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Analysis and Control of Multi-Robot Systems

Elements of Port-Hamiltonian Modeling

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Introduction to Port-Hamiltonian Systems

- Port-Hamiltonian Systems (PHS): strong link with passivity

\[ E_{\text{in}} \rightarrow \sum \rightarrow E_{\text{out}} \]

\[ E_{\text{out}} \leq E_{\text{in}} \]

- Passivity:
  - I/O characterization
  - “Constraint” on the I/O energy flow
  - Many desirable properties
    - Stability of free-evolution
    - Stability of zero-dynamics
    - Easy stabilization with static output-feedback
    - Modularity: passivity is preserved under proper compositions

- However, no insights on the structure of a passive system
- PHS: focus on the structure behind passive systems
Review of the mass-spring-damper example

\[ m\ddot{x} + b\dot{x} + kx = f \]

This system was shown to be passive w.r.t. the pair \((u, y)\) with \(u = f\), \(y = \dot{x}\), and as storage function the total energy (kinetic + potential)

\[ V = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \]

Indeed, it is

\[ \dot{V} = f\dot{x} - b\dot{x}^2 = yu - by^2 \leq yu \]

But why is it passive? We must investigate its internal structure...
The spring-mass system is made of 2 components (2 states)

- Assume for now no damping $b = 0$

- Mass = kinetic energy
  \[ K = \frac{1}{2} m \dot{x}^2 = \frac{p^2}{2m} \]
  \[ p = m \dot{x} \]

- Spring = elastic energy
  \[ V = \frac{1}{2} k x^2 \]

Let us consider the 2 components separately

**Kinetic energy storing**
\[ \mathcal{K}: \begin{cases} 
  \dot{p} &= f_p \\
  v_p &= \frac{\partial K}{\partial p} = \frac{p}{m} (= \dot{x}) 
\end{cases} \]

**Potential energy storing**
\[ V: \begin{cases} 
  \dot{x} &= v_x \\
  f_x &= \frac{\partial V}{\partial x} = kx 
\end{cases} \]

Note that these (elementary) systems are the "integrators with nonlinear outputs" we have seen before

We know they are passive w.r.t. $(v_p, f_p)$ and $(v_x, f_x)$, respectively
Mass-spring-damper vs. PHS

\[ K : \begin{cases} \dot{p} = f_p \\ v_p = \frac{\partial K}{\partial p} = \frac{p}{m} (= \dot{x}) \end{cases} \]

Kinetic energy storing

\[ V : \begin{cases} \dot{x} = v_x \\ f_x = \frac{\partial V}{\partial x} = kx \end{cases} \]

Potential energy storing

• Let us interconnect them in “feedback”

\[ v_x = v_p, \ f_p = -f_x + f \]

• The resulting system can be written as

\[
\begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial x} \\
\frac{\partial H}{\partial p}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix} f
\]

where \( H(x, p) = K(p) + V(x) \) is the total energy (Hamiltonian)

• Prove that (■) is equivalent to \( m\ddot{x} + kx = f \)
Mass-spring-damper vs. PHS

- How does the energy balance look like?

\[
\dot{H} = \begin{bmatrix}
\frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p}
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial x} \\
\frac{\partial H}{\partial p}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p}
\end{bmatrix} \begin{bmatrix}
0 & 1
\end{bmatrix} f = \frac{\partial H^T}{\partial p} f = f^T v_p
\]

\[\equiv 0\]

- We find again the passivity condition w.r.t. the pair \((f, v_p)\)

- The subsystems \(\mathcal{K}\) and \(\mathcal{V}\) exchange energy in a power-preserving way - no energy is created/destroyed

- The subsystem \(\mathcal{K}\) exchanges energy with the “external world” through the pair \((f, v_p)\)

- Total energy \(H\) can vary only because of the power flowing through \((f, v_p)\)
What if a damping term $b > 0$ is present in the system?

By interconnecting $K$ and $\mathcal{V}$ as before (feedback interconnection), we get

$$
\begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} - 
\begin{bmatrix}
0 & 0 \\
0 & b
\end{bmatrix} 
\begin{bmatrix}
\frac{\partial H}{\partial x} \\
\frac{\partial H}{\partial p}
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} f
$$

Skew-symmetric  Positive semi-def.

