Self assessment - 00B (with solutions)

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1 Exercise

Consider the Mass-Spring-Damper system with parameters m, μ and k, find analytically the natural modes for the special case $\mu = 2\sqrt{k\,m}$.

Sol. MSD system characteristic polynomial and eigenvalues

$$p_A(\lambda) = \lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} \qquad \rightarrow \qquad \lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{\mu}{m}\right)^2 - 4\frac{k}{m}}$$

which, for $\mu = 2\sqrt{k\,m}$, become coincident and equal to $\lambda_1 = -\frac{\mu}{2m}$ which therefore has algebraic muyltiplicity equal to 2. One real eigenvalue $\lambda_1 =$ with algebraic multiplicity 2. We need to find the geometric multiplicity and therefore the dimension of the eigenspace associated to λ_1. From the $A - \lambda_1 I$ matrix, being $\mu = 2\sqrt{km}$,

$$A - \lambda_1 I = \begin{pmatrix} \frac{\mu}{2m} & 1 \\ -\frac{k}{m} & -\frac{\mu}{2m} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{k}{m}} & 1 \\ -\frac{k}{m} & -\sqrt{\frac{k}{m}} \end{pmatrix}$$

it is evident (being the matrix with rank 1) that the nullspace has dimension 1

$$\mathcal{N}(A - \lambda_1 I) = \text{gen}\left\{ \begin{pmatrix} 1 \\ -\sqrt{\frac{k}{m}} \end{pmatrix} \right\}$$

and therefore the geometric multiplicity is 1. The natural modes are therefore $e^{\lambda_1 t}$ and $t\,e^{\lambda_1 t}$.

2 Exercise

Given the system

$$A = \begin{pmatrix} 2 & -1.5 \\ 2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

- Find a "sensor", that is the C matrix, such that the unstable mode will never result in the output free response.
- What is the corresponding impulse response?
- Is the system asymptotically stable?

Sol. Matrix A has characteristic polynomial and eigenvalues/eigenvectors

$$p_A(\lambda) = \lambda^2 - 1 \qquad \rightarrow \qquad \lambda_1 = -1, u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 1, u_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So there is an unstable aperiodic natural mode $e^{\lambda_2 t} = e^t$. In order to make this mode not appear in the output free response, that is for any initial condition, the C has to be such that

$$Cu_2 = 0, \qquad \rightarrow \qquad C = \begin{pmatrix} -2 & 3 \end{pmatrix}$$

Being the left eigenvalue associated to λ_1

$$v_1^T = \begin{pmatrix} -0.5 & 0.75 \end{pmatrix}$$

the corresponding impulse response is

$$Ce^{At}B = e^{\lambda_1 t} C u_1 v_1^T B + e^{\lambda_2 t} C u_2 v_2^T B = e^{\lambda_1 t} C u_1 v_1^T B = e^{-t} \begin{pmatrix} -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -0.5 & 0.75 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2e^{-t}$$

Note that it is not just a coincidence that $C \parallel v_1^T$.

3 Exercise

Given the system

$$A = \begin{pmatrix} 0 & -0.5 \\ -2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Compute the system eigenvalues and corresponding eigenspaces. Draw a phase plane plot of the typical qualitative state free evolutions (starting from different initial conditions that you choose and motivate).
- Let the input be an impulse, compute the corresponding state response assuming zero initial state.
- Is the previous state impulse response diverging? Interpret the result in terms instantaneous state transfer and eigenspaces.
- Denote by λ_2 the resulting positive eigenvalue and assume the input does not contain $e^{\lambda_2 t}$, will the diverging exponential $e^{\lambda_2 t}$ appear in any forced output response?

Sol. Eigenvalues and eigenvectors are (other choices are possible for the eigenvectors, but all parallel to these)

$$\lambda_1 = -1, u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_1^T = \begin{pmatrix} 0.5 & 0.25 \end{pmatrix}, \qquad \lambda_2 = 1, u_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, v_2^T = \begin{pmatrix} 0.5 & -0.25 \end{pmatrix},$$

The state impulsive response is given by

$$H(t) = e^{At}B = e^{\lambda_1 t} u_1 v_1^T B + e^{\lambda_2 t} u_2 v_2^T B = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and is converging. Since we know that the impulse instantaneously transfers the 0-state in a state which has the same values as B and noticing that $B \parallel u_1$, the state impulse response coincides

