Self assessment - 02 - Solutions highlights

Updated 30/12/2019 - 03/01/2020 - 30/01/2020 - 03/01/2024

1 Exercise

Consider the plant

$$\begin{aligned}\dot{x}_1 &= x_1 + x_3 + u \\ \dot{x}_2 &= u \\ \dot{x}_3 &= -2x_3 \\ y &= x_1 + x_2 + \alpha x_3\end{aligned}$$

with $\alpha \in \mathbb{R}$ a real parameter.

1. Study the controllability property and, if necessary, do a Kalman decomposition w.r.t. controllability.
2. Find the value(s) of α such that there exists an unobservable asymptotically stable subsystem. Decompose w.r.t. observability. From now on use this value of α.
3. Using the previous decomposition, is it possible to find an output stabilizing dynamic controller of dimension 2? Why?
4. Find the plant's transfer function and determine if the system is stabilizable with a simple static output feedback.
5. Determine an output dynamic controller which assigns the closed-loop poles in -1, -2 and -3.
6. How does the previous closed-loop system behaves at steady-state w.r.t. a constant reference and to an unknown constant output disturbance?

2 Exercise

Let the open-loop system be

$$F(s) = \frac{K(s+1)}{s(s+100)^2}$$

1. Study, as $K \in \mathbb{R}$ varies, the stability of the unit feedback closed-loop system both using the Nyquist criterion and the root-locus plot.
2. Determine if there is a closed-loop dominant pole and, if it exists, discuss its contribution as K increases (positive values only).

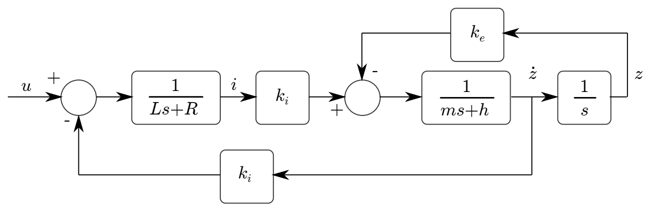


Figure 1: Ex. 5, loudspeaker block diagram

3 Exercise

Let the open-loop system be

$$F(s) = \frac{10}{s(s+11)}$$

Determine the frequency range (in rad/s or Hertz) where the closed-loop system guarantees an attenuation of at least 20 dB to sinusoidal disturbances acting on the feedback loop. An answer based on an approximate behavior is accepted.

4 Exercise

Let the open-loop system be

$$P(s) = \frac{K(s+z)}{(s+2)(s+3)}, \quad z > 0$$

Determine for the closed-loop system the different types of stability as both $K \in \mathbb{R}$ and $z > 0$ vary. Illustrate the different corresponding root-locus plots.

5 Exercise

In a magnetic loudspeaker, a cone of mass m and position $z(t)$ is kept in place by an elastic suspension characterized by an elastic constant k_e. During its movement, the cone is subject to some viscous damping (acoustic coupling with the air) which depends linearly, through the coefficient h, on the cone's velocity $\dot{z}$. The mobile coil is represented by an electrical circuit with a resistor R and an inductance L while the electroacoustic coupling due to the magnetic flux in the air gap is given by k_i. Let $i(t)$ be the current through the mobile coil and $u(t)$ the applied input voltage. The dynamic equations are

$$\begin{aligned} L\frac{di(t)}{dt} + Ri(t) + k_i\frac{dz(t)}{dt} &= u \qquad \text{electric components} \\ m\frac{d^2z(t)}{dt^2} + h\frac{dz(t)}{dt} + k_e z(t) &= k_i i(t) \qquad \text{forces equilibrium} \end{aligned}$$

- Show that the block diagram reported in Fig. 1 corresponds to the system under consideration.
- Find the transfer function $u(s) \to \dot{z}(s)$.
- Is there a physical interpretation of the particular numerator found in the previous question?

6 Exercise

Let the plant be modeled by the following transfer function

$$P(s) = \frac{10}{s(s+0.1)}$$

Design a control scheme which guarantees a steady-state error in magnitude smaller than 1% w.r.t. a unit reference ramp, a phase margin of at least 30° and a crossover frequency of 1 rad/s.

7 Exercise

Study, as $K \in \mathbb{R}$ varies, the stability of the unit feedback closed-loop system having $F(s)$ as open-loop. Use both the Nyquist criterion and the root-locus plot.

$$F(s) = \frac{K(s^2+20s+100)}{s^2(s+1)}$$

Finally check with the Routh criterion.

