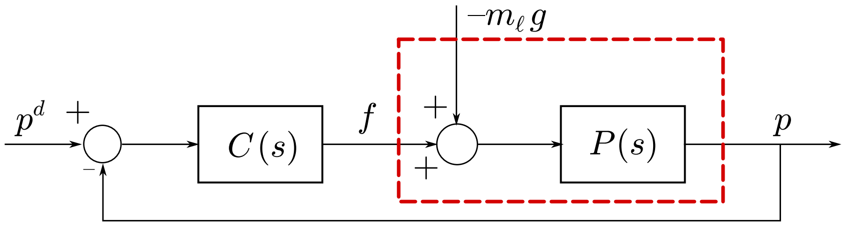


Figure 1: Control scheme for the lift

In order to make the lift reach the desired position at steady state regardless of the constant gravity (we cannot compensate for the gravity term since we do not know the value of m_ℓ, only the platform contribution has been compensated with a counterweight) we need a pole in $s = 0$ in the controller (necessary condition). The system is then of type 2 (provided the closed loop system is asymptotically stable) and therefore one would obtain a zero steady state error w.r.t. a constant reference p^d (when we want the lift to reach the height p^d, the reference changes as a step input from 0 to p^d). The modified plant becomes

$$\widehat{P}(s) = \frac{1}{s}\frac{1}{s(ms+d)}$$

which does not give an asymptotic closed loop system since the pole polynomial would be $s(ms^2 + ds) + 1$ (no term in s so the necessary condition is not verified); same result if we use the controller K_c/s. We need to stabilize the control system.

A first possible choice would be to just add a negative zero in $-z$ (the resulting controller would then be proper since it has already a pole) so to obtain $n - m = 2$. In this case the closed loop system is stabilizable with high positive gain provided the center of asymptotes is negative. The loop function becomes

$$L_1(s) = \frac{K'(s+z)}{s}P(s) = \frac{K'}{m}\frac{(s+z)}{s^2(s+d/m)} = K\frac{(s+z)}{s^2(s+d/m)}, \qquad z > 0, \quad K = \frac{K'}{m}$$

with the center of asymptotes

$$s_0 = \frac{-d/m + z}{2}$$

Since we want to choose $z > 0$ (negative real zero) such that s_0 is negative for all possible values of m, we need

$$0 < z < \frac{d}{m^{\max}} = \frac{d}{m_p + m_\ell^{\max}}$$

We can now choose the value of K by using the Routh criterion to the closed loop pole polynomial

$$p(s, K) = s^2\left(s + \frac{d}{m}\right) + K(s+z) = s^3 + \frac{d}{m}s^2 + Ks + Kz$$

which gives the following Routh table

$$\begin{array}{|cc} 1 & K \\ \frac{d}{m} & Kz \\ \alpha & \\ K & \end{array}$$

with

$$\alpha = K\frac{d}{m} - Kz = K\left(\frac{d}{m} - z\right)$$

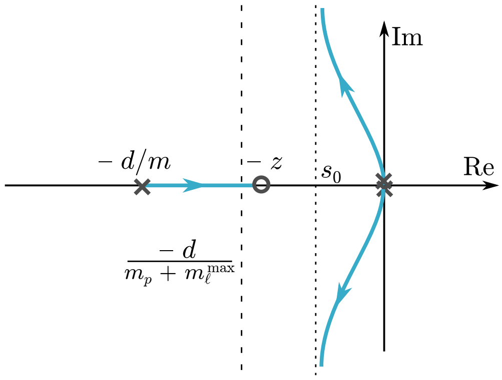


Figure 2: Positive root locus

being positive for the given choice of z, that is $0 < z < \frac{d}{m_p + m_\ell^{\max}}$. In other words, any $K > 0$ would stabilize the closed loop system (with the properly chosen z) and therefore any $K' > 0$ can be chosen. The final controller is therefore

$$C_1(s) = \frac{K'(s+z)}{s}, \qquad \text{with} \qquad 0 < z < \frac{d}{m_p + m_\ell^{\max}} \quad \text{and} \quad K' > 0.$$

The corresponding root locus is shown in Fig. 2.