Prove that (■) is equivalent to

$$
\dot{m} \ddot{x} + b \dot{x} + k x = f
$$

The energy balance now reads

$$
\dot{H} = - \left[ \frac{\partial H^T}{\partial x} \quad \frac{\partial H^T}{\partial p} \right] 
\begin{bmatrix}
0 & 0 \\
0 & b
\end{bmatrix} 
\begin{bmatrix}
\frac{\partial H}{\partial x} \\
\frac{\partial H}{\partial p}
\end{bmatrix} + 
\begin{bmatrix}
\frac{\partial H^T}{\partial x} \\
\frac{\partial H^T}{\partial p}
\end{bmatrix} 
\begin{bmatrix}
0 \\
1
\end{bmatrix} f \leq \frac{\partial H^T}{\partial p} f = f^T v_p
$$
Mass-spring-damper vs. PHS

\[ \dot{H} = - \left[ \begin{array}{cc} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial p} \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{array} \right] + \left[ \begin{array}{cc} \frac{\partial H^T}{\partial x} & \frac{\partial H^T}{\partial p} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial p} \end{array} \right] \left[ \begin{array}{c} f \\ 0 \end{array} \right] \leq 0 \]

\[ f \leq \frac{\partial H^T}{\partial p} f = f^T v_p \]

- Again the passivity condition w.r.t. the pair \((f, v_p)\)

- Total energy \(H\) can now
  - vary only because of the power flowing through \((f, v_p)\)
  - decrease because of internal dissipation

- But still, power-preserving exchange of energy between \(K\) and \(V\)

Robuffo Giordano P., Multi-Robot Systems: Port-Hamiltonian Modeling
Mass-spring-damper vs. PHS

• Summarizing, this particular passive system is made of:
  • Two atomic energy storing elements $\mathcal{K}$ and $\mathcal{V}$
  • A power-preserving interconnection among $\mathcal{K}$ and $\mathcal{V}$
  • An energy dissipation element $b$
  • A pair $(f, v_p)$ to exchange energy with the “external world”

• Why passivity of the complete system?

• $\mathcal{K}$ and $\mathcal{V}$ are passive (and “irreducible”)
  • Their power-preserving interconnection is a feedback interconnection (thus, preserves passivity)
  • The element $b$ dissipates energy
  • Therefore, any increase of the total energy $H$ is due to the power flowing through $(f, v_p)$. For this reason, this pair is also called power-port

• How general are these results?
In the linear time-invariant case

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\( (\mathbb{I}) \)

Passivity implies existence of a storage function

\[ H(x) = \frac{1}{2} x^T Q x, \quad Q = Q^T \geq 0 \]

such that \( A^T Q + QA \leq 0 \) and \( C = B^T Q \)

If \( \ker Q \subset \ker A \) (always true if \( Q > 0 \) )

then \( (\mathbb{I}) \) can be rewritten as

\[
\begin{align*}
\dot{x} &= (J - R)Qx + Bu, \quad J = -J^T, \quad R = R^T \geq 0 \\
y &= B^T Q x
\end{align*}
\]

and energy balance

\[
\dot{H} = -x^T QRQx + x^T QBu \leq y^T u
\]

\( H(x) \) is called the Hamiltonian function
Similarly, most nonlinear passive system can be rewritten as

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u, \quad J(x) = -J^T(x), \quad R(x) \geq 0 \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

with \( H(x) \geq 0 \) being the Hamiltonian function (storage function) and

\[
\dot{H} = -\frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x} g(x)u \leq y^T u
\]

showing the passivity condition

- **Roles:**
  - \( H(x) \) represents the energy stored by the system
  - \( R(x) \) represents the internal dissipation in the system
  - \( J(x) \) represents an internal power-preserving interconnection among different components
  - \((u, y)\) represents a “power-port”, allowing energy exchange (in/out) with the external world
In the mass-spring-damper case, the generic Port-Hamiltonian formulation specializes into:

\[
\begin{align*}
    \dot{x} &= \left[ J(x) - R(x) \right] \frac{\partial H}{\partial x} + g(x)u, \\
    y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

specializes into:

\[
J(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
In the (more abstract) example we have seen during the Passivity lectures, we showed that

\[
\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + u
\end{cases}
\]

is a passive system with passive output \( y = x_2 \) and Storage function

\[
V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \geq 0
\]

Can it be recast in PHS form with \( H(x) = V(x) \) being the Hamiltonian?