8 Exercise

We want to control the temperature $T(t)$ inside a closed tank containing a fluid. Using the energy conservation principle we obtain the following differential equation which describes the temperature $T(t)$ time evolution

$$C\dot{T}(t) + qc_v\left[T(t) - T_i\right] + \frac{1}{R}\left[T(t) - T_a\right] = Q_{in}(t)$$

where $Q_{in}(t)$ can be manipulated.

Symbol	Units	Description
C	J/K	Thermal capacity
q	kg/s	Fluid flow in transit
c_v	J/(kg . K)	Fluid specific heat
T_i	K	Constant input fluid temperature
R	K . s/J	Thermal resistance due to the tank's wall
T_a	K	External constant temprature
$Q_{in}(t)$	J/s	Input heat flux

- Give a state-space representation of the system. Let $T(t)$ be measurable.
- How does the dynamic behavior change as C varies? Give a physical interpretation.
- Draw a control scheme to regulate the internal temperature $T(t)$.
- Discuss which specifications would you require and how to solve them.

9 Exercise

For the interconnected system in Fig. 2, find the transfer function $d_2 \to y$.

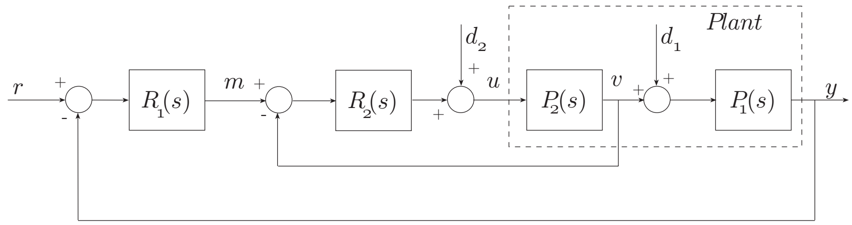


Figure 2: Ex. 9, interconnected system

10 Exercise

Let the plant be

$$P(s) = \frac{1}{s + 0.1}$$

Design a control system such that the following specifications are met:

a) the output asymptotically tracks the reference signal $r(t) = t\delta_{-1}(t)$, with a maximum allowed error in magnitude equal to 1;

b) no steady-state influence of a constant disturbance acting on the plant's output;

c) phase margin of at least 30°;

d) crossover frequency equal to $\omega_c^* = 0.1$ rad/sec.

A Exercise 1

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & \alpha \end{pmatrix}$$

Triangular A matrix, therefore $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -2$.

1. The controllability matrix is singular and has rank $= 2$, therefore there exists an uncontrollable subsystem of dimension 1.

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{Im}(P) = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

   and the change of coordinates can be chosen with T^{-1} given by

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{identity matrix}$$

   which means the system is already decomposed w.r.t. controllability and the uncontrollable system is characterized by the eigenvalue $\lambda_3 = -2$

   Note that since the A matrix has an upper block triangular structure and the B has its last element equal to 0,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \quad \text{with} \quad A_{22} = -2, \quad B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   we could have immediately stated that the eigenvalue $\lambda_3 = -2$ was uncontrollable. Moreover knowing from the controllability rank test that the uncontrollable subsystem has dimension 1, then it necessarily has to be that both λ_1 and λ_2 are controllable that is (A_{11}, B_1) controllable.

2. Since $\lambda_3 = -2$ is the only asymptotically stable eigenvalue, we can use the PBH test to find the value(s) of α for which this eigenvalue is unobservable

$$\mathrm{rank}\begin{pmatrix} C \\ A - \lambda_3 I \end{pmatrix} = \mathrm{rank}\begin{pmatrix} 1 & 1 & \alpha \\ 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 < 3 \quad \text{iff} \quad 6\alpha - 2 = 0 \quad \Leftrightarrow \quad \alpha = 1/3$$

   Note that it could happen that for the same value of α other eigenvalues become unobservable and therefore the unobservable subsystem would not be asymptotically stable. We have the observability matrix given by

$$\mathcal{O} = \begin{pmatrix} 1 & 1 & 1/3 \\ 1 & 0 & 1/3 \\ 1 & 0 & 1/3 \end{pmatrix}, \quad \mathrm{Ker}(\mathcal{O}) = \mathrm{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \right\}$$

   which shows that the unobservable subsystem has dimension 1 so it is fully characterized by the eigenvalue $\lambda_3 = -2$. Alternatively one could check the observability PBH test for the other eigenvalues but since we are asked to do the Kalman decomposition, checking the observability matrix is a necessary first step.