Note that we cannot choose the zero so to cancel the asymptotically stable pole of the plant since we do not know the value of this pole being the load mass m_ℓ uncertain. However we could choose the zero in

$$z = \frac{d}{m^{\max}} = \frac{d}{m_\ell + m_w^{\max}}$$

that is the one with the smallest cut-off frequency $d/m^{\max}$. With this choice we would obtain a center of asymptotes which is $s_0 \leq 0$. The case $s_0 = 0$ corresponds to the case of zero/pole cancellation when $m = m^{\max}$ so that only the two poles in $s = 0$ remain. In order to avoid $s_0 = 0$ we should add an extra pair of zero/pole. The final controller would be

$$C_2(s) = \frac{(s + d/m^{\max})(s + z_2)}{s(s + p_2)}, \qquad \text{with} \qquad p > z > 0$$

A possible alternative to the root locus reasoning would be to do some sort of loop shaping (that is working on the open loop magnitude and phase) in order to guarantee a positive phase margin and the applicability of Bode's stability theorem. The modified plant in Bode's canonical form is

$$\widehat{P}(s) = \frac{1}{m}\frac{1}{s^2(s + d/m)}$$

In this case a possible Bode plot is shown in Fig. 3 where the smallest smallest value of the gain $1/m$ is reported

$$\min \frac{1}{m} = \frac{1}{m^{\max}} = \frac{1}{m_p + m_\ell^{\max}}$$

together with the smallest crossover frequency $\omega_c^{\min}$ for the range of possible values of the total weight m.

From the phase plot it is clear that if we add a zero in $-d/m^{\max}$ the phase will always be greater or equal to $-\pi$ (equal when the weight m is at its maximum value). This corresponds to the analysis done previously when we added a zero in $-d/m^{\max}$. Adding a zero with cutoff frequency to the left of $d/m^{\max}$ will give a positive phase margin at any frequency (recall that the drawn plots of the phase are approximate) confirming the previous analysis done using the root locus since this corresponds to asking

$$0 < z < \frac{d}{m^{\max}} = \frac{d}{m_\ell + m_w^{\max}}$$

Typical errors:

- Some have considered $f - m_\ell g$ as a known constant which enters in the B matrix as

$$B = \begin{pmatrix} 0 \\ \frac{f - m_\ell g}{m} \end{pmatrix}$$

  clearly not understanding that f is the control input and $m_\ell g$ a constant disturbance acting at the plant's input.

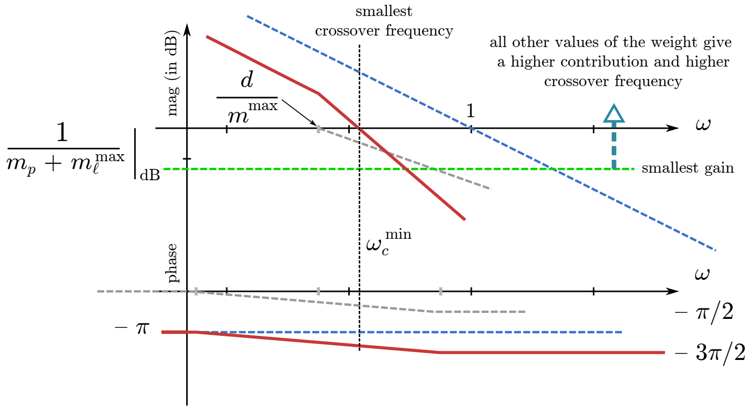


Figure 3: Possible Bode plots for the lift control problem

- It is tempting to denote $u = f - m_\ell g$ and consider u as the control input. However we then lose the fact that $m_\ell g$ is a disturbance and we want the system to be astatic. You could argue that if u is the output of the controller then the real force that should be applied is $f = u + m_\ell g$ but this requires the perfect knowledge of the mass m_ℓ.
- This solution has been given in the general case; some have directly used the numerical values and evaluated the effect of the unknown mass.

