RANDOMIZED ROUNDING

SET COVER :

$$\min \sum_{S \in \mathcal{S}} c(S)\, x_S$$

$$\text{st} \quad \sum_{S : e \in S} x_S \geq 1 \qquad \forall e \in U$$

$$x_S \in \{0,1\} \overset{\text{RELAX}}{\Longrightarrow} \qquad x_S \geq 0$$

F-apx: SOLVE LP RELAXATION

ROUND TO 1 ALL

$$x_S \geq \frac{1}{f}$$

$$f = \max_{e \in U} |\{S \in \mathcal{S} : e \in S\}|$$

PROOF: $\forall e, \exists S \in \mathcal{S} : y_S \geq \frac{1}{f} \Rightarrow$ SOLUTION FEASIBLE

$$c(SOL) \leq f \; OPT_{LP}$$

RANDOMIZED ROUNDING:

PICK SET S WITH PB $p_S = x_S$

$$E[\text{COST}(\text{SOL})] = \sum_{S \in \mathcal{S}} \Pr[S \text{ is picked}]\, c(S)$$

$$= \sum_{S \in \mathcal{S}} p_S\, c(S) = OPT_{LP}$$

IS THE SOLUTION FEASIBLE?

$\forall e$, $S_1, S_2, \dots, S_k = \{S : e \in S\}$

$\Pr[e$ IS COVERED BY SOL$]$

$$\geq 1 - (1 - p_{S_1})(1 - p_{S_2}) \cdots (1 - p_{S_k})$$

$$\geq 1 - \left(1 - \frac{1}{k}\right)^k \geq 1 - \frac{1}{e}$$

SINCE $p_{S_1} + p_{S_2} + \dots + p_{S_k} \geq 1$

EACH ELEMENT IS COVERED WITH PB $\geq 1 - \frac{1}{e}$

PICK $d \log n$ SUBCOLLECTIONS WHERE

$$\left(\frac{1}{e}\right)^{d\log n} \leq \frac{1}{4n} \qquad C' = C_1 \cup \dots \cup C_{d\log n}$$

$$\Pr[a \text{ IS NOT COVERED}] \leq \left(\frac{1}{e}\right)^{d\log n} \leq \frac{1}{4n}$$

SUMMING OVER ALL n elements:

$$\Pr[C' \text{ IS NOT FEASIBLE}] \leq n \frac{1}{4n} \leq \frac{1}{4}$$

$$E[\mathrm{COST}(C')] \leq OPT_{LP}\, d \log n$$

$$\Pr[c(C') \geq OPT_{LP} \cdot 4 d \log n] \leq \frac{1}{4}$$

$$\Downarrow$$

$$\Pr[C' \text{ IS FEASIBLE AND COST} \leq OPT_{LP} 4 d\log n]$$

$$\geq \frac{1}{2}$$

$$E[\#\text{ OF REPETITIONS}] \leq 2$$

3.1

L.P. RELAXATION AND RANDOMIZED ROUNDING FOR MAX-WEIGHTED SAT

- USE RANDOMIZATION IN THE ROUNDING STEP
- THE **VALUE OF** THE OBJECTIVE FUNCTION OBTAINED BY THE ALGORITHM IS A RANDOM VARIABLE
- $E[ALG(I)] \geq \frac{1}{2} OPT(I)$

  $\forall I$
- DERANDOMIZATION IS OFTEN POSSIBLE, MAY BE ALWAYS ?

MAX WEIGHTED SAT

$C = \{C_1, C_2, \ldots, C_t\}$ COLLECTION OF CLAUSES IN DISJUNCTIVE FORM

w_i : WEIGHT OF CLAUSE C_i

$C_i = \ell_{i,1} \vee \ell_{i,2} \vee \ldots \vee \ell_{i,k_i}$

$\ell_{i,j} \in \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$

n BOOLEAN VARIABLES

FIND AN ASSIGNMENT OF

$x_1, \ldots, x_n \in \{0,1\}^n$ TO

MAXIMIZE THE WEIGHT OF SATISFIED CLAUSES

2- APPROX ALGORITHM

[JOHNSON, '74]

- SET x_j TO 1 WITH Pb $\frac{1}{2}$
  $\emptyset$ OTHERWISE

- C_i IS SATISFIED WITH PB $\geq 1-\left(\frac{1}{2}\right)^{K_i}$
  $\geq \frac{1}{2}$

- $E[ALG] \geq \frac{1}{2} \sum_{i=1}^{t} w_i \geq \frac{1}{2}$ OPT

  WE USE $\sum_{i=1}^{t} w_i$ AS UPPER BOUND OVER OPT.

