For the $U$ function we use the logarithmic penalty function derived in Section 3.1. Thus $s(r) = r$, and $I(x) = -\sum_{i=1}^{m} \ln g_i(x)$. The choice of $s$ and $I$ satisfies the requirements stated before. Then $L(x, r) = x_1 + x_2 - r \ln (-x_1^2 + x_2) - r \ln x_1$. This simple problem can be solved analytically using the fact that it is twice differentiable. Using the first-order necessary condition (2.12),

$$1 + \frac{r(2x_1)}{-x_1^2 + x_2} - \frac{r}{x_1} = 0$$

and

$$1 - \frac{r}{-x_1^2 + x_2} = 0.$$ 

Solving yields

$$x_1(r) = \frac{-1 \pm \sqrt{1 + 8r}}{4}.$$ 

Since $x_1(r)$ must be positive, only the root $x_1(r) = \frac{-1 + \sqrt{1 + 8r}}{4}$ is of interest. Then $x_2(r)$ becomes

$$x_2(r) = \frac{(-1 + \sqrt{1 + 8r})^2}{16} + r.$$ 

That these values for $x_1(r)$ and $x_2(r)$ are local minima readily follows by observing that they satisfy the sufficient conditions (2.39) and (2.41) of Section 2.3.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Values of $r$, $x(r)$ for Example</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>1.000</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.500</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0.250</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.100</td>
</tr>
</tbody>
</table>

In Table 1 are shown the computed values of $x(r)$ for four different values of $r$. In Figure 3 the problem is shown geometrically and indicates the points corresponding to these values of $r$. In the limit, the minimizing points approach the solution $(0, 0)^T$ as $r_k \to 0$.

In this problem there is only one unconstrained local minimum for each value of $r$. The problem happens to have a unique solution. It turns out that in problems with many local minima there is (subject to a mild regularity condition) a sequence of local unconstrained minima converging to each set of
Figure 3 Solution of example by the unconstrained logarithmic function. The shaded area is the feasible region. The problem is:

\[
\begin{align*}
\text{minimize } & \ f(x_1, x_2) \equiv x_1 + x_2 \\
\text{subject to } & \ g_1(x_1, x_2) \equiv -x_1^2 + x_2 \geq 0 \\
& \ g_2(x_1, x_2) \equiv x_1 \geq 0.
\end{align*}
\]

constrained local minima. The precise statement and proof of this fact are given in the next section.

3.3 CONVERGENCE PROOFS FOR INTERIOR POINT ALGORITHMS

In this section the existence of unconstrained U-function minima converging to local constrained minima of Problem B, assuming certain topological properties, is proved. The definitions of the usual terms of analysis are assumed known to the reader and can be found in such texts as [102] and [3]. The following definitions and lemmas are intended to make more precise the concepts of “local” minima, constrained and unconstrained.

We state without proof the following well-known result (see [102], page 67).
(ii) \( \lim_{k \to \infty} s(r_k)I[x(r_k)] = 0 \),
(iii) \( \lim_{k \to \infty} f[x(r_k)] = v^* \),
(iv) \( \lim_{k \to \infty} U[x(r_k), r_k] = v^* \),
(v) \( \{f[x(r_k)]\} \) is a monotonically decreasing sequence, and
(vi) \( \{I[x(r_k)]\} \) is a monotonically increasing sequence.

**Proof.** The proof of Parts i–iv is a by-product of the proof of Theorem 10. Let \( f^k \) denote \( f[x(r_k)] \), and so on. Then, because each \( x^k \) is a global unconstrained minimum in the interior of the compact set \( S \) about \( A^* \),

\[
    f^k + s(r_k)I^k \leq f^{k+1} + s(r_k)I^{k+1},
\]

\[
    f^{k+1} + s(r_{k+1})I^{k+1} \leq f^k + s(r_{k+1})I^k.
\]

Adding \( s(r_k)/s(r_{k+1}) \) times the first inequality to the second, rearranging; and dividing [by our assumptions on \( \{r_k\} \) and \( s(r), s(r_k) > s(r_{k+1}) \)] yields

\[
    f^k \geq f^{k+1}.
\]

This proves Part v. Using this and the first inequality above proves Part vi. Q.E.D.