2 - Sol.) Due to the block triangular structure of the matrix the eigenvalues are

$$\lambda_1 = -1, \qquad \lambda_{2,3} = \mathrm{eig}\begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix} = \{1, -2\}$$

The controllability and observability matrix are respectively

$$P = \begin{pmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \end{pmatrix}, \quad \mathrm{rank}[P] = 1, \qquad\qquad O = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{rank}[O] = 1$$

therefore we have 2 uncontrollable natural modes and 2 unobservable natural modes (from this analysis we do not know yet if the uncontrollable and unobservable modes coincide).

From the structure of the A and C matrices we can for sure say the eigenvalue $\lambda_3 = -2$ is unobservable, but since the nullspace of O has dimension 2, another eigenvalue is unobservable.

We can therefore compute (although not necessary) the image of P and the kernel of O

$$\mathrm{Im}[P] = \mathrm{gen}\left\{ \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} \right\}, \qquad \mathrm{Ker}[O] = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

but it is not necessary to do the Kalman decompositions; it is sufficient to check the structural properties (controllability and observability) through the PBH test:

- for $\lambda_1 = -1$ we have

$$\mathrm{rank}\begin{pmatrix} 0 & 1 & 0 & | & 3 \\ 0 & 2 & 0 & | & 6 \\ 0 & 2 & -1 & | & 4 \end{pmatrix} = 2, \qquad \mathrm{rank}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & -1 \\ -- & -- & -- \\ 0 & 1 & 0 \end{pmatrix} = 2$$

- for $\lambda_2 = 1$ we have

$$\mathrm{rank}\begin{pmatrix} -2 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & | & 6 \\ 0 & 2 & -3 & | & 4 \end{pmatrix} = 3, \qquad \mathrm{rank}\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -3 \\ -- & -- & -- \\ 0 & 1 & 0 \end{pmatrix} = 3$$

- for $\lambda_3 = -2$ we have

$$\text{rank}\begin{pmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 3 & 0 & | & 6 \\ 0 & 2 & 0 & | & 4 \end{pmatrix} = 2, \qquad \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \\ -- & -- & -- \\ 0 & 1 & 0 \end{pmatrix} = 2$$

- Note that once we find that λ_1 is uncontrollable and λ_2 is controllable, there is no need to check λ_3 which necessarily will be uncontrollable since the rank of the controllability matrix is 2. Similarly for the PBH test for observability. Even better, thinking about the next question, we could just verify the controllability and observability of the only unstable eigenvalue.

We can conclude that the system is characterized by an asymptotically stable uncontrollable and unobservable subsystem with eigenvalues λ_1 and λ_3 and therefore the system is both stabilizable with state feedback (if the state is available) and stabilizable with output feedback (for example through the separation principle since the system is also detectable) recalling that state stabilizability is a necessary condition for output stabilizability.

To compute the output impulse response, we need first to compute the matrix exponential e^{At}; however we can exploit the particular structure of A and C as

$$w(t) = Ce^{At}B = \begin{pmatrix} C_1 & | & 0 \end{pmatrix} e^{\left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right)t} \begin{pmatrix} B_1 \\ \hline 4 \end{pmatrix} = \begin{pmatrix} C_1 & | & 0 \end{pmatrix} \left(\begin{array}{c|c} e^{A_{11}t} & 0 \\ \hline \# & e^{A_{22}t} \end{array}\right) \begin{pmatrix} B_1 \\ \hline 4 \end{pmatrix}$$

$$= C_1 e^{A_{11}t} B_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} e^{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}t} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & ** \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 6e^{t}$$

which is what we expected since λ_2 is the only controllable and observable eigenvalue and therefore the only one to appear in the impulse response. The transfer function, given by the Laplace transform of the impulse response, is

$$W(s) = \mathcal{L}\left[w(t)\right] = \frac{6}{s-1}$$

Using a feedback control scheme, we need to both stabilize the closed loop system which can evidently be achieved with a simple gain $C(s) = K_c$ and assign the required bandwidth. The resulting closed loop pole polynomial would then be

$$p(s, K_c) = s - 1 + 6K_c$$

We then recall that a transfer function with a single pole (and no zeros) has the bandwidth coincident with the pole cut-off frequency. Therefore in order to achieve the required bandwidth of $B_3 = 10$ rad/s for the closed loop we just need to choose $K_c = 11/6$ so that

$$p(s, 11/6) = (s + 10).$$

The closed loop complementary sensitivity function is then

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{K_c P(s)}{1 + K_c P(s)} = \frac{11}{s + 10}$$

Note that the gain of the complementary sensitivity is not 1 (there are no poles in $s = 0$ in the loop function).