For the Kalman decomposition wrt observability, choose (the inverse is not complicated)

$$\tilde{T}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \Rightarrow \quad \tilde{T} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

which leads to

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; \tilde{B} = \tilde{T}B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; \tilde{C} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

but with the special feature that $\tilde{A}_{21} = 0$.

Note that this special "block-diagonal" structure does not depend on the particular choice of the $\mathrm{Im}(P)$ base. For example, if one chooses instead of the identity matrix for T, the matrix

$$T_{2c}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad T_{2c} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the decomposition wrt controllability leads to

$$A_{2c} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad B_{2c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C_{2c} = \begin{pmatrix} 2 & 1 & 1/3 \end{pmatrix}$$

The corresponding observability matrix is

$$\mathcal{O}_{2c} = \begin{pmatrix} 2 & 1 & 1/3 \\ 1 & 1 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix}, \quad \mathrm{Ker}(\mathcal{O}_{2c}) = \mathrm{gen}\left\{ \begin{pmatrix} 0 \\ 1/3 \\ -1 \end{pmatrix} \right\}$$

and therefore

$$T_{2i}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad T_{2i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & -1 \end{pmatrix}$$

The decomposition wrt observability is finally

$$A_{2o} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad B_{2o} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C_{2o} = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$$

with the same "block-diagonal" structure.

3. From the previous decomposition wrt observability (any) we always obtain a system of the form

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix}$$

with $(\tilde{A}_{11}, \tilde{B}_1)$ controllable and $(\tilde{A}_{11}, \tilde{C}_1)$ observable. The asymptotic stable dynamics associated to the eigenvalue $\lambda_3 = -2$ is both uncontrollable and unobservable (check with PBH tests).

For such a system, using the separation principle consists in

- choosing a state feedback which stabilizes the system as if the whole state was available
- designing an observer to reconstruct the state
- using the reconstructed state in the previous feedback instead of the actual state.

In general the resulting controller has the dimension of the observer.

Note, however, that in this case, partitioning the state according to the controllability decomposition (with $\tilde{T}$ being either the identity matrix or, for example, T_{2c})

$$z = \tilde{T}x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \text{ with } z_1 \in I\!R^2$$

the state feedback would be of the form $u = F_1 z_1$, so, if possible, we need to observe/reconstruct only the z_1 components. In all choices this corresponds to a linear combination of the first two state components of the original system. Since the observable subsystem coincides with the controllable one, we only need to design the observer for the observable subsystem

$$\begin{aligned} \dot{\xi}_1 &= \tilde{A}_{11}\xi_1 + \tilde{B}_1 u + K_1(y - y_r) \\ y_r &= \tilde{C}_1 \xi \end{aligned}$$

In other words, since the state feedback requires only $z_1 \in I\!R^2$ which coincides with the (or is a linear combination of the) state of the observable subsystem, we can reconstruct only z_1. By the separation principle, using the feedback $u = F_1\xi_1$ together with the designed observer of dimension 2, the closed-loop system is asymptotically stable (provided the eigenvalues are chosen appropriately).

Note that this does not happen in general since we may need also unobservable components of the state in the state feedback and thus the observer would be of full dimension.

4. From the block-diagonal structure of $\tilde{A}$, the same structure is inherited by $(sI - \tilde{A})$ and being in general (for square matrices M_{11} and M_{22})

$$\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}^{-1} = \begin{pmatrix} M_{11}^{-1} & 0 \\ 0 & M_{22}^{-1} \end{pmatrix}$$

we directly have, due to the special structure of $\tilde{B}$ and $\tilde{C}$,

$$P(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 = \frac{2s - 1}{s(s-1)}$$