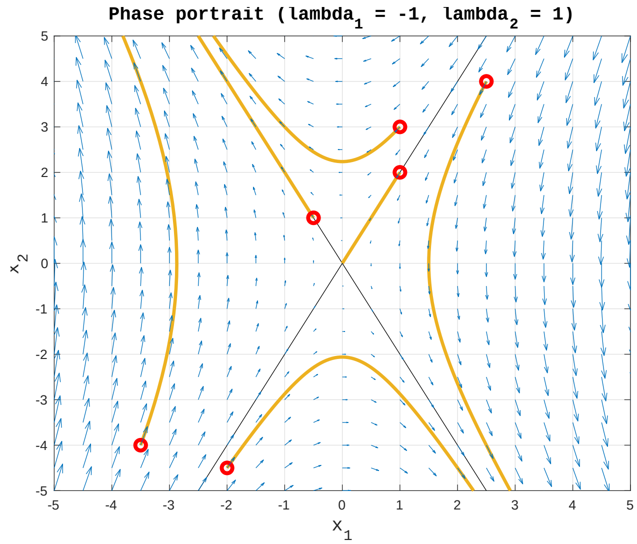

Figure 1: Phase plane trajectories

with the state evolution from an initial state which belongs to the eigenspace generated by u_1 and therefore tends to the origin, being $\lambda_1 = -1$, exponentially.

Since the output impulse response is a linear combination ($W(t) = CH(t)$) of the state impulse response, it will not contain the diverging natural mode $e^{\lambda_2 t}$. Moreover, being any forced output a convolution integral of the impulse response with the input

$$y_{ZSR}(t) = \int_0^t W(t-\tau)u(\tau)d\tau$$

no diverging component $e^{\lambda_2 t}$ will ever appear in $y_{ZSR}(t)$.

4 Exercise

Consider the system matrix

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{pmatrix}$$

Find the particular change off coordinates T (which may have elements with complex numbers) that makes the system matrix become

$$TAT^{-1} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

Sol. Let's start from what we know: given the real matrix

$$A_{cc} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

with complex eigenvalues $-1 \pm j$, the diagonalizing change of coordinates T_{cc} is given by the eigenvectors associated to $-1+j$ and $-1-j$

$$A - (-1+j)I = \begin{pmatrix} -j & 1 \\ -1 & -j \end{pmatrix} \qquad \rightarrow \qquad u_1 = \begin{pmatrix} 1 \\ j \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 1 \\ -j \end{pmatrix}$$

and therefore

$$T_{cc} = \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix}^{-1}$$

which, as stated by the theory, gives

$$T_{cc} A_{cc} T_{cc}^{-1} = \begin{pmatrix} -1+j & 0 \\ 0 & -1-j \end{pmatrix} \qquad \rightarrow \qquad T_{cc}^{-1} \begin{pmatrix} -1+j & 0 \\ 0 & -1-j \end{pmatrix} T_{cc} = A_{cc}$$

We can therefore state that the change of coordinates which transforms the diagonal matrix in A_{cc} is given by

$$T_r = T_{cc}^{-1}.$$

Being the original matrix block diagonal, we can directly write

$$T = \begin{pmatrix} 1 & 0 \\ 0 & T_r \end{pmatrix}$$

5 Exercise

Given the dynamic matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

- Determine the eigenvalues and their multiplicities (algebraic and geometric).
- Is the corresponding system asymptotically stable, marginally stable or unstable?

Sol. The characteristic polynomial is

$$\det \begin{pmatrix} \lambda & -1 & -1 \\ 0 & \lambda+1 & 1 \\ 0 & -1 & \lambda-1 \end{pmatrix} = \lambda^3$$

so there is one eigenvalue $\lambda_1 = 0$ with algebraic multiplicity $m_a(\lambda_1) = 3$. The geometric multiplicity is given by the dimension of the nullspace

$$\dim(\mathcal{N}(A - \lambda_1 I)) = \dim(\mathcal{N}(A)) = 2$$

since

$$\mathcal{N}(A) = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

so $m_g(\lambda_1) = 2 < m_a(\lambda_1)$ which implies that the system is unstable.

6 Exercise

Given the dynamic matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Compute the matrix exponential e^{At}.
- Is there a particular choice of the input matrix B which will not lead to a diverging state impulse response?
- For a generic input matrix B, is there a particular choice of the output matrix C which will not lead to a diverging impulse response?