As a possible alternative solution one could reason on the loop function crossover frequency and the approximation (it's however an approximation) that the crossover frequency of L and the bandwidth of T are "close". So one could look for a gain that moves the crossover frequency to 10 rad/s (the required closed loop bandwidth). This value is not evident from the handwritten Bode plot of $|P(j\omega)|$ to understand the required amplification $-|P(j10)|_{dB}$. More importantly even if we read correctly this value, we still need to verify closed loop stability (here the Bode theorem cannot be applied since we have an unstable pole in $P(s)$). We can either compute the closed loop polynomial (but then we go back to the first solution) or do the Nyquist plot (do it as an exercise).

For completeness Fig. 4 shows some magnitude plots of interest.

Typical errors:

- Once you have found the eigenvalues (thanks to the particular form of the A matrix) and you notice that the rank of the controllability matrix is 1, you just have to check if the unique unstable eigenvalue is controllable (similarly for the observability).

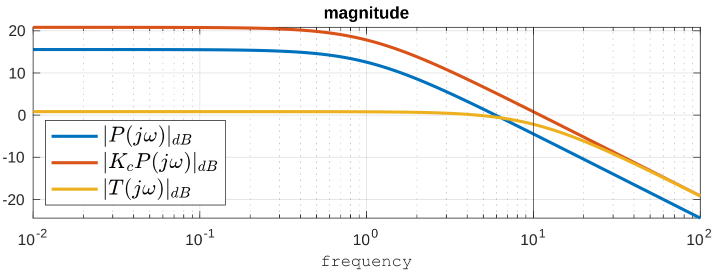


Figure 4: Problem 2: magnitude of $P(j\omega)$, $K_cP(j\omega)$ and $T(j\omega)$ in dB

- Some have found a transfer function (or impulse response) which is not compatible with the controllability and observability analysis (this is considered a serious flaw).
- Writing just "the unobservable subsystem is asymptotically stable and therefore the system can be stabilized via output feedback" is incomplete: you should recall that the stabilizability with state feedback is a necessary condition.
- Recalling the connection between the impulse response and the transfer function, once you compute on of them obtaining the other one is straightforward; no need to do unnecessary computation all over again.
- Some have computed the step response instead of the impulse response.

3 - Sol.) We know that in order to guarantee zero tracking error with respect to a sinusoidal signal, the loop function needs to have (necessary condition) a pair of imaginary poles corresponding to the reference signal frequency; in our case the loop function needs to have the poles $\pm j$ and since they are not present in the plant, we introduce them throught the controller

$$C_{temp}(s) = \frac{1}{s^2+1} \qquad \Longrightarrow \qquad \widehat{P}(s) = C_{temp}(s)P(s) = \frac{1}{(s+1)(s^2+1)}$$

We then have to stabilize the closed loop system.

The first attempt could be done looking at the root locus of $K_c\widehat{P}(s)$ shown in Fig. 5 which indicates that the closed loop system can be stabilized with negative and sufficiently small in magnitude values of K_c.

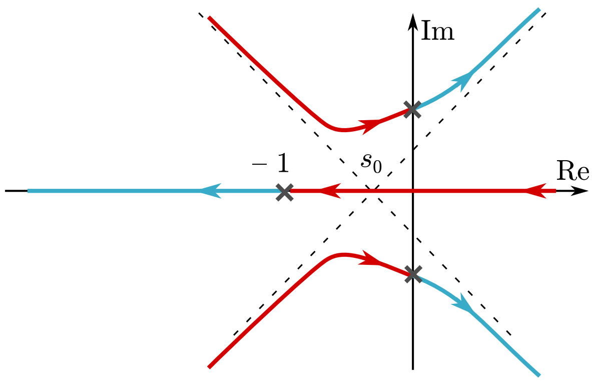


Figure 5: Problem 3: a possible root locus for $C(s)P(s) = K_c/((s^2+1)(s+1))$

Writing the corresponding closed loop polynomial we obtain

$$p(s, K_c) = (s^2+1)(s+1) + K_c = s^3 + s^2 + s + K_c + 1$$

so the necessary condition requires $K_c > -1$. The Routh table is

$$\left|\begin{array}{cc} 1 & 1 \\ 1 & 1+K_c \\ -K_c & \\ 1+K_c & \end{array}\right.$$