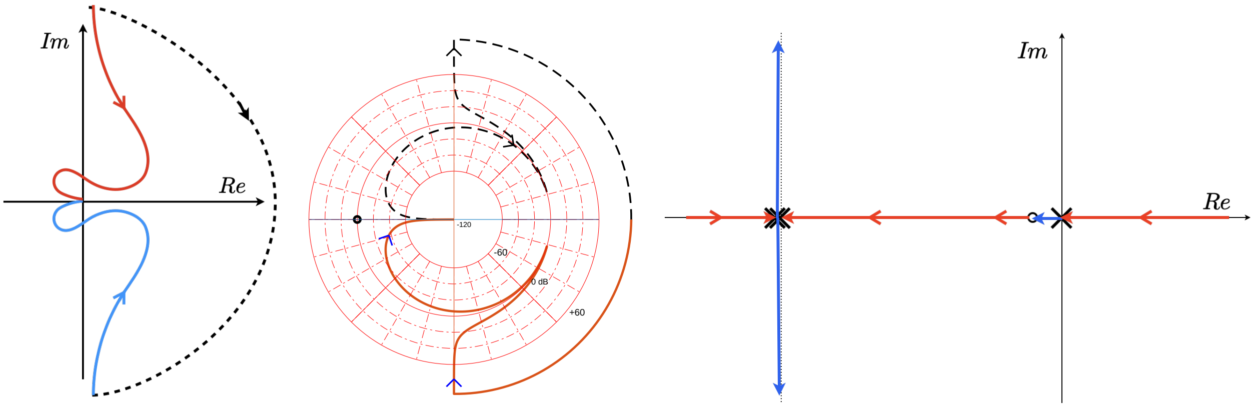
With a static feedback $u = Ky$, we obtain the following closed loop pole polynomial

$$p(s, K) = s(s-1) + K(2s-1) = s^2 + s(2K-1) - K$$

which is clearly unstable for any value of K. Note that being the system non-minimum phase, we can only say it is not stabilizable with high-gain feedback; here we checked for all values of the gain K.

5. We can solve the problem with a simple pole assignment design. Being $n = 2$, a parametric controller of dimension $n - 1 = 1$ (the closed loop system will then have $n + n - 1 = 2 + 1 = 3$ poles) of the form

$$C(s) = \frac{as + b}{s + c}$$



(a) Nyquist plot by hand (b) Nyquist plot (log scale) (c) Root locus

Figure 3: Exercise 2 - Nyquist plot for positive K and root locus.

will allow to assign, for example, the following desired closed loop polynomial

$$p^*(s) = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

The actual closed loop system polynomial is

$$p(s) = (s+c)s(s-1) + (as+b)(2s-1) = s^3 + (2a+c-1)s^2 + (2b-a-c)s - b$$

and therefore, comparing the two polynomials gives $a = 30$, $b = -6$ and $c = -53$. The final output feedback controller is then

$$C(s) = \frac{30s - 6}{s - 53}$$

6. The previous output feedback controller leads to an asymptotically stable closed loop system (and the hidden dynamics is characterized by $\lambda_3 = -2$). Moreover the plant has a pole is $s = 0$ in the forward loop, so the system is of type 1 and the steady-state response to a constant reference will coincide with the reference itself.

B Exercise 2

The Nyquist plot is quite standard and, after drawing the Bode plots, looks like the one reported in Fig. 3 for positive values of K

The corresponding Nyquist plot for negative K is obtained with a 180° degrees rotation. It is straightforward to conclude that the Nyquist criterion is verified only for positive values of the gain K. The root-locus is also shown in Fig. 3

From the root locus it is quite evident that a closed loop pole will lie between 0 and -1 while the other two poles will be further to the right of the imaginary axis (real part close to -100). Therefore there is a clear dominant pole, the slower (closer to the origin) one. However it should also be noted that as K increases this closed loop pole will approach the open loop zero in $s = -1$ and therefore its effect will become smaller and smaller. One could compute the step response and analyze how the residue of the closed loop pole closer to the origin (the apparent dominant one) tends to zero as K increases.

C Exercise 3

The effect of a disturbance entering the feedback loop on the controlled variable is represented by the complementary sensitivity function (with a minus sign)

$$W_{ny}(s) = -\frac{F(s)}{1+F(s)} = -\frac{10}{s(s+11)+10} = -\frac{10}{(s+1)(s+10)}$$

From the corresponding magnitude Bode diagram or equivalently from its approximation it is straightforward to note that the required attenuation (-20 dB) is attained at frequencies greater than approximately 10 rad/s (or $10/(2\pi) \approx 1.6$ Hertz).

D Exercise 4

By letting the zero in $s = -z$ move on the real axis to the left of the origin (z is positive), we have the situations illustrated in Fig. 4. The only critical situation is when the singular point moves from the positive to the negative real axis i.e. from the situation (a) to the (c). In particular we need to understand when the singular point is in $s = 0$ as in the situation (b).

