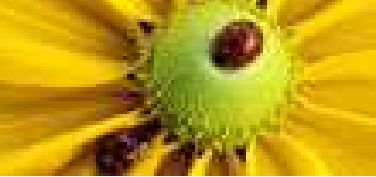


# LP-based Approximation Algorithms

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Sapienza Università di Roma

UPC, Barcelona - Spring 2010



# Lecture Outline

## ● Lecture Outline

LP-based Relaxation and Rounding

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The Primal-Dual Method

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Primal-dual Method for Vertex Cover

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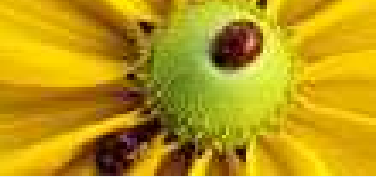
Set Cover via Dual Lifting

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Randomized Rounding

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- $f$ -approximation for Set Cover
- More basic techniques

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Randomized Rounding

# LP-based Relaxation and Rounding

# Optimization Problems

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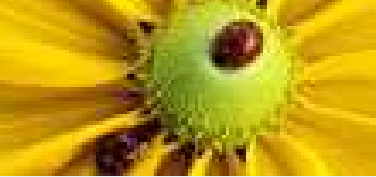
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## Approximation algorithm for problem $P$ :

- $I$  : Instance of problem  $P$
- $A(I)$ : value of  $A$ 's solution on  $I$
- $OPT(I)$ : value of the optimal solution on  $I$
- $A$  is  $r$ -approximate if
  - ◆ **Minimization:**  $\forall I, A(I) \leq r \times OPT(I)$
  - ◆ **Maximization:**  $\forall I, A(I) \geq \frac{1}{r} \times OPT(I)$

# Relate to the Optimum



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- The optimum may be hard to compute or to characterize
- However in some cases we can relate to the optimum, e.g., Christofides

$$ALG(I) \leq MST(I) + \text{Mathing}(I) \leq \left(1 + \frac{1}{2}\right) OPT(I)$$

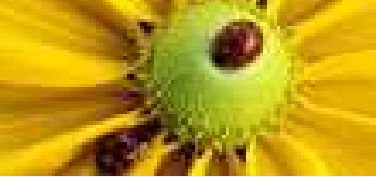
Relate to the optimum of a relaxed problem:

$$OPT(I) = \min_{x \in S(I)} f(x)$$

Relaxation:  $S(I) \subseteq R(I)$

$$LB(I) = \min_{x \in R(I)} g(x)$$

$$\forall I, x \in S(I), g(x) \leq f(x)$$



# Use the solution of the relaxed problem

If  $\forall I, ALG(I) \leq r \times LB(I)$   
then  $\forall I, ALG(I) \leq r \times OPT(I)$

Optimum of  $LB(I)$ :

$$x^* : g(x^*) = \min_{x \in R(I)} g(x)$$

The optimum to the relaxed problem must be easy to compute.

Rounding:

Round  $x^*$  to a  $\bar{x} \in S(I)$ :  $f(\bar{x}) \leq r \times g(x^*)$

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## Vertex Cover

- $G = (V, E)$ ,  $w(u) \in \mathbb{R}^+$ ,  $u \in V$
- Find a set  $U \subseteq V$  of min total cost  $\sum_{u \in U} w(u)$  such that:
- $\forall e = (u, v) \in E$ , either  $u \in U$  or  $v \in U$

$$\min \sum_{v \in V} x(v)w(v)$$

$$\text{s.t. } \begin{aligned} x(u) + x(v) &\geq 1 && \forall (u, v) \in E \\ x(u) &\in \{0, 1\} && u \in V \end{aligned}$$