so we obtain that the necessary and sufficient condition for closed loop stability is $K_c \in (-1, 0)$. Clearly $K_c = -1$ corresponds to a closed loop real pole being in the origin since

$$p(s, -1) = s(s^2+s+1).$$

One can even compute the candidate singular points as the solutions of

$$\frac{1}{s+1} + \frac{1}{s+j} + \frac{1}{s+j} = 0 \qquad \Leftrightarrow \qquad 3s^2 + 2s + 1 = 0$$

which has complex roots and these roots correspond to complex values of K_c so these are not singular values. Note however that with a different plant, for example

$$P_2(s) = \frac{1}{s+10}$$

all the reasoning is similar but the candidate singular points should instead verify the equation

$$3s^2 + 20s + 1 = 0$$

which has real roots $s_1^* \approx -6.6$ and $s_2^* \approx -0.05$ which are therefore singular points. The corresponding root locus becomes that of Fig. 6.

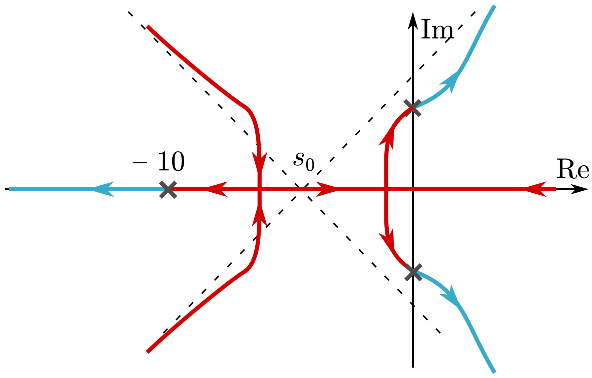


Figure 6: Problem 3: a possible root locus for $C(s)P_2(s) = K_c/((s+10)(s^2+1))$

As an alternative solution, closed loop stability can be achieved, for example, by adding a zero (we can cancel the pole in $s = -1$) and move the center of asymptotes into the left half-plane with a pole/zero pair. For example we could choose

$$C(s) = K_c \frac{(s+1)(s+4)}{(s^2+1)(s+10)}$$

which gives a center of asymptotes $s_0 = -3$. The resulting root locus is shown in Fig. 7.