- EASY TO DERANDOMIZE

4/3 - APPROX [YANNAKAKIS, '92]

TREAT DIFFERENTLY CLAUSES OF 1 LITERAL

4/3 APPROX ALG. BASED ON L.P. [GOEMANS, WILLIAMSON, '93]

- THE SOLUTION OF THE LP RELAXATION GIVES A SET OF PROBABILITIES $x_1^*, \ldots, x_n^*$

- SET x_j TO $\begin{cases} 1 \\ 0 \end{cases}$ WITH PB $\begin{cases} x_j^* \\ 1 - x_j^* \end{cases}$

T_i : SET OF BOOLEAN VARIABLES THAT OCCUR UNNEGATED IN C_i

F_i : SET OF BOOLEAN VARIABLES THAT OCCUR NEGATED IN C_i

Z_i : 1 if C_i is SATISFIED

$$\text{MAX} \sum_{i=1}^{t} Z_i \, \omega_i$$

$$\sum_{j \in T_i} x_j + \sum_{j \in F_i} (1 - x_j) \geq Z_i$$

$$x_j, Z_i \in \{0, 1\} \qquad j = 1, \ldots, n \quad i = 1, \ldots, t$$

ROUNDING SCHEME

- x^*, z^* : OPTIMAL SOLUTION OF THE RELAXED LP

$$x_j \geq 0 \,, \quad z_i \leq 1$$

- SET $\quad x_j = 1$ WITH Pb x_j^*

  $\quad\quad\; x_j = 0$ WITH Pb $1 - x_j^*$

LEMMA: Pb [C_i is SATISFIED] $\geq$

$$\geq 1 - \prod_{j \in I_i} (1 - x_j^*) \prod_{j \in F_i} x_j^*$$

$$\geq \alpha_{K_i} z_i^*$$

$$\alpha_{K_i} = 1 - \left(1 - \frac{1}{K_i}\right)^{K_i}$$

PROOF: $C_i = l_{i,1} \vee l_{i,2} \vee \quad \vee\, l_{i,k_i}$

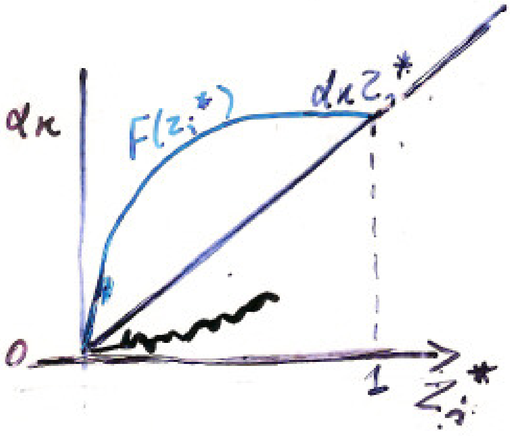
ASSUME EVERY VARIABLE IS UNNEGATED (OTHERWISE SUBSTITUTE $\bar{x}_j$ BY x_j AND x_j BY $\bar{x}_j$ IN ANY CLAUSE)

$$P_B[C_i \text{ SATISFIED}] \geq 1 - \prod_{j=1}^{k_i}(1 - x_j^*)$$

SINCE $\dfrac{x_1 + \dots + x_{k_i}}{k_i} \geq \sqrt[k_i]{x_1 \cdot x_2 \cdots x_{k_i}}$

$$\geq 1 - \left(\frac{\sum_{j=1}^{k_i}(1 - x_j^*)}{k_i}\right)^{k_i}$$

$$\geq 1 - \left(1 - \frac{\sum_{j=1}^{k_i} x_j^*}{k_i}\right)^{k_i}$$



$$\geq 1 - \left(1 - \frac{z_i^*}{k_i}\right)^{k_i}$$

$$\geq \left(1 - \left(1 - \frac{1}{k_i}\right)^{k_i}\right) z_i^*$$

$$= \alpha_{k_i} z_i^*$$

$\left(\frac{e}{e-1}\right) \sim 1.57$ APPROX ALG.