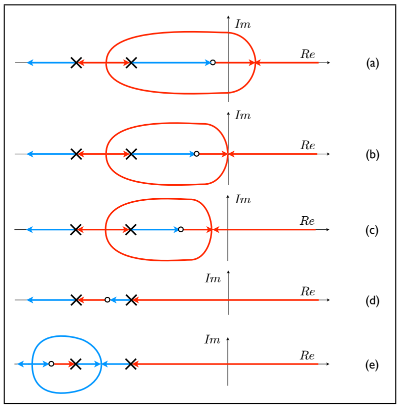


Figure 4: Exercise 4 - Root locus as z varies

To find the singular points we use the formula

$$\frac{1}{s+2} + \frac{1}{s+3} - \frac{1}{s+z} = 0 \quad \Rightarrow \quad s^2 + 2zs + 5z - 6 = 0$$

and therefore one singular point will be in $s = 0$ when $z = 6/5$. The discussion on how K and z affect the stability of the closed loop system follows directly.

E Exercise 5

Taking the Laplace transform of the two equations we obtain

$$\begin{aligned} (Ls+R)i(s) &= u(s) - k_i \dot{z}(s) \qquad \text{electric components} \\ (ms+h)\dot{z}(s) &= k_i i(s) - k_e z(s) \qquad \text{forces equilibrium} \end{aligned}$$

These relations can be directly verified on the block diagram scheme as shown in Fig. 5.

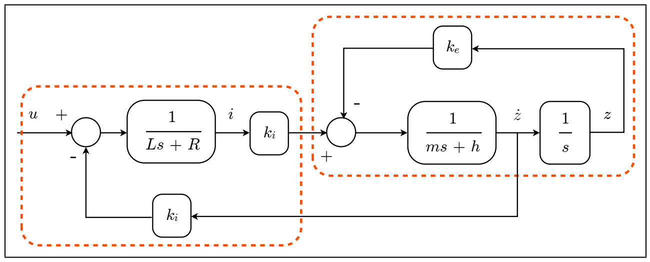


Figure 5: Exercise 5 - Loudspeaker

By manipulating the previous equations and being $\dot{z}(s) = sz(s)$ (by $\dot{z}(s)$ we denote the Laplace transform of $\dot{z}(t)$), we have

$$\left[((ms+h)(Ls+R)+k_i^2)s + k_e(Ls+R)\right] z(s) = k_i u(s)$$

and therefore

$$F(s) = \frac{\dot{z}(s)}{u(s)} = \frac{k_i s}{((ms+h)(Ls+R)+k_i^2)s + k_e(Ls+R)}$$

This transfer function has gain equal to $F(0) = 0$ and therefore a constant voltage cannot produce a non-zero velocity at steady-state which confirms the physical intuition.

F Exercise 6

Standard loop shaping exercise. The plant is

$$P(s) = \frac{10}{s(s+0.1)} = \frac{100}{s(1+10s)} \quad \Rightarrow \quad K_p = 100$$

Due to the presence of a pole in $s = 0$ in the forward path, the system is of type 1 and therefore, if the closed loop system is asymptotically stable, the steady state error will be

$$e_1 = \frac{1}{K_L} = \frac{1}{K_c K_p} \quad \Rightarrow \quad |e_1| \le 0.01 \Leftrightarrow K_c \ge \frac{1}{0.01 K_p} = 1$$

We choose $K_c = 1$. From the approximate Bode diagrams of $K_c P(j\omega)$ we see that we need to increase the phase and attenuate the magnitude at the desired crossover frequency of 1 rad/s. We can therefore choose a lead/lag function combination. We can select the lead with $m_a = 8$ and a normalized frequency of 0.9. This choice will give an amplification of approximately 2 dB and a phase lead of 35°. Note that the actual phase at the desired frequency will be greater than the approximated $-180°$. We therefore choose the lag function in order to attenuate 22 dB and introduce at most 5° of phase lag. For example a possible choice is $m_i = 12$ and a normalized frequency of 100. To achieve these effects at the desired crossover frequency of 1 rad/s, we obtain $\tau_a = 0.9$ and $\tau_i = 100$. The overall controller is finally given by

$$C(s) = K_c \frac{1+\tau_a s}{1+\tau_a/m_a s} \frac{1+\tau_i/m_i s}{1+\tau_i s} = \frac{1+0.9s}{1+0.9/8s} \frac{1+100/12s}{1+100s}$$

The closed loop system asymptotic stability is guaranteed by the Bode stability theorem. For completeness the Bode diagrams of interest are reported in Fig. 6.