# LP-relaxation for Vertex Cover

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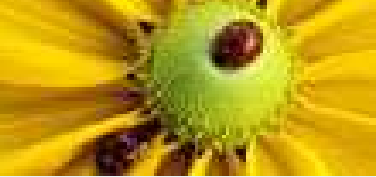
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$$\min \sum_{v \in V} x(v)w(v)$$

$$\text{s.t. } \begin{aligned} x(u) + x(v) &\geq 1 && \forall (u, v) \in E \\ x(u) &\in [0, 1] && u \in V \end{aligned}$$

- The fractional LP program can be computed in polynomial time
- All vertex covers are still feasible solution to the LP relaxation
- The optimum to the LP relaxation is a lower bound to the optimum vertex cover

# Rounding



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### The Primal-Dual Method

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### Randomized Rounding

■ Find an optimal solution  $x^* \in R$  to the relaxed problem in polynomial time

■ Round  $x^* \in R$  to a  $\bar{x} \in S$

- ◆  $\bar{x}(u) = 1$  if  $x^*(u) \geq \frac{1}{2}$
- ◆  $\bar{x}(u) = 0$  if  $x^*(u) < \frac{1}{2}$

■ The solution  $\bar{x}$  is feasible:

$$\forall e = (u, v), \bar{x}(u) + \bar{x}(v) \geq 1$$

since either  $x^*(u) \geq \frac{1}{2}$  or  $x^*(v) \geq \frac{1}{2}$

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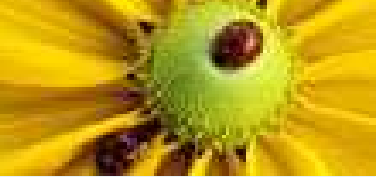
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- The solution is 2-approximated:

$$\sum_{u \in U} w(u) \bar{x}(u) \leq 2 \times \sum_{u \in V} w(u) x^*(u) \leq 2 \times OPT,$$

since  $\bar{x}(u) \leq 2 \times x^*(u)$ .



# Integrality Gap

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#### Randomized Rounding

**Definition:** largest ratio on all instances between the optimum integral solution and the optimum relaxed solution.

One cannot hope to achieve an approximation ratio better than the integrality gap of the relaxation

The rounding step should pay a factor at least equal to the integrality gap of the relaxation

# Integrality Gap for Vertex Cover

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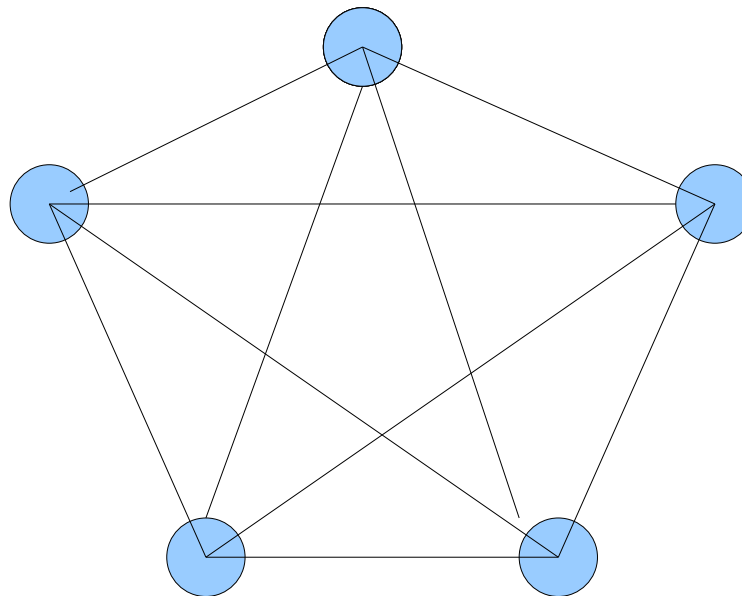
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- On a clique graph the optimal vertex cover is of size  $n - 1$
- $x^*(u) = \frac{1}{2}$  is a feasible fractional solution of value  $n/2$
- The integrality gap is equal to  $2 \left(1 - \frac{1}{n}\right)$
- It is not possible to prove better than 2 approximation for Vertex Cover with this LP