Yes:

\[
\begin{cases}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
&= 

\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\frac{\partial H}{\partial x}
+ 

\begin{bmatrix}
0 \\
1
\end{bmatrix}
u \\
y
&= 

\begin{bmatrix}
0 & 1
\end{bmatrix}
\frac{\partial H}{\partial x}
\end{cases}
\]
What is then Port-Hamiltonian modeling?

It is a cross-domain energy-based modeling philosophy, generalizing Bond Graphs

Historically, network modeling of lumped-parameter physical systems (e.g., circuit theory)

Main insights: all the physical domains deal, in a way or another, with the concept of Energy storage and Energy flows

- Electrical
- Hydraulical
- Mechanical
- Thermodynamical

Dynamical behavior comes from the exchange of energy

The “energy paths” (power flows) define the internal model structure
Introduction to Port-Hamiltonian Systems

• Port-Hamiltonian modeling

• Most (passive) physical systems can be modeled as a set of simpler subsystems (modularity!) that either:
  • Store energy
  • Dissipate energy
  • Exchange energy (internally or with the external world) through power ports

• Role of energy and the interconnections between subsystems provide the basis for various control techniques

• Easily address complex nonlinear systems, especially when related to real “physical” ones
Port-Hamiltonian systems can be formally defined in an abstract way.

Everything revolves about the concepts of
- **Power ports** (medium to exchange energy)
- **Dirac structures** (“pattern” of energy flow)
- **Hamiltonian** (storage of energy)

We will now give a (very brief and informal) introduction of these concepts.

Big guys in the field:
- Arjan van der Schaft
- Romeo Ortega
- Bernard Maschke
- Mark W. Spong
- Stefano Stramigioli
- Alessandro Astolfi
- and many more (maybe one of you in the future?)
A power port is a pair of variables \((e, f)\) called “effort” and “flow” that mediates a power exchange (energy flow) among 2 physical components.

<table>
<thead>
<tr>
<th>Physical domain</th>
<th>Flow (f)</th>
<th>Effort (e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>electric</td>
<td>Current</td>
<td>Voltage</td>
</tr>
<tr>
<td>magnetic</td>
<td>Voltage</td>
<td>Current</td>
</tr>
<tr>
<td>Potential (mechanics)</td>
<td>Velocity</td>
<td>Force</td>
</tr>
<tr>
<td>Kinetic (mechanics)</td>
<td>Force</td>
<td>Velocity</td>
</tr>
<tr>
<td>Potential (hydraulic)</td>
<td>Volume flow</td>
<td>Pressure</td>
</tr>
<tr>
<td>Kinetic (hydraulics)</td>
<td>pressure</td>
<td>Volume flow</td>
</tr>
<tr>
<td>chemical</td>
<td>Molar flow</td>
<td>Chemical potential</td>
</tr>
<tr>
<td>thermal</td>
<td>Entropy flow</td>
<td>temperature</td>
</tr>
</tbody>
</table>
Introduction to Port-Hamiltonian Systems

- A generic port-Hamiltonian model is then
  - A set of energy storage elements (with their power ports \((e_S, f_S)\))
  - A set of resistive elements (with their power ports \((e_R, f_R)\))
  - A set of open power-ports (with their power ports \((e_P, f_P)\))
  - An internal power-preserving interconnection \(\mathcal{D}\), called Dirac structure

- An explicit example of a “Dirac structure” is the power-preserving interconnection represented by the skew-symmetric matrix \(J(x)\)
Any mechanical system (also constrained) described by the Euler-Lagrange equations can be recast in a Port-Hamiltonian form.