The following example demonstrates an application of the theorem.

**Example.**

minimize \( x_2 \)

subject to

\[
    x_2 - \sin x_1 - \frac{x_1}{2} \geq 0.
\]

Consider the \( U \) function to be the logarithmic function

\[
    L(x, r) = x_2 - r \ln \left[ x_2 - \sin x_1 - \frac{x_1}{2} \right].
\] (3.11)

Since these functions are twice differentiable we use first and second derivatives to determine the local unconstrained minima of (3.11). Using (2.12),

\[
    \frac{r[\cos (x_1) + \frac{1}{2}]}{[x_2 - \sin (x_1) - x_1/2]} = 0
\]

and

\[
    1 - \frac{r}{[x_2 - \sin (x_1) - x_1/2]} = 0.
\]
There are two sets of possible solutions:

\[ x_1(r) = \frac{2\pi}{3} \pm 2n\pi, \quad n = 0, 1, 2, 3, \ldots, \quad (3.12) \]

\[ x_2(r) = \sin \left[ \frac{2\pi}{3} \pm 2n\pi \right] + \frac{\pi}{3} \pm n\pi + r, \quad n = 0, 1, \ldots \quad (3.13) \]

and

\[ x_1(r) = \frac{4\pi}{3} \pm 2n\pi, \quad n = 0, 1, 2, \ldots, \quad (3.14) \]

\[ x_2(r) = \sin \left( \frac{4\pi}{3} \pm 2n\pi \right) + \frac{2\pi}{3} \pm n\pi + r, \quad n = 0, 1, \ldots \quad (3.15) \]

The matrix of the second partial derivatives of (3.11) is

\[
\begin{bmatrix}
-\sin [x_1(r)] & 0 \\
0 & \frac{1}{r}
\end{bmatrix}
\]

Figure 4  Convergence to local unconstrained minima. The problem is:

minimize \( x_2 \)

subject to

\[ x_2 - \sin x_1 - \frac{x_1}{2} \geq 0. \]
Let $O_i(g_i) = [(g_i - |g_i|)/2]^2$, let $O(x) = \sum O_i[g_i(x)]$, and let $p(t) = t$. Then

$$T = -x_1x_2 + t \left[ \left( -x_1 - x_2^2 + 1 - |x_1 - x_2^2 + 1| \right)^2 \right.$$  
$$+ t \left[ \left( x_1 + x_2 - \frac{|x_1 + x_2|}{2} \right)^2 \right]$$  \hspace{1cm} (4.9)$$

is an exterior point unconstrained minimization function.

In Table 3 are given the values of $x(t)$ for four different values of $t$. These are plotted in Figure 5 and are seen to approach the optimum from the region of infeasibility as $t$ increases.

![Diagram](image-url)

Figure 5  Solution of example by an exterior unconstrained algorithm. The shaded area is the feasible region. The problem is:

minimize $-x_1x_2$

subject to

$$-x_1 - x_2^2 + 1 \geq 0,$$

$$x_1 + x_2 \geq 0.$$
4.2 Convergence Theorem for Exterior Point Algorithms

Table 3 Values of $x(t)$ for Example

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x_1(t)$</th>
<th>$x_2(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>1.0</td>
<td>0.67</td>
</tr>
<tr>
<td>$t_2$</td>
<td>2.0</td>
<td>0.62</td>
</tr>
<tr>
<td>$t_3$</td>
<td>3.0</td>
<td>0.61</td>
</tr>
<tr>
<td>$t_4$</td>
<td>10.0</td>
<td>0.58</td>
</tr>
<tr>
<td>Optimum</td>
<td>2/3</td>
<td>$\sqrt{3}/3$</td>
</tr>
</tbody>
</table>

There are several important differences between interior point and exterior point algorithms other than the obvious one that the former are always feasible and the latter are always infeasible. These differences are discussed at length in Section 4.3. In the next section a very important similarity to the interior point algorithm—the convergence to compact sets of local minima of the exterior point algorithm—is proved. The proof is based on the fundamental Theorem 7 and needs only the continuity of the problem functions.