# LP rounding for Set Cover

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## Given:

- $U$ : universe of  $n$  elements  $\{e_1, \dots, e_n\}$
- $\mathcal{S} = \{S_1, \dots, S_m\}$ : collection of  $m$  subsets of  $U$
- $c : S_i \rightarrow \mathbb{R}^+$ : cost function for sets

## Goal:

Find a subcollection of minimum cost that covers  $U$

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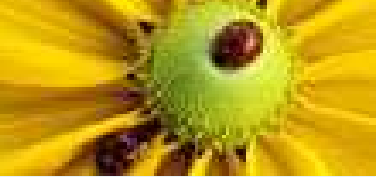
- Randomized Rounding

$$\begin{aligned} \min \quad & \sum_{S \in \mathcal{S}} c(S)x(S) \\ \text{s.t.} \quad & \sum_{S: e \in S} x(S) \geq 1 \quad \forall e \in U \\ & x(S) \in \{0, 1\} \quad S \in \mathcal{S} \end{aligned}$$

LP relaxation:

$$\begin{aligned} \min \quad & \sum_{S \in \mathcal{S}} c(S)x(S) \\ \text{s.t.} \quad & \sum_{S: e \in S} x(S) \geq 1 \quad \forall e \in U \\ & x(S) \in [0, 1] \quad S \in \mathcal{S} \end{aligned}$$

# $f$ -approximation for Set Cover



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### The Primal-Dual Method

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### Randomized Rounding

Let  $f = \max_{e \in U} |\{S \in \mathcal{S} | e \in S\}|$

1. Round to 1 all variables  $x(S) \geq \frac{1}{f}$
2. The solution is feasible since every elements appears in at least one set with  $x(S) \geq \frac{1}{f}$
3.  $ALG \leq f \times OPT^{LP}$

For Vertex cover we have  $f = 2$ .

An  $O(\log n)$  approximation algorithm for Weighted Set Cover will follow in this lecture

# More basic techniques

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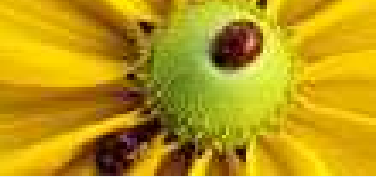
- Primal-dual scheme: consider the dual of a relaxation of the problem:

$$\max\{h(y) : y \in D\} \leq \min\{g(x) : x \in R\}$$

Construct a feasible solution  $x \in S$  for the primal from  $y \in D$  such that

$$f(x) \leq r \times h(y) \leq r \times h(y^*) \leq r \times f(x^*) \leq r \times OPT$$

- Use Randomization in the rounding step: Interpret  $x^*$  as a set of probabilistic values to guide the rounding step. E.g.,  $\bar{x}(u) = 1$  with pb  $x^*(u)$ .
- Use different relaxations: e.g., semi-definite programming relaxations



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**The Primal-Dual Method**

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# The Primal-Dual Method

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$$\min \quad 7x_1 + x_2 + 5x_3$$

$$\text{s.t.} \quad x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

$$x_1, x_2, x_3 \geq 0$$

Lower bounds on OPT:

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3$$

$$+ 5x_1 + 2x_2 - x_3 \geq 16$$

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## Best lower bound on OPT

$$\min \quad 7x_1 + x_2 + 5x_3$$

$$\text{s.t.} \quad x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

$$x_1, x_2, x_3 \geq 0$$

$$\max \quad 10y_1 + 6y_2$$

$$\text{s.t.} \quad y_1 + 5y_2 \leq 7$$

$$-y_1 + 2y_2 \leq 1$$

$$3y_1 - y_2 \leq 5$$

$$y_1, y_2 \geq 0$$

# Primal-Dual Formulation

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$$\begin{aligned} \min \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \geq b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{i=1}^m b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \leq c_j \quad j = 1, \dots, n \\ & y_i \geq 0 \quad i = 1, \dots, m \end{aligned}$$