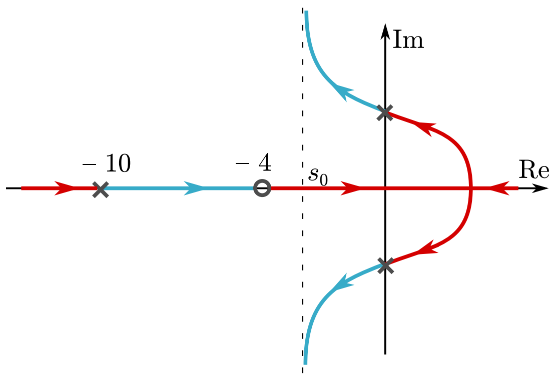


Figure 7: Problem 3: a possible root locus with $C(s) = K_c(s+1)/(s^2+1)$

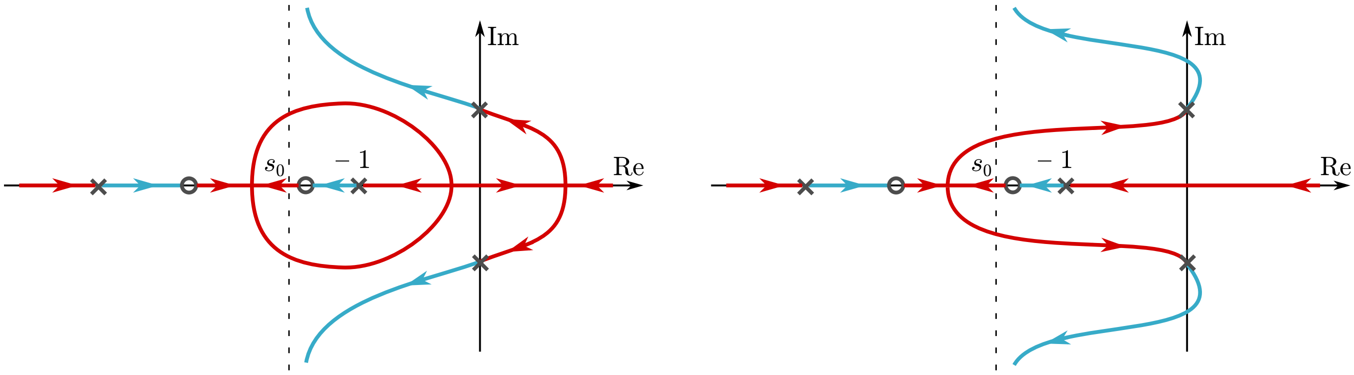
In order to verify that the closed loop system is asymptotically stable for any $K_c > 0$, we can compute the closed loop polynomial

$$L(s) = \frac{K_c(s+4)}{(s^2+1)(s+10)} \qquad \Longrightarrow \qquad p(s, K_c) = s^3 + 10s^2 + (K_c+1)s + 10 + 4K_c$$

The necessary condition for all poles to be with real part negative is $K_c > -1$, while from the following Routh table (where some rows have been multiplied by a positive number to simplify the computation) it is clear that the necessary and sufficient condition is satisfied for $K_c > 0$.

1	K_c+1
10	$10+4K_c$
$6K_c$	
$10+4K_c$	

If the asymptotically stable pole is not cancelled by the additional zero we obtain two possible root locus plots shown in Fig. 8. Both cases are possible, depending on the location of the additional zero in $C(s)$. In the first case the closed loop system would be asymptotically stable for any $K_c > 0$ while in the second case only for $K_c > K_{\text{crit}} > 0$. The closed loop polynomial is now a 4th order polynomial and the corresponding Routh table is more computationally intensive.



(a) A possible root locus

(b) Another possible root locus

Figure 8: Problem 3 - Two alternative root loci whene adding an extra pole/zero pair.

These are some of the possible controllers which all guarantee exact tracking of the particular sinusoidal reference signal at steady state. These controllers differ with respect to the transient behavior as shown in Fig. 9 where the tracking error $e(t) = y(t) - r(t)$ is shown fro the three following controllers

$$C_1(s) = -\frac{1}{2}\frac{1}{s^2+1}, \qquad C_2(s) = 26\frac{(s+1)(s+5)}{(s^2+1)(s+10)}, \qquad C_3(s) = 300\frac{(s+5)(s+10)}{(s^2+1)(s+20)}$$

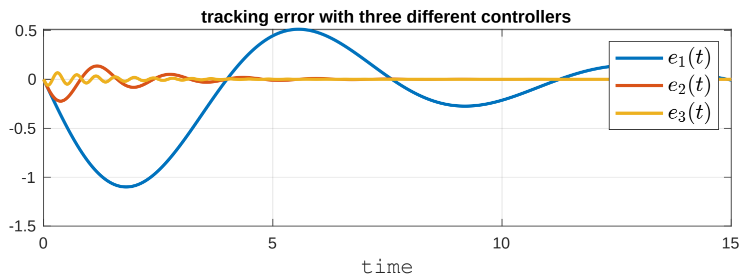
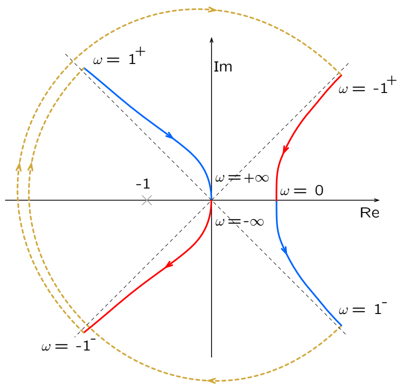