Start with a set of generalized coordinates:

\[ q = [q_1^T \ldots q_n^T]^T \]

Define the Lagrangian:

\[ L = K(q, \dot{q}) - V(q) \text{ with } K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} \]

being the kinetic energy, \( V(q) \) the potential energy, and \( M(q) > 0 \) the positive definite Inertia matrix.

Apply a change of coordinates:

\[ (q, \dot{q}) \rightarrow (q, p) \]

where \( p = M(q) \dot{q} \) are usually called "generalized momenta.”

The kinetic energy in the new coordinates is:

\[ K(q, p) = \frac{1}{2} p^T M^{-1}(q)p \]
General Mechanical System

- Define the Hamiltonian (total energy) of the system as

\[
H(q, p) = K(q, p) + V(q)
\]

- The Euler-Lagrange equations for the system are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \tau
\]  

(Ⅱ)

- Since \( p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial K}{\partial \dot{q}} \) we can rewrite (Ⅱ) as

\[
\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial q} + \tau
\end{align*}
\]

\[
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau
\]
General Mechanical System

- It follows that
  \[ \dot{H} = \frac{\partial H^T}{\partial p} \tau = \dot{q}^T \tau \]

- If \( H(q, p) \) (i.e., \( V(q) \)) is bounded from below, the system is passive w.r.t. the power port \((\dot{q}, \tau)\)

- Similarly, a mechanical system with collocated inputs and outputs (also underactuated) is generally described by
  \[
  \begin{align*}
  \dot{q} &= \frac{\partial H}{\partial p} \\
  \dot{p} &= -\frac{\partial H}{\partial q} + B(q)u \\
  y &= B^T(q) \frac{\partial H}{\partial p} 
  \end{align*}
  \]
  \( (= B^T(q)\dot{q}) \)

- Again, passivity w.r.t. \((y, u)\)
Modularity

- As one can expect, the “proper” interconnection of a number of Port-Hamiltonian Systems

\[(\mathcal{M}_i, \mathcal{D}_i, H_i), i = 1 \ldots k\]

through a Dirac structure \(\mathcal{D}_I\) is again a Port-Hamiltonian System \((\mathcal{M}, \mathcal{D}, H)\) with

- Hamiltonian \(H = H_1 + \ldots H_k\)

- State manifold \(\mathcal{M} = \mathcal{M}_1 \times \ldots \mathcal{M}_k\)

- Dirac structure \(\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_k, \mathcal{D}_I\)

- This allows for modularity and scalability
Modularity

- Example: given two Port-Hamiltonian System

\[
\begin{align*}
\dot{x}_1 &= (J_1(x_1) - R_1(x_1)) \frac{\partial H_1}{\partial x_1} + g_1(x_1)u_1 \\
y_1 &= g_1^T(x_1) \frac{\partial H_1}{\partial x_1} \\
\end{align*}
\begin{align*}
\dot{x}_2 &= (J_2(x_2) - R_2(x_2)) \frac{\partial H_2}{\partial x_2} + g_2(x_2)u_2 \\
y_2 &= g_2^T(x_2) \frac{\partial H_2}{\partial x_2}
\end{align*}
\]

- Define an interconnecting Dirac structure \( D_I \) as (for example)

\[
u_1 = y_2, \quad u_2 = -y_1
\]

- The composed system is again Port-Hamiltonian

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \left( \begin{bmatrix}
J_1(x_1) & g_1(x_1)g_2^T(x_2) \\
-g_2(x_2)g_1^T(x_1) & J_2(x_2)
\end{bmatrix}
- \begin{bmatrix}
R_1(x_1) & 0 \\
0 & R_2(x_2)
\end{bmatrix} \right)
\begin{bmatrix}
\frac{\partial H_1}{\partial x_1} \\
\frac{\partial H_2}{\partial x_2}
\end{bmatrix}
\]

with Hamiltonian function \( H(x_1, x_2) = H_1(x_1) + H_2(x_2) \)
Further generalizations

- Much more could be said on Port-Hamiltonian System....