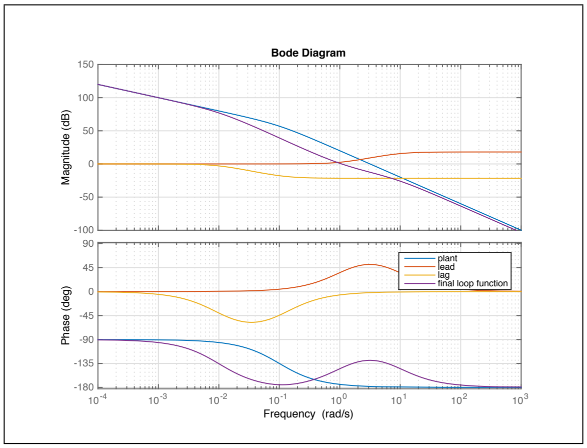


Figure 6: Exercise 6 - Bode diagrams

Just as an example, one could have obtained the lead with two coincident lead functions having $m_a = 3$ and $\tau_a = 0.4$ thus requiring a smaller attenuation (20 dB instead of 22 dB) which can be obtained with $m_i = 10$ and normalized frequency 100. The resulting controller is

$$C_2(s) = \left(\frac{1 + 0.4s}{1 + 0.4/3s}\right)^2 \left(\frac{1 + 100/10s}{1 + 100s}\right)$$

and the corresponding Bode diagrams are shown in Fig. 7.

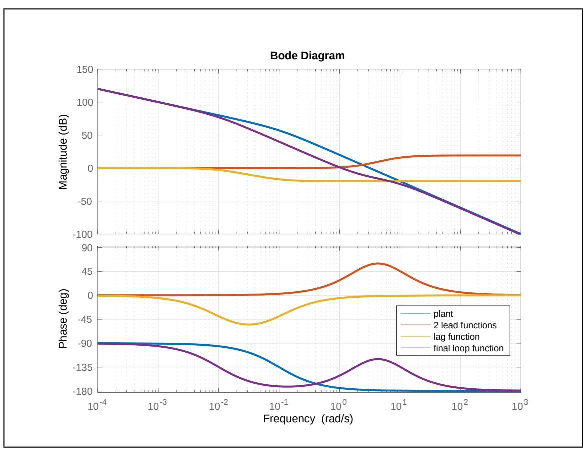


Figure 7: Exercise 6 - Bode diagrams with the alternative choice $C_2(s)$

G Exercise 7

The Bode canonical form is

$$F(s) = \frac{K(s^2 + 20s + 100)}{s^2(s+1)} = \frac{K(s+10)^2}{s^2(s+1)} = \frac{100K(1 + s/10)^2}{s^2(1+s)}$$

and the resulting Nyquist plot, after having also plotted the Bode diagrams, is approximately as in Fig. 8. Since the open loop system has no poles with positive real part, the closed loop will be asymptotically stable iff the Nyquist plot makes no encirclements around the point $(-1, 0)$. It is therefore fundamental to understand where the crossing of the real axis occurs. For negative values of the gain K, the Nyquist obtained by a 180° rotation indicates that the closed loop system will be unstable.

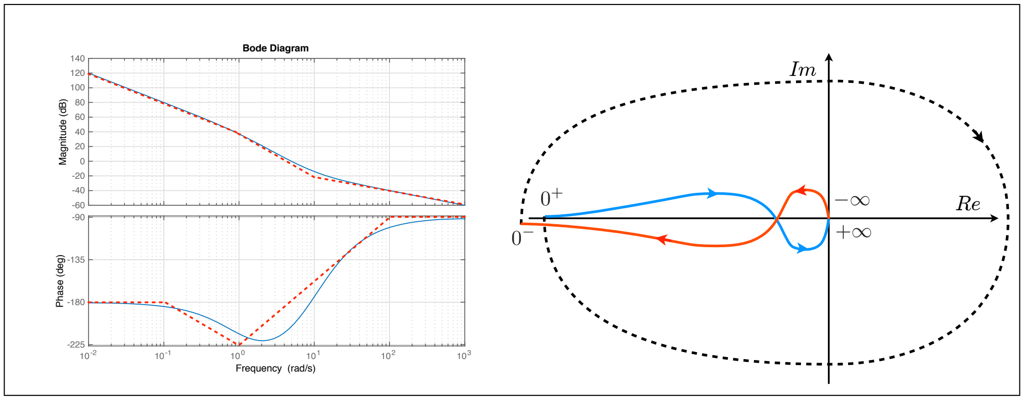


Figure 8: Exercise 7 - Bode plot (approximate and exact) and Nyquist plot for positive K