Sol. By definition of matrix exponential (in this case matrix A is nilpotent since $A^2 = 0$)

$$e^{At} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly any matrix B of the form

$$B = \begin{pmatrix} * \\ 0 \\ * \end{pmatrix}$$

i.e. with a second component equal to 0, will always lead to the state impulse response

$$e^{At}B = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * \\ 0 \\ * \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ * \end{pmatrix}$$

Similarly, for a generic B choosing the output vector with the first element equal to 0 leads to a non-diverging impulse response

$$C = \begin{pmatrix} 0 & * & * \end{pmatrix} \qquad \rightarrow \qquad Ce^{At}B = \begin{pmatrix} 0 & * & * \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & * & * \end{pmatrix} B$$

7 Exercise

Let the dynamic matrix be

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

- Find the spectral decomposition of A.
- Compute the exponential e^{At}.
- Draw some illustrative phase plane trajectories.
- Verify on Matlab.

Sol. Being

$$p_A(\lambda) = (\lambda + 1)(\lambda - 2)$$

we have

$$\lambda_1 = -1, \qquad u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_1^T = \begin{pmatrix} 1/3 & 1/3 \end{pmatrix}$$

$$\lambda_2 = 2, \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2^T = \begin{pmatrix} 2/3 & -1/3 \end{pmatrix}$$

so that the spectral decomposition is

$$A = \lambda_1 u_1 v_1^T + \lambda_2 u_2 v_2^T = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \end{pmatrix}$$

$$= -\begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} + 2 \begin{pmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{pmatrix}$$

and the exponential is

$$e^{At} = e^{-t} \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} + e^{2t} \begin{pmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} (e^{-t} + 2e^{2t})/3 & (e^{-t} - e^{2t})/3 \\ 2(e^{-t} - e^{2t})/3 & (2e^{-t} + e^{2t})/3 \end{pmatrix}$$

The phase plot is similar (with different eigenspaces) to Fig. 1. Possible Matlab code follows.

```
clear all
A = [1, -1; -2, 0];  % dynamic matrix
B = [1;0];              % whatever, since it is not specified
C = [1,1];              % whatever, since it is not specified
D = [0];                 % whatever, since it is not specified
System = ss(A,B,C,D);  % state space representation of the system
[Y1,T1,X1] = initial(System,[1;1]);  % free evolution from initial state [1;1]
[Y2,T2,X2] = initial(System,[-1;1]); % free evolution from initial state [-1;1]
[Y3,T3,X3] = initial(System,[1;-1]); % and so on
[Y4,T4,X4] = initial(System,[-1;-1]);
[Y5,T5,X5] = initial(System,[0;2]);
[Y6,T6,X6] = initial(System,[0;-2]);
[Y7,T7,X7] = initial(System,[1;2]);
[Y8,T8,X8] = initial(System,[-1;-2]);

% the three dots at the end of the following line of code is to split
% a long line of code between multiple lines
Plot = plot(X1(:,1),X1(:,2),X2(:,1),X2(:,2),X3(:,1),X3(:,2),X4(:,1),X4(:,2),...
     X5(:,1),X5(:,2),X6(:,1),X6(:,2),X7(:,1),X7(:,2),X8(:,1),X8(:,2)), grid
set(Plot,'LineWidth', 2);    % to make the lines thick enough
axis([-4,4,-4,4])  % to see only this portion of the plane
xlabel('x1')       % label for the x-axis
ylabel('x2')       % label for the y-axis
title('Phase plane trajectories')
```

8 Exercise

Consider the Mass-Spring-Damper system (MSD).

- Choose the parameters such that the eigenvalues are real and distinct. Compute the maximum extension of the mass when a force impulse is applied.
- Same problem with a different choice of the parameters leading to a complex pair of eigenvalues.