- Can model distributed parameters physical systems (wherever energy plays a role)
  - Transmission line
  - Flexible beams
  - Wave equations
  - Gas/fluid dynamics

- Are modular (re-usability)
  - Network structure (.... -> multi-agent)

- Are flexible
  - State-dependent (time-varying) interconnection structure $J(x)$
Summary

• PHS are a powerful way to model a very large class of physical systems
  • For instance, every physical system admitting an Energy concept (the whole physics?)

• In PHS, the emphasis is on the internal structure of a system. A PHS system is a network of
  • Power ports: medium to exchange energy
  • Elementary/irreducible energy storing elements endowed with their power ports
  • Dissipating elements endowed with their power ports
  • “External world” power ports for external interaction
  • A power-preserving interconnection structure (Dirac structure) among the internal power ports

• The total energy of a PHS is called Hamiltonian. If the Hamiltonian is bounded from below, a PHS is passive w.r.t. its external ports

• Proper compositions of PHS are PHS
Control of PHS

• How to control a Port-Hamiltonian System?

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

• A PHS is still a dynamical system in the general form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

hence, one could use any of the available (nonlinear) control techniques

• However, in closed-loop, we want to retain and to exploit the PHS structure
  • PHS plant and controller
  • Power-preserving interconnection among them
Control of PHS

- The general idea is: assume a **plant** and **controller** in PHS form, and interconnected through a suitable $D_I$

$$
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\begin{align*}
\dot{x}_c &= (J_c(x_c) - R_c(x_c)) \frac{\partial H_c}{\partial x_c} + g_c(x_c)u_c \\
y_c &= g^T_c(x_c) \frac{\partial H_c}{\partial x_c}
\end{align*}
$$

where we split the plant port $(u, y)$ into $(u_1, y_1)$ and $(u_2, y_2)$, and use $(u_1, y_1)$ for the **interconnection with the controller port $(u_c, y_c)$**.

- In general, one can imagine two distinct control goals
  - **Regulation** to $x^*$ or **tracking** of $x^*(t)$ for the plant state variables $x(t)$
  - Desired (closed-loop) behavior of the plant at the interaction port $(u_2, y_2)$
  - The latter is for instance the case of Impedance Control for robot manipulators

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Energy Transfer Control

• Consider two PHS

\[
\begin{align*}
\dot{x}_1 &= J_1(x_1) \frac{\partial H_1}{\partial x_1} + g_1(x_1) u_1 \\
y_1 &= g_1^T(x_1) \frac{\partial H_1}{\partial x_1}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 &= J_2(x_2) \frac{\partial H_2}{\partial x_2} + g_2(x_2) u_2 \\
y_2 &= g_2^T(x_2) \frac{\partial H_2}{\partial x_2}
\end{align*}
\]

• And assume we want to transfer some amount of energy among them by keeping the total energy \( H_1(x_1) + H_2(x_2) \) constant

• This can be done by interconnecting the two PHS as

\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
0 & -\alpha y_1(x_1) y_2^T(x_2) \\
\alpha y_2(x_2) y_1^T(x_1) & 0
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}, \quad \alpha \in \mathbb{R}
\]

Skew-symmetric

• Note that this is an example of a state-modulated power preserving interconnection

\[
J(x) =
\begin{bmatrix}
0 & -\alpha y_1(x_1) y_2^T(x_2) \\
\alpha y_2(x_2) y_1^T(x_1) & 0
\end{bmatrix}
\]
Energy Transfer Control

- Since the interconnection is power-preserving, it follows that the total Hamiltonian $H(x_1, x_2) = H(x_1) + H(x_2)$ stays constant, i.e.,

  \[
  \dot{H}(x_1, x_2) = 0
  \]

- However, what happens to the individual energies?