The root-locus is shown in Fig. 9 and is consistent with the previous analysis on the Nyquist plot. The critical value K_{crit} of the positive K which corresponds to the crossing of the imaginary axis can be determined through the Routh criterion

$$\begin{array}{|cc} 1 & 20K \\ 1+K & 100K \\ \frac{20K}{1+K}(K-4) & \\ 100K & \end{array}$$

From the Routh table we have $K_{crit} = 4$. Note that the value $K = -1$ which zeroes a first element in the Routh table is not a valid candidate for being a K_{crit}. First, from the root locus plot we see that the critical value should be positive (but we could be unsure about the plot). The real reason is that zeroing an element on the first column gives a potential critical value of the gain. The real critical value will arise only when the number of positive roots – and therefore the number of sign changes – differs as K goes through that critical value. For K slightly smaller than -1, the first column has signs "$+ - --$" and for slightly larger values than -1 the signs are "$+ + +-$". The number of sign changes remains the same so -1 is not a critical value.

H Exercise 8

We have a first order equation in the temperature $T(t)$ which will clearly be the state of our system (only variable which is differentiated). Moreover it is indicated that $Q_{in}(t)$ can be manipulated and therefore it is the control input. The two other inputs T_i and T_a are to be considered as disturbances. We can therefore rewrite the system equation as

$$\dot{T}(t) = -\frac{1}{C}\left(qc_v + \frac{1}{R}\right)T(t) + \frac{1}{C}\left(Q_{in}(t) + qc_vT_i + \frac{1}{R}T_a\right)$$

which is of the form

$$\dot{T}(t) = A\,T(t) + B\,(Q_{in}(t) + d) \quad \text{with} \quad d = qc_vT_i + \frac{1}{R}T_a$$

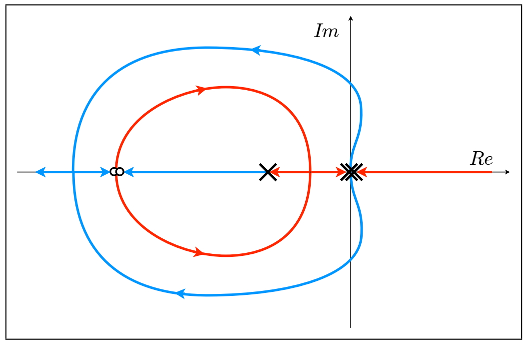


Figure 9: Exercise 7 - Root locus

Since $T(t)$ is measurable, the output can be chosen as $y = T(t)$. From the dynamic equation, the one dimensional system is characterized by the real negative eigenvalue

$$\lambda = -\frac{1}{C}\left(qc_v + \frac{1}{R}\right)$$

Note that the disturbance d, as shown in the equation, enters at the same level as the input so it can be considered as an input disturbance. The transfer function from $Q_{in}(t) + d$ to $T(t)$ is directly obtained as

$$P(s) = \frac{1}{C}\frac{1}{s-\lambda} = \frac{1}{Cs + qc_v + \frac{1}{R}}$$

There are a number of interesting remarks.

- As C increases, λ becomes smaller and the system is slower consistently with having a larger thermal capacity (as an inertia in a mechanical system).
- As the thermal resistance increases the system becomes slower.
- If the fluid flow is larger (larger q) the system is faster.
- Note that the system gain is independent from the thermal capacity, i.e. if a constant variation is applied through $Q_{in}(t)$ (for example from Q_{in1} to Q_{in2}), the temperature will change more or less rapidly depending on the thermal capacity (eigenvalue changes) but the final reached temperature will be independent from C.

A typical control scheme aimed at controlling the temperature $T(t)$ is shown in Fig. 10. The discussion on the required specifications is free but one has to highlight the following important aspects:

- never forget to analyze/check the closed loop stability.
- Since the open loop has no poles in $s = 0$ we have to decide if we want a type 0 system but with a high gain in order to guarantee a small steady state temperature error or a type 1 system (by introducing a pole in $s = 0$ in the controller) but with worse transient behavior. In order to analyze this last situation a quick root locus (1 pole in 0 and one in λ) helps. In both cases the closed system is asymptotically stable.
- An approximate analysis on the control sensitivity function can also be easily carried out in both situations.