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## Strong Duality Theorem

**Theorem 1** *If the Primal has finite optimum then the Dual has finite optimum. Let  $x^*$  and  $y^*$  be the primal and the dual optimum solutions. Then*

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

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## Weak Duality Theorem

**Theorem 2** *If  $x$  is feasible for the Primal and  $y$  is feasible for the Dual, then*

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i$$

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i \right) x_j = \tag{1}$$

$$= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i \tag{2}$$

**Corollary 3**  *$x$  and  $y$  are optimal for the Primal and the Dual if and only if (1) and (2) hold with equality.*

# Complementary Slackness Conditions

$x$  and  $y$  are optimal solutions if and only if:

## (1) Primal Complementary Slackness Condition

$$\forall 1 \leq j \leq n \quad \text{either} \quad x_j = 0$$
$$\text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

## (2) Dual Complementary Slackness Condition

$$\forall 1 \leq i \leq m \quad \text{either} \quad y_i = 0$$
$$\text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i$$

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# The Primal-Dual Schema

Ensure Primal Complementary Slackness Condition:

$$\forall 1 \leq j \leq n \quad \text{either} \quad x_j = 0$$
$$\text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

Relax Dual Complementary Slackness Condition

$$\forall 1 \leq i \leq m \quad \text{either} \quad y_i = 0$$
$$\text{or} \quad \sum_{j=1}^n a_{ij} x_j \leq r \times b_i$$

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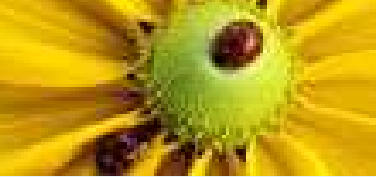
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# The Primal-Dual Schema

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**Theorem 4** *If  $x$  and  $y$  satisfy the conditions of the Primal-Dual schema then*

$$\sum_{j=1}^n c_j x_j \leq r \times \sum_{i=1}^m b_i y_i$$

**Proof:**

$$\begin{aligned} \sum_{j=1}^n c_j x_j &= \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i \right) x_j = \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j \right) y_i \leq r \times \sum_{i=1}^m b_i y_i \end{aligned}$$

□

# Application of the Primal-Dual schema

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- Application of the Primal-Dual schema

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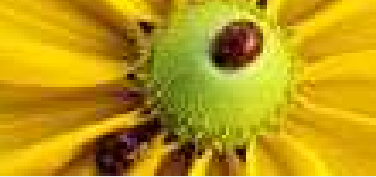
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- Primal is a relaxation of a problem  $P$ .
- $x$  is integral feasible for  $P$
- It follows:

$$\begin{aligned} \sum_{j=1}^n c_j x_j &\leq r \times \sum_{i=1}^m b_i y_i \leq r \times \sum_{i=1}^m b_i y_i^* \\ &= r \times \sum_{j=1}^n c_j x_j^* \leq r \times OPT \end{aligned}$$

Primal-dual schema gives  $r$ -approximation algorithm



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**Primal-dual Method for Vertex Cover**

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# Primal-dual Method for Vertex Cover

# LP formulation for Vertex Cover

Given a graph  $G = (V, E)$ ,  $\delta(v) = \{e = (v, u) \in E\}$

$$\begin{aligned} \text{Primal: } \min \quad & \sum_{v \in V} x(v)w(v) \\ \text{s.t.} \quad & x(u) + x(v) \geq 1 \quad \forall e = (u, v) \in E \\ & x(u) \geq 0 \quad u \in V \end{aligned}$$

$x(u) \leq 1$  in the fractional relaxation can be omitted

$$\begin{aligned} \text{Dual: } \max \quad & \sum_{e \in E} y(e) \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y(e) \leq w(v) \quad \forall v \in V \\ & y(e) \geq 0 \quad \forall e \in E \end{aligned}$$