- **Exercise: show that**

  \[
  \dot{H}_1(x_1) = -\alpha \|y_1\|^2 \|y_2\|^2 \quad \dot{H}_2(x_2) = \alpha \|y_1\|^2 \|y_2\|^2
  \]

- Thus, depending on the parameter $\alpha$, energy is extracted/injected from system 1 to system 2 (no energy transfer with $\alpha = 0$)

- If $H_1(x_1)$ is lower-bounded, a finite amount of energy will be transferred to system 2. Indeed, at the minimum, $y_1 = 0 \implies \dot{H}_1 = 0$ and $\dot{H}_2 = 0$

- The same of course holds for $H_2(x_2)$

- We will use these ideas in some of the following developments
Energy Tanks

- Let us examine a concrete example of the Energy Transfer Control technique

- To this end, we introduce the concept of “Energy Tank”

- Assume the usual PHS

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

- We know it is passive w.r.t. \((u, y)\) since

\[
\dot{H} = -\frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x} g(x)u \leq y^T u
\]
Energy Tanks

• In its integral form, the passivity condition reads

\[ H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, d\tau - \int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \, d\tau \leq 0 \]

• Let \( E_{in}(t) = H(t) - H(t_0) \) and \( E_{ext}(t) = \int_{t_0}^{t} y^T u \, d\tau \)

• Over time, \( E_{in}(t) \leq E_{ext}(t) \)
Energy Tanks

- Why this gap over time between $E_{\text{ext}}(t)$ and $E_{\text{in}}(t)$?

- Because of the integral of the dissipation term

\[ H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, d\tau - \int_{t_0}^{t} \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \, d\tau \leq 0 \]

- However, we would be happy (from the passivity point of view) by just ensuring a lossless energy balance

\[ H(t) - H(t_0) = \int_{t_0}^{t} y^T u \, dt \quad \iff \quad E_{\text{in}}(t) = E_{\text{ext}}(t) \]
Energy Tanks

- Dissipation term: passivity margin of the system
- Imagine we could recover this “passivity gap”
- This recovered energy can be freely used for whatever goal without violating the passivity constraint
- This idea is at the basis of the Energy Tank machinery

- Energy Tank: an atomic energy storing element with state $x_t \in \mathbb{R}$ and energy function $T(x_t) = \frac{1}{2} x_t^2 \geq 0$

$$\begin{cases} 
\dot{x}_t &= u_t \\
\dot{y}_t &= \frac{\partial T}{\partial x_t} ( = x_t ) 
\end{cases}$$
Energy Tanks

• We want to exploit the tank for:
  • storing back the natural dissipation of a PHS
  • allowing to use the stored energy for implementing some action on the PHS
  • this tank-based action will necessarily meet the passivity constraint

• How to achieve these goals? Let us consider again the PHS and Tank Energy element

\[
\begin{align*}
\dot{x} & = \left[ J(x) - R(x) \right] \frac{\partial H}{\partial x} + g(x)u \\
y & = g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_t & = u_t \\
y_t & = x_t
\end{align*}
\]

• Let \( D(x) = \frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} \) represent the (scalar) dissipation rate of the PHS

• We start by choosing \( u_t = \frac{1}{x_t} D(x) + \tilde{u}_t \) in the Tank dynamics
Energy Tanks

• The choice \( u_t = \frac{1}{x_t} D(x) + \tilde{u}_t \) allows to store back the dissipated energy.

• In fact, \( \dot{T}(x_t) = x_t \left( \frac{1}{x_t} D(x) + \tilde{u}_t \right) = D(x) + x_t \tilde{u}_t \)

• In order to exploit this stored energy to implement an action on the PHS system, we must design a suitable (power-preserving) interconnection among the PHS and Tank element.

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

\[
\begin{align*}
\dot{x}_t &= \frac{1}{x_t} D(x) + \tilde{u}_t \\
y_t &= x_t
\end{align*}
\]

• We will make use of the ideas seen in the Energy Transfer Control technique!

• Implement the desired action as a “lossless energy transfer” between Tank and PHS.

• This action will always preserve passivity by construction.
Assume we want to implement the action \( w \in \mathbb{R}^m \) on the PHS (\( m = \dim(u) \))

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

\[\begin{cases}
\dot{x}_t &= \frac{1}{x_t} D(x) + \tilde{u}_t \\
y_t &= x_t
\end{cases}\]

We then interconnect the PHS and the Tank element by means of this state-modulated power-preserving interconnection

\[
\begin{bmatrix}
   u \\
   \tilde{u}_t
\end{bmatrix} =
\begin{bmatrix}
   0 & \frac{w}{x_t} \\
   -\frac{w^T}{x_t} & 0
\end{bmatrix}
\begin{bmatrix}
   y \\
   y_t
\end{bmatrix}
\]