4.2 CONVERGENCE THEOREM FOR EXTERIOR POINT ALGORITHMS

In this section a theorem on the existence of unconstrained $T$-function minima converging to local solutions of Problem B analogous to Theorem 8 for interior point algorithms is given. The essential difference in the requirements on the problem is that in this instance the interior of the feasible region ($R^0$) need not be nonempty.

Theorem 9 [Convergence to Compact Sets of Local Minima by Exterior Point Algorithms]. If (a) the functions $f, g_1, \ldots, g_m$ are continuous, (b) the function $T = f(x) + p(t)O(x)$ is an exterior point unconstrained minimization function (that is, $p$ and $O$ satisfy the properties stated in Section 4.1), (c) a set of points $A^*$, which are local minima with minimum value $v^*$ of Problem B, is a nonempty, isolated compact set, and (c) $\{t_k\}$ is a strictly increasing unbounded non-negative sequence then

(i) there exists a compact set $S$ as given in Theorem 7 such that $A^* \subset S^0$ and for $t_k$ large enough the unconstrained minima of $T(x, t_k)$ in $S^0$ exist and every limit point of any subsequence $\{x^{k}\}$ of the minimizing points is in $A^*$;
(ii) $\lim_{k \to \infty} p(t_k)O[x(t_k)] = 0$;
(iii) $\lim_{k \to \infty} f[x(t_k)] = v^*$;
(iv) $\lim_{k \to \infty} T[x(t_k), t_k] = v^*$, $T[x(t_k), t_k] \leq v^*$, all $k$;
(v) $\{f[x(t_k)]\}$ is a monotonically increasing sequence; and
(vi) $\{O[x(t_k)]\}$ is a monotonically decreasing sequence.
Proof. Parts i–iv are a by-product of Theorem 10. Let $f^k$ denote $f[x(t_k)]$, and so on. Then, because each $x^k$ is a global unconstrained minimum in the interior of a compact set containing $A^*$,

\[ f^k + p(t_k)O^k \leq f^{k+1} + p(t_k)O^{k+1} \]

and

\[ f^{k+1} + p(t_{k+1})O^{k+1} \leq f^k + p(t_{k+1})O^k. \]

Figure 6  Solution of example by a mixed algorithm. The problem is:

\[
\begin{align*}
\text{minimize } & \ln x_1 - x_2 \\
\text{subject to } & x_1 - 1 \geq 0, \\
& x_1^2 + x_2^2 - 4 = 0.
\end{align*}
\]
Adding \( p(t_{k+1})/p(t_k) \) times the first inequality to the second, and canceling, yields

\[
\left[ \frac{p(t_{k+1})}{p(t_k)} - 1 \right] f^k \leq \left[ \frac{p(t_{k+1})}{p(t_k)} - 1 \right] f_{k+1}.
\]

Using our assumptions on \( \{t_k\} \) and \( p(t) \), Part v follows. Using (v) and the second inequality above proves (vi).

Q.E.D.

As with the interior point algorithm, several corollaries follow from this theorem.

**Corollary 11.** If a set of points that are global minima of \( f \) in \( R \) constitute a bounded isolated set, then the conclusions of Theorem 9 apply for this set.

It is interesting that unlike the corresponding result for the interior point method, the minimizing sequence \( \{x^k\} \) need not be global minima.