Sol: Compute the state impulse response $e^{At}B$, the velocity response – second component of the state impulse response – will be of the form

$$v(t) = c_{21}e^{-\lambda_1 t} + c_{22}e^{-\lambda_2 t}$$

find the time at which the velocity first is 0, this will be

$$t_{\max} = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{-c_{21}}{c_{22}}\right)$$

put it in the position expression of the impulse response and obtain

$$p_{\max} = c_{11}e^{-\lambda_1 t_{\max}} + c_{12}e^{-\lambda_2 t_{\max}}$$

For the complex conjugate case, the eigenvalues are (λ_1, λ_1^*) with

$$\lambda_1 = -\frac{\mu}{2m} + j\sqrt{\frac{k}{m} - \frac{\mu^2}{4m^2}} = \alpha + j\omega$$

Therefore we have

$$A - \lambda_1 I = \begin{pmatrix} -\alpha - j\omega & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} - \alpha - j\omega \end{pmatrix} \qquad u_1 \parallel \begin{pmatrix} 1 \\ \alpha + j\omega \end{pmatrix}$$

so we can choose

$$u_a = \text{real}(u_1) = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \qquad u_b = \text{imag}(u_1) = \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

The generic state impulse response $H(t) = e^{At}B$ is obtained from the generic expression of the state free evolution choosing as initial condition exactly B. Since we should establish when the velocity becomes zero, we should look only at the second component of the generic expression

$$e^{At}B = m_R e^{\alpha t}\left(\sin(\omega t + \varphi_R)u_a + \cos(\omega t + \varphi_R)u_b\right)$$

where we have set $x(0) = B$. Since

$$T_R = \begin{pmatrix} 1 & 0 \\ \alpha & \omega \end{pmatrix}^{-1} \quad \to \quad T_R B = \begin{pmatrix} c_a \\ c_b \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{m\omega} \end{pmatrix}, \qquad c_a = 0, \quad c_b = \frac{1}{m\omega}$$

so $m_R = 1/m\omega$ and $\varphi_R = 0$. We have

$$e^{At}B = \frac{1}{m\omega}e^{\alpha t}\left[\sin\omega t \begin{pmatrix} 1 \\ \alpha \end{pmatrix} + \cos\omega t \begin{pmatrix} 0 \\ \omega \end{pmatrix}\right]$$

The velocity is then

$$v(t) = \frac{1}{m\omega}e^{\alpha t}\left[\alpha \sin\omega t + \omega\cos\omega t\right]$$

which becomes zero at $t_{\max}$ if one the two holds

$$\begin{aligned} \sin\omega t_{\max} &= -\omega \quad &\text{and}\quad &\cos\omega t_{\max} = \alpha \\ \sin\omega t_{\max} &= \omega \quad &\text{and}\quad &\cos\omega t_{\max} = -\alpha \end{aligned}$$

i.e.

$$t_{\max} = \text{atan2}(-\omega, \alpha)/\omega, \qquad \text{or} \qquad t_{\max} = \text{atan2}(\omega, -\alpha)$$

Finally, from the first component of the state impulse response, we get the maximum position as

$$p_{\max} = p(t_{\max}) = \frac{1}{m\omega} e^{\alpha t_{\max}} \sin\omega t_{\max}$$

9 Exercise

Consider the chemical reaction between two components described by the equations given in the slides.

- Find the change of coordinates that diagonalizes the dynamic matrix and interpret the result (conservation of some quantity relative to the 0 eigenvalue).
- Draw the phase plane plots highlighting the two eigenspaces.
- The Mass-Spring-Damper system with no spring ($K = 0$) has a similar dynamic behavior; what quantity is conserved in this case?

Sol. The dynamic matrix A is

$$A = \begin{pmatrix} -k_d & k_i \\ k_d & -k_i \end{pmatrix}$$

with an evident zero eigenvalue (being the matrix singular). The characteristic polynomial is

$$p_A(\lambda) = \lambda(\lambda + k_d + k_i)$$

so $\lambda_1 = 0$ and $\lambda_2 = -(k_d + k_i) < 0$. Eigenvectors are

$$\lambda_1 = 0, \quad u_1 = \begin{pmatrix} k_i \\ k_d \end{pmatrix}, \qquad \lambda_2 = -(k_d + k_i), \quad u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and therefore

$$\mathcal{U} = \begin{pmatrix} k_i & 1 \\ k_d & -1 \end{pmatrix} \quad \mathcal{U}^{-1} = \frac{1}{k_d + k_i} \begin{pmatrix} 1 & 1 \\ k_d & -k_i \end{pmatrix}$$

which shows that in the new coordinates

$$z = Tx = \mathcal{U}^{-1} x = \frac{1}{k_d + k_i} \begin{pmatrix} C_A + C_B \\ k_d C_A - k_i C_B \end{pmatrix}$$