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# The Primal-Dual Algorithm for Vertex Cover

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1. Increase variable  $y(e)$  for an edge  $e = (u, v)$  until

$$\sum_{e \in \delta(u)} y(e) = w(u) \quad \left( \text{or} \quad \sum_{e \in \delta(v)} y(e) = w(v) \right)$$

2. Set  $x(u) = 1$  ( or  $x(v) = 1$  )

3. Remove all edges adjacent to  $u$  ( or  $v$  )

Repeat until all edges are removed

# Proof of 2-approximation

Primal complementary slackness condition holds

$$\forall u \in V \quad \text{either} \quad x(u) = 0$$
$$\quad \quad \quad \text{or} \quad \sum_{e \in \delta(u)} y(e) = w(u)$$

Dual complementary slackness condition is 2-relaxed

$$\forall e \in E \quad \text{either} \quad y(e) = 0$$
$$\quad \quad \quad \text{or} \quad x(u) + x(v) \leq 2$$

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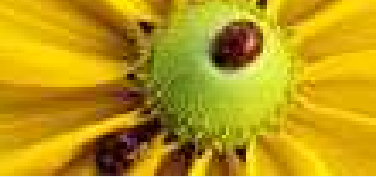
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# Set Cover via Dual Lifting

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## Given:

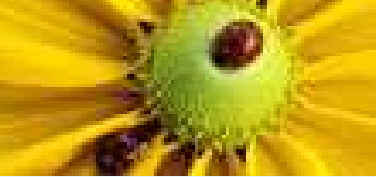
- $U$ : universe of  $n$  elements  $\{e_1, \dots, e_n\}$
- $\mathcal{S} = \{S_1, \dots, S_m\}$ : collection of  $m$  subsets of  $U$
- $c : S_i \rightarrow \mathbb{R}^+$ : cost function for sets

## Goal:

Find a subcollection of minimum cost that covers  $U$

The greedy algorithm achieves an  $O(\log n)$  approximation.

# Greedy algorithm for Set Cover



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- Pick at any iteration the most cost-effective set
- $C_i$ : set of elements yet not covered before set  $S_i$  is selected by Greedy
- $c(S)/(C_i \cap S)$ : cost-effectiveness of set  $S$

1.  $C_0 = U$
2. While  $C_i \neq \emptyset$  do

Find the set  $S$  with  $\min \alpha = c(S)/(C_i \cap S)$

Pick set  $S$  and  $\forall e \in S \cap C_i, price(e) = \alpha$

$$C_{i+1} = C_i / S$$

3. Output the picked sets

# Analysis of Greedy

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Assume  $U$  covered by Greedy in order  $\{e_1, \dots, e_n\}$

**Lemma 5**  $price(e_j) \leq \frac{OPT}{n-j+1}$

**Proof:**

■ At any iteration the optimal solution covers  $C_i$  at cost at most  $OPT$

■ There exists a set of OPT with  $\alpha \leq \frac{OPT}{C_i}$

■ When  $e_j$  is covered at iteration  $i$ ,  $C_i \geq n - j + 1$

■ Since  $e_j$  is covered by the most cost-effective set:

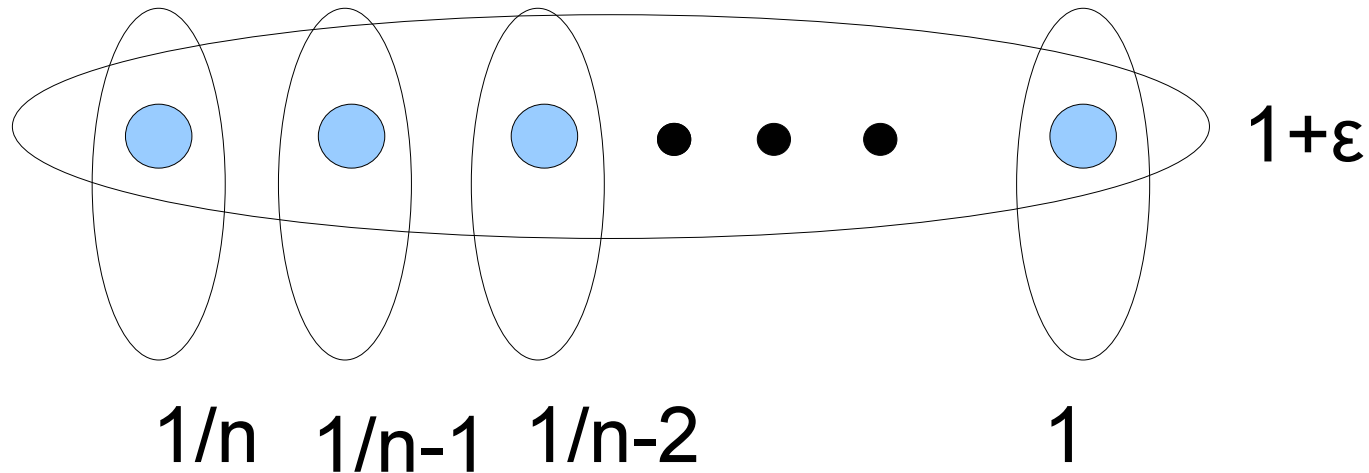
$$price(e_j) \leq \frac{OPT}{C_i} \leq \frac{OPT}{n-j+1}$$

□

**Theorem 6**

$$ALG = \sum_{j=1}^n price(e_j) \leq OPT \times \sum_{j=1}^n \frac{1}{n-j+1} = OPT \times H_n$$

# A tight example for Greedy



Greedy outputs all singleton sets with cost

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Optimum cost is  $1 + \epsilon$ .

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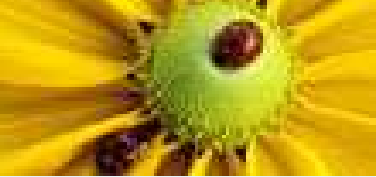
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# Dual-fitting analysis of Greedy Set Cover



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## Dual-fitting method:

- Show that the integral solution is fully paid by an unfeasible dual solution
- The dual solution can be made feasible by scaling down each variable by a factor  $f$



$$ALG = DUAL^{unf} = f \times DUAL^{feas} \leq f \times OPT^{LP} \leq f \times OPT$$

- Alternative to argue about complementary slackness conditions

# LP formulation for Set Cover

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$$\begin{aligned} \text{Primal: } \quad & \min \sum_{S \in \mathcal{S}} c(S)x(S) \\ & \text{s.t. } \sum_{S: e \in S} x(S) \geq 1 \quad \forall e \in U \\ & \quad \quad \quad x(S) \geq 0 \quad S \in \mathcal{S} \end{aligned}$$

Constraint  $x(S) \leq 1$  can be omitted

$$\begin{aligned} \text{Dual: } \quad & \max \sum_{e \in U} y(e) \\ & \text{s.t. } \sum_{e \in S} y(e) \leq c(S) \quad \forall e \in U \\ & \quad \quad \quad y(e) \geq 0 \quad \forall e \in U \end{aligned}$$

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Interpret the Greedy solution as an unfeasible dual:

$$y(e) = price(e), ALG = \sum_{e \in U} y(e)$$

**Lemma 7**  $y'(e) = \frac{price(e)}{H_n}$  is dual feasible.

- Consider any set  $S = \{e_1, \dots, e_k\}$  with elements numbered by the order they are covered by Greedy.

- $S$  can cover  $e_i$  at price  $\leq \frac{c(S)}{k-i+1}$  when  $i$  is covered.