Since this coupling is skew-symmetric, no energy is created/lost during the transfer.
After this coupling the individual dynamics become

\[ \dot{x} = \left[ J(x) - R(x) \right] \frac{\partial H}{\partial x} + g(x) \left( \frac{w}{x_t} y_t \right) = \left[ J(x) - R(x) \right] \frac{\partial H}{\partial x} + g(x) w \]

and

\[ \dot{x}_t = \frac{1}{x_t} D(x) - \frac{w^T}{x_t} y = \frac{1}{x_t} D(x) - \frac{w^T}{x_t} g^T(x) \frac{\partial H}{\partial x} \]

And altogether, a new PHS with Hamiltonian \( \mathcal{H}(x, x_t) = H(x) + T(x_t) \)

\[
\begin{bmatrix}
\dot{x} \\
\dot{x}_t
\end{bmatrix} = \left( \begin{bmatrix}
J(x) & w \\
-w^T & 0
\end{bmatrix} \right) - \left( \begin{bmatrix}
R(x) & 0 \\
-1 \frac{\partial H^T}{\partial x} & 0
\end{bmatrix} \right) \left( \begin{bmatrix}
\frac{\partial \mathcal{H}}{\partial x} \\
\frac{\partial \mathcal{H}}{\partial x_t}
\end{bmatrix} \right)
\]

Skew-symmetric
Energy Tanks

- **Fact 1:** action $w$ is correctly implemented on the original PHS

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x} + g(x)w$$

- **Fact 2:** the composite PHS is (altogether) a passive (lossless) system whatever the expression of $w$

- **Proof:** evaluating $\dot{H}$ along the system trajectories, we obtain a lossless energy balance

$$\begin{bmatrix} \dot{x} \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} J(x) & w \\ -\frac{w^T}{x_t} & 0 \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ -\frac{1}{x_t} \frac{\partial H^T}{\partial x} R(x) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial x_t} \end{bmatrix}$$

$$\dot{H} = -\frac{\partial H^T}{\partial x} R(x) \frac{\partial H}{\partial x} + \frac{\partial H^T}{\partial x_t} \frac{1}{x_t} \frac{\partial H}{\partial x} R(x) \frac{\partial H}{\partial x} = 0$$
Energy Tanks

- **Fact 3**: the machinery proposed so far becomes singular when \( x_t = 0 \)

- What does the condition \( x_t = 0 \) represent?

- From the definition of the Tank energy function \( T(x_t) = \frac{1}{2} x_t^2 \geq 0 \) we have that \( x_t = 0 \iff \) the Tank energy is depleted

- Therefore, this singularity represents the impossibility of passively perform the desired action \( w \)

- One can always imagine some (safety) switching parameter \( \alpha(t) \) such that

\[
\begin{cases}
    \alpha = 1 & \text{if } T(x_t) \geq \epsilon > 0 \\
    \alpha = 0 & \text{if } T(x_t) < \epsilon
\end{cases}
\]

and implement \( \alpha(t)w \) instead of \( w \) (i.e., implement \( w \) only if you can in a “passive way”). If cannot implement \( w \), **wait for better times (the Tank gets replenished)**
Energy Tanks

• Note that the Tank dynamics is made of two terms

\[
\dot{x}_t = \frac{1}{x_t} D(x) - \frac{w^T}{x_t} g T (x) \frac{\partial H}{\partial x}
\]

• The first term is always non-negative, and represents the “refilling” action due to the dissipation present in the PHS plant

• The second term can have any sign, also negative. It is then possible for the action \( w \) to actually refill the tank!

• Finally, note that no condition is present on \( x_t(t_0) \)! This can be chosen as any \( x_t(t_0) > 0 \)

• In other words, complete freedom in choosing the initial amount of energy in the tank \( T(x_t(t_0)) \)

• In fact, passivity ultimately is: bounded amount of extractable energy, but for whatever initial energy in the system (only needs to be finite)

Robuffo Giordano P., Multi-Robot Systems: Port-Hamiltonian Modeling