the dynamic equations become

$$\dot{z}_1 = 0 \qquad (1)$$
$$\dot{z}_2 = \lambda_2 z_2 \qquad (2)$$

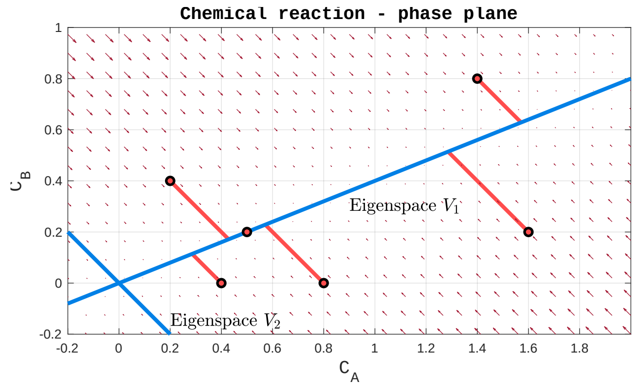

Figure 2: Phase plane trajectories for the chemical reaction example

so the quantity $C_A + C_B$ remains constant in time. The variable z_1 remains constant in time and therefore highlights a *conserved quantity*.

For the Mass-Damper system characterized by the dynamic matrix

$$A_{MD} = \begin{pmatrix} 0 & 1 \\ 0 & -\mu/m \end{pmatrix}$$

we have

$$\lambda_1 = 0, \qquad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -\frac{\mu}{m}, \qquad u_2 = \begin{pmatrix} 1 \\ -\mu/m \end{pmatrix}$$

and therefore the diagonalizing change of coordinates is

$$T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -\mu/m \end{pmatrix} \qquad \rightarrow \qquad T = \begin{pmatrix} 1 & m/\mu \\ 0 & -m/\mu \end{pmatrix}$$

Since the first new coordinate z_1 is the one relative to the 0 eigenvalue and the corresponding dynamic equation is $\dot{z}_1 = 0$, this is the conserved quantity which, in terms of the position x_1 and velocity x_2, is expressed as

$$z_1 = x_1 + \frac{m}{\mu} x_2$$

10 Exercise

Consider the electrical circuit in Fig. 3. Find the dynamic model and discuss its behavior when the two capacitors have an initial charge, i.e. when we have initial condition $v_{C1}(0)$ and $v_{C2}(0)$ and no input voltage v_i is applied.

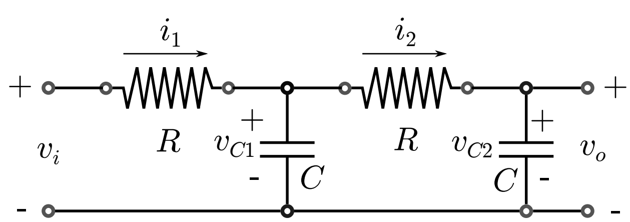

Figure 3: Electrical circuit

11 Exercise

Consider the electrical circuit in Fig. 4.

- Find the dynamic model and discuss its behavior.
- Compare this system with the Mass-Damper system (i.e. MSD with no elastic spring).

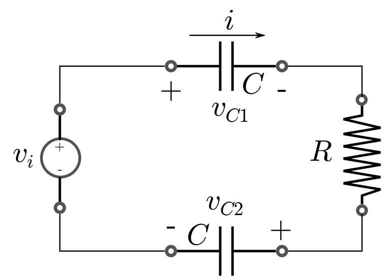

Figure 4: Electrical circuit

Sol. We can write

$$i = \frac{!}{R}(v_i - v_{C1} - v_{C2}) \tag{3}$$

$$\dot{v}_{C1} = \frac{1}{C} i \tag{4}$$

$$\dot{v}_{C2} = \frac{1}{C} i \tag{5}$$

from which

$$\dot{v}_{C1} = -\frac{1}{RC}(v_{C1} + v_{C2} - v_i) \tag{6}$$

$$\dot{v}_{C2} = -\frac{1}{RC}(v_{C1} + v_{C2} - v_i) \tag{7}$$

i.e.

$$A = \begin{pmatrix} -1/RC & -1/RC \\ -1/RC & -1/RC \end{pmatrix}, \quad B = \begin{pmatrix} 1/RC \\ 1/RC \end{pmatrix},$$