Figure 9: Problem 3: tracking errors with $C_1(s)$, $C_2(s)$ and $C_3(s)$

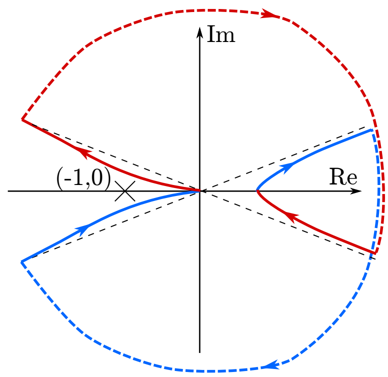
The Nyquist plot for the pure gain controller is shown in Fig. 10a (this is one of the examples in the slides `Nyquist.pdf`). When K_c is negative (as the case for the stabilizing values of K_c) the plot rotates of $-\pi$ and therefore there are no encirclements of the point $(-1, 0)$ for small (in absolute value) negative values of K_c. With the other controller $C_2(s)$ we obtain the Nyquist plot shown in Fig.10b. The Nyquist plot with $C_3(s)$ is similar.

Typical errors:

- Some have chosen the necessary part of the controller to be $3/(s^2+1)$ and not $1/(s^2+1)$ since the reference was $3\sin t$.
- The presence of poles in $\pm j$ is a necessary condition; stability still needs to be guaranteed.
- Some have just added a pole in $s = 0$ ...
- Usual problems with the Nyquist plot (absence of the closure at infinity, ...).



(a) Problem 3: Nyquist plot for $C_1(s)$



(b) Problem 3: Nyquist plot for $C_2(s)$

4 - Sol.) The eigenvalues of the dynamic matrix A are

$$p_A(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda+1 & 1 \\ 1 & \lambda+1 \end{pmatrix} = \lambda(\lambda+2) \qquad \Longrightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -2$$

and therefore the eigenvectors are

- for $\lambda_1 = 0$, the eigenvectors are such that $(A - \lambda_1 I)u_1 = Au_1 = 0$

$$Au_1 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} u_1 = 0 \qquad \Longrightarrow \qquad \mathcal{U}_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- for $\lambda_2 = -2$, the eigenvectors are such that $(A - \lambda_2 I)u_2 = 0$

$$(A - \lambda_2 I)u_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} u_2 = 0 \qquad \Longrightarrow \qquad \mathcal{U}_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Choosing the right eigenvectors as

$$u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

the left eigenvectors v_i^T which will be used in the spectral form should be such that $v_i^T u_j = \delta_{ij}$ that is

$$\begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad v_1^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad v_2^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We can now write the spectral form of the matrix exponential e^{At}

$$\begin{aligned} e^{At} &= e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T = e^{0t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{-2t} \\ &= \frac{1}{2} \begin{pmatrix} 1 + e^{-2t} & -1 + e^{-2t} \\ -1 + e^{-2t} & 1 + e^{-2t} \end{pmatrix} \end{aligned}$$

We can now represent the two eigenspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ in the (x_1, x_2) plane together with some typical free evolutions from different initial conditions. The result is shown in Fig. 11. We clearly have that initial conditions belonging to $\mathcal{U}_1$ lead to degenerate state evolutions (the state remains at the initial condition). This is consistent with the fact that every state belonging to $\mathcal{U}_1$ is an equilibrium point.

Typical errors:

- The choice for the left eigenvectors v_i^T, when using the spectral decomposition, should be such that $v_i^T u_j = \delta_{ij}$.
- Some have obtained at the end of the computation a scalar function for e^{At} while it is a 2×2 matrix since A is a square 2-dimensional matrix.
- Recall that the product u_i (column vector) with v_i^T (row vector) gives a matrix.

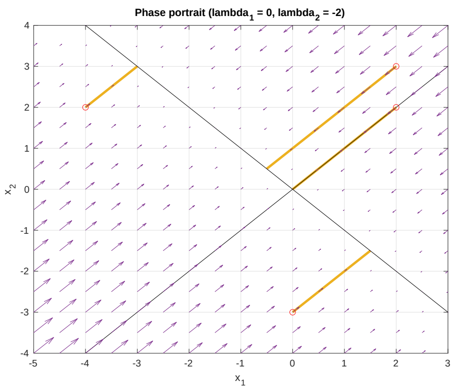


Figure 11: Free state trajectories in state space