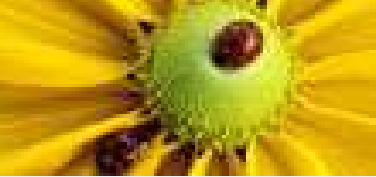
- Since Greedy picks the most cost-effective set:

$$price(e_i) \leq \frac{c(S)}{k-i+1}$$

- Dual variables  $y'(e) \leq \frac{1}{H_n} \frac{c(S)}{k-i+1}$

- Dual constraint for set  $S$ :

$$\sum_{i=1}^k y'(e_i) \leq \frac{c(S)}{H_n} \left( \frac{1}{k} + \frac{1}{k-1} + \dots + 1 \right) = c(S) \frac{H_k}{H_n} \leq c(S)$$



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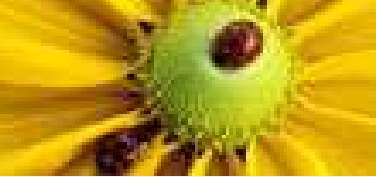
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**Randomized Rounding**

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- Randomized Rounding for Set Cover
- Make the solution feasible
- Make the solution feasible

# Randomized Rounding for Set Cover



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- Make the solution feasible

- Make the solution feasible

- Interpret primal variables in the fractional relaxation as probabilities
- Obtain a primal solution by setting variables to 1 independently with probability equal to the fractional value
- The expected cost of the solution is equal to the fractional optimum ..... but
- **Solution may not be feasible**
- Repeat as many times as needed to enforce feasibility

# Randomized Rounding for Set Cover

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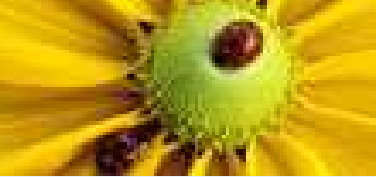
- Make the solution feasible

- Make the solution feasible

$$\begin{aligned} \min \quad & \sum_{S \in \mathcal{S}} c(S)x(S) \\ \text{s.t.} \quad & \sum_{S: e \in S} x(S) \geq 1 \quad \forall e \in U \\ & x(S) \in [0, 1] \quad S \in \mathcal{S} \end{aligned}$$

- Pick set  $S$  with probability  $p(S) = x^*(S)$
- $E[ALG] = \sum_{S \in \mathcal{S}} c(S)p(S) = \sum_{S \in \mathcal{S}} c(S)x^*(S) = OPT^{LP} \leq OPT$
- Is the solution feasible?

# Make the solution feasible



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- **Make the solution feasible**

- Make the solution feasible

- For an element  $a$ :  $\{S_1, \dots, S_k\} = \{S \in \mathcal{S} : a \in S\}$

$$\begin{aligned} \Pr[e \text{ is covered}] &= 1 - (1 - p(S_1)) \times \dots \times (1 - p(S_k)) \\ &\geq 1 - \left(1 - \frac{1}{k}\right)^k \\ &\geq 1 - \frac{1}{e} \end{aligned}$$

since  $p(S_1) + \dots + p(S_k) \geq 1$

- Each element  $a \in U$  is covered with  $Pb \geq 1 - \frac{1}{e}$

# Make the solution feasible

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- Make the solution feasible

- Make the solution feasible

- Pick  $d \log n$  subcollections  $C' = C_1 \cup \dots \cup C_{d \log n}$  with  $d$  such that:

$$Pr[a \text{ not covered}] \leq \left(\frac{1}{e}\right)^{d \log n} \leq \frac{1}{4n}$$

- $E[COST(C')] \leq d \times \log n OPT^{LP}$
- $Pr[COST(C') \geq 4d \times \log n OPT^{LP}] \leq \frac{1}{4}$
- $Pr[C' \text{ not feasible}] \leq n \times \frac{1}{4n} \leq \frac{1}{4}$
- $Pr[COST(C') \leq 4d \times \log n OPT^{LP} \text{ AND } C' \text{ feasible}] \geq \frac{1}{2}$
- Expected[number of repetitions] = 2