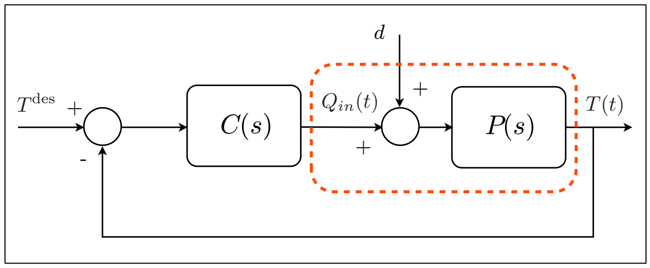


Figure 10: Exercise 8 - Control scheme

I Exercise 9

The most direct solution is through some algebraic manipulations. Setting all the other inputs (r and d_1) to zero, from the block diagram we have the following relationships where all the terms refer either to transfer functions or the Laplace transform of the signal

$$\begin{aligned} m &= -R_1 y \\ u &= d_2 + R_2(m - P_2 u) \\ y &= P_1 P_2 u \end{aligned}$$

Manipulating these equations we finally obtain

$$F_{d_2 y}(s) = \frac{y(s)}{d_2(s)} = \frac{P_1(s)P_2(s)}{1 + R_2(s)P_2(s)[1 + R_1(s)P_1(s)]}$$

J Exercise 10

Clearly specification **a)** overrules **b)** since it requires a pole in $s = 0$ and a gain K_c of at least 0.1. The pole in $s = 0$ in the controller makes the astatism requirement automatically satisfied (provided the closed loop system will be asymptotically stable). The necessary part of the controller is given by

$$C_1(s) = \frac{0.1}{s}$$

From the Bode diagrams of the modified plant $C_1(s)P(s)$, we see that we need an attenuation of 20 dB while we can stand a lag of at most 15°. For example choosing the lag function having $m_i = 10$ and with normalized frequency 30 (i.e. $\tau_i = 30/0.1 = 300$) leads to the final controller

$$C(s) = \frac{0.1}{s} \frac{1 + 30s}{1 + 300s}$$

The different Bode diagrams of interest are reported in Fig. 11 together with a simulation. Stability is finally ensured by Bode's stability theorem.

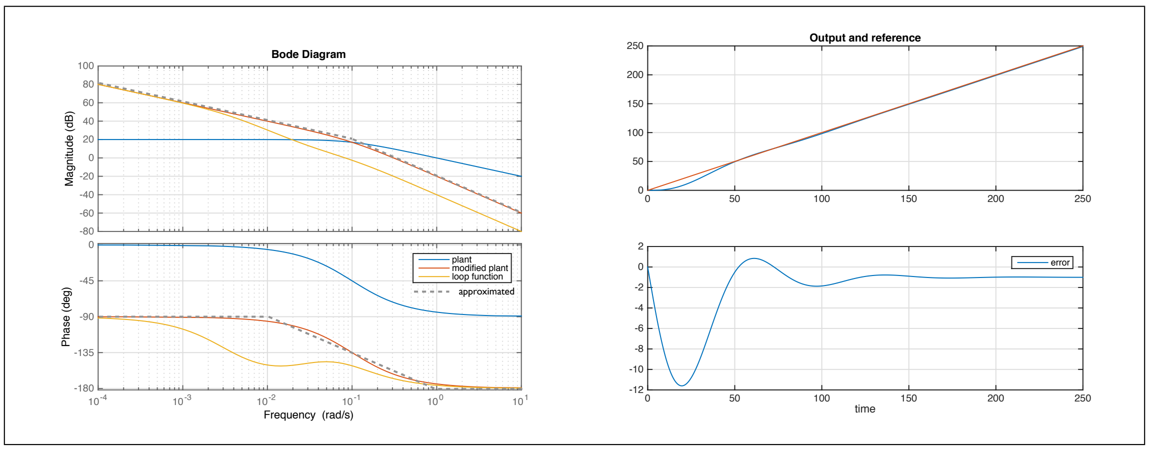


Figure 11: Exercise 10 - Bode diagrams of interest and simulation