**Corollary 12.** If \( x^* \) is an isolated local minimum with local minimum value \( v^* \), then the sequence \( \{x(t_k)\} \) of Theorem 9 is such that \( x(t_k) \) converges to \( x^* \).

Note that interiority is not needed. Thus the results derived here are even more general than the results of Theorem 8.

Note also that equality constraints can be handled by simply rewriting each equality as two inequalities, in the obvious manner, assuming only that the resulting system subscribes to the conditions required for Theorem 9. (This handling of equalities and the resulting form of the unconstrained function is indicated in the example of Figure 6.)

### 4.3 MIXED INTERIOR POINT–EXTERIOR POINT ALGORITHMS

For certain problems, for example, problems with equality constraints, interior methods do not apply since there is no "interior" to the region of feasibility. For such problems we would like to handle that subset of constraints differently. For other reasons, for example, when continued satisfaction of certain constraints is demanded, it is useful to develop a mixed algorithm, one that maintains strict satisfaction of some of the constraints as computations proceed, requiring others to be satisfied only as the solution is approached. In the following discussion a straightforward mixed algorithm is developed, combining the methods developed above and the theoretical convergence proved under the same conditions of compactness and continuity as for the two "pure" methods.
dove $\mu^*$ è tale che

$$\left\{ \frac{1}{\epsilon_k} (g(x_k) + Y_k y_k) \right\}_K \rightarrow \mu^*. \quad (34)$$

Dalla (33) segue intanto che

$$\nabla f(x^*) + \nabla g(x^*) \mu^* = 0, \quad (35)$$

$$- (y^*)_i^2 \mu^*_i = 0. \quad (36)$$

La (36) insieme alla (33) danno

$$g_i(x^*) \mu^*_i = 0, \quad i = 1, \ldots, m$$

ovvero le condizioni di complementarietà. Infine, notiamo che dalla (34) considerato che $y^*_k = \max\{0, -g_i(x_k)\}$, segue che

$$\left\{ \frac{1}{\epsilon_k} \max\{g(x_k), 0\} \right\}_K \rightarrow \mu^*. \quad (34)$$

Inoltre, siccome ogni elemento della successione $\{(1/\epsilon_k) \max\{g(x_k), 0\}\}$ è positivo o nullo, segue anche che

$$\mu^* \geq 0,$$

il che conclude la prova.

Dall’analisi condotta fin’ora, sembrerebbe emergere che l’algoritmo SEQPEN, a meno di alcune situazioni patologiche, è un algoritmo tutto sommato robusto, ammesso che per ogni indice $k$ al passo 3 si è in grado di determinare una soluzione del problema (17) (anche non esatta ma con una precisione sempre maggiore). Purtroppo questa conclusione è falsa. Infatti, trovare una soluzione del problema (17) al passo 3 dell’algoritmo SEQPEN diventa sempre più difficile mano a mano che il parametro di penalità $\epsilon_k \downarrow 0$. Questa crescente complessità è dovuta essenzialmente al crescente mal condizionamento della funzione di penalità come mostrato dall’esempio che segue.

**Esempio 4** Si consideri il seguente problema non lineare

$$\min \quad -x - y$$

$$x^2 + y^2 = 1.$$ 

Questo problema ammette $(x^*, y^*) = (\sqrt{2}/2, \sqrt{2}/2)$ come unico punto di minimo globale.

La funzione di Penalità esterna per questo problema è la seguente.

$$P(x, y; \epsilon) = -x - y + \frac{1}{\epsilon} (1 - x^2 - y^2)^2.$$ 

In Figura 4 sono riportate le curve di livello della $P(x, y; \epsilon)$ per $\epsilon = 1, 1/2, 1/20$. Valgono le medesime osservazioni dell’Esempio 1.
Figure 4: Curve di livello di $P(x; \epsilon)$ ($\epsilon = 1, 0.5, 0.05$) per il problema dell'Esempio 4. Il cerchio tratteggiato rappresenta il vincolo del problema.