$$E[ALG] \geq \sum_{i=1}^{t} \left(1 - \left(1 - \frac{1}{K_i}\right)^{K_i}\right) w_i z_i^*$$

since $\left(1-\frac{1}{K}\right)^K \leq \frac{1}{e}$

$$\geq \left(1 - \frac{1}{e}\right) \underline{\sum_{i=1}^{t} w_i z_i^*}$$

$$\geq \left(\frac{e-1}{e}\right) OPT$$

- LP RELAXATION GIVES AN UPPER BOUND BETTER THAN $\sum_{i=1}^{t} w_i$

4/3 - APPROX ALG.

JOHNSON : $E[ALG] \geq \sum_{i=1}^{t} \left(1 - \frac{1}{2^{k_i}}\right) w_i Z_i^*$

GOOD FOR BIG K_i

GOEMANS AND WILLIAMSON : $E[ALG] \geq \sum_{i=1}^{t} \left(1 - \left(1 - \frac{1}{K_i}\right)^{K_i}\right) w_i Z_i^*$

GOOD FOR SMALL K_i

COMBINE TWO ALGORITHMS WITH DISJOINT BAD CASES

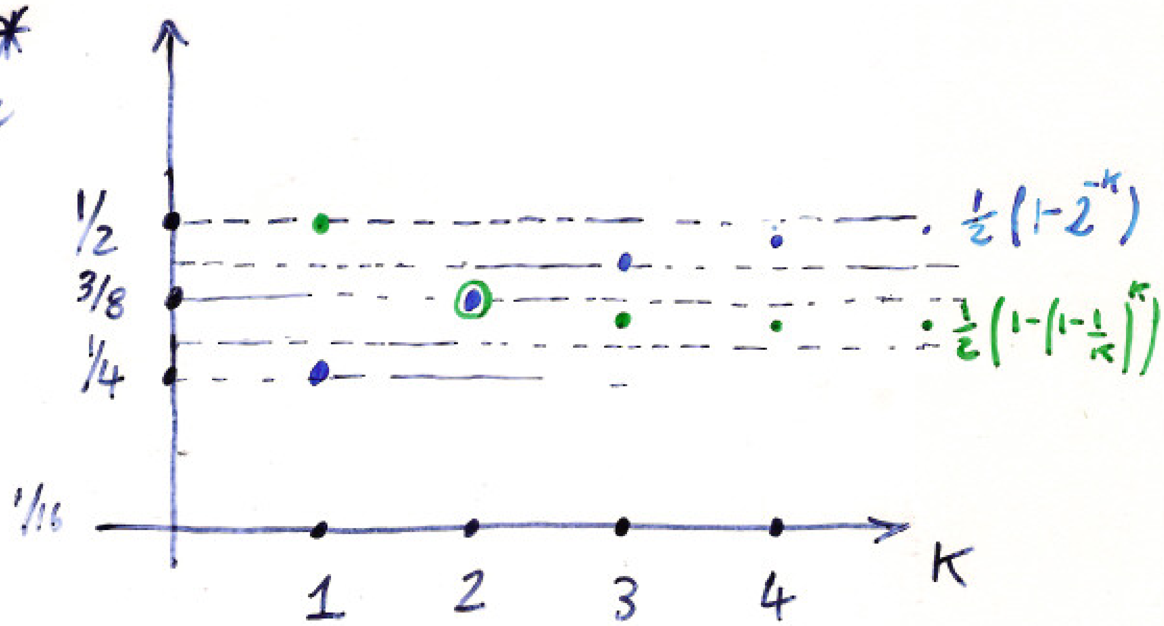
- RUN JOHNSON WITH PB $\frac{1}{2}$
  G & W WITH PB $\frac{1}{2}$

PROOF :

$$\Pr[C_i \text{ is satisfied}] \geq$$

$$\frac{1}{2}\left[\left(1-2^{-k_i}\right)+\left(1-\left(1-\frac{1}{k_i}\right)^{k_i}\right)\right] z_i^*$$

$$\geq \frac{3}{4} z_i^*$$



$$E[ALG] \geq \frac{3}{4}\sum_{i=1}^{t} w_i z_i^* \geq \frac{3}{4} OPT$$

∎