Università di Roma Tre

Complementi di Controlli Automatici

Controllo dei robot mobili

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Wheeled Mobile Robots (WMRs)

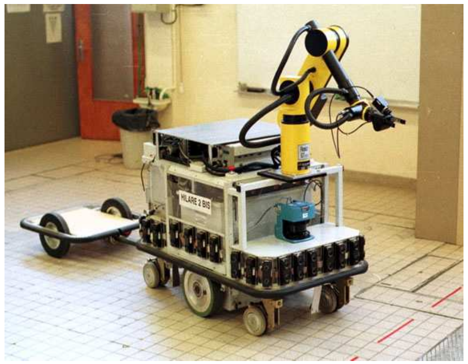
a growing population

Yamabico

MagellanPro

Sojourner

ATRV-2

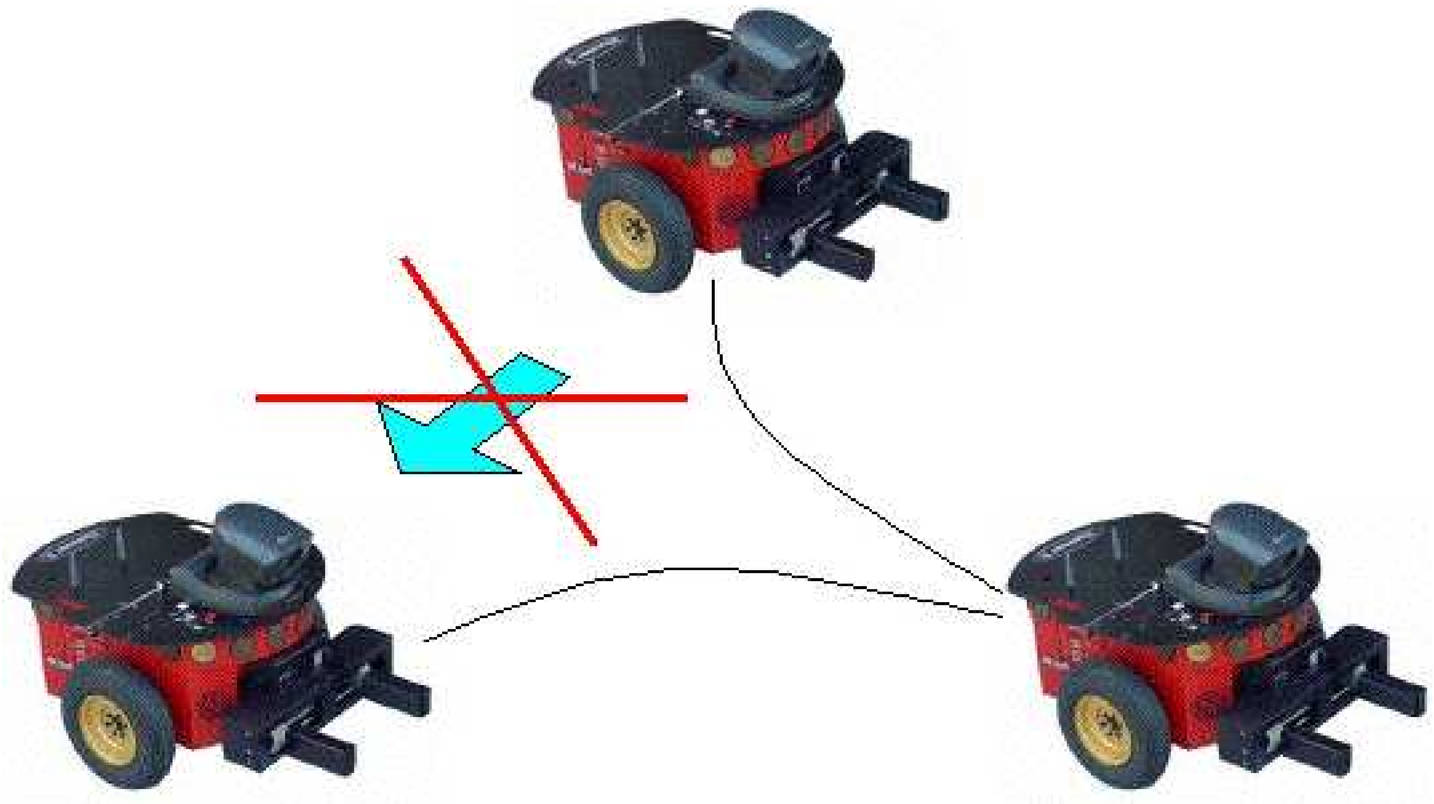


Hilare 2-Bis

Koy

The Central Issue

due to the presence of wheels, a WMR **cannot move sideways**



this is the **rolling without slipping** constraint, a special case of **nonholonomic** behavior problems:

- our everyday experience indicates that WMRs are controllable, but can it be proven?

  $\hookrightarrow$ we need a mathematical characterization of nonholonomy

- in any case, if the robot must move between two configurations, a **feasible** path is required (i.e., a motion that complies with the constraint)

  $\hookrightarrow$ we need appropriate path planning techniques

- the feedback control problem is much more complicated, because:

  ◇ a WMR is **underactuated**: less control inputs than generalized coordinates

  ◇ a WMR is **not smoothly stabilizable** at a point

  $\hookrightarrow$ we need appropriate feedback control techniques

INTRODUCTION

- the configuration of a mechanical system can be uniquely described by an n-dimensional vector of **generalized coordinates**

$$q = (q_1 \quad q_2 \quad \dots \quad q_n)^T$$

- the configuration space $\mathcal{Q}$ is an n-dimensional smooth manifold, locally represented by $I\!R^n$

- the **generalized velocity** at a generic point of a trajectory $q(t) \subset \mathcal{Q}$ is the tangent vector

$$\dot{q} = (\dot{q}_1 \quad \dot{q}_2 \quad \dots \quad \dot{q}_n)^T$$

- **geometric constraints** may exist or be imposed on the mechanical system

$$h_i(q) = 0 \qquad i = 1, \dots, k$$

  restricting the possible motions to an $(n-k)$-dimensional submanifold

- a mechanical system may also be subject to a set of **kinematic constraints**, involving generalized coordinates and their derivatives; e.g., first-order kinematic constraints

$$a_i^T(q,\dot{q}) = 0 \qquad i = 1,\dots,k$$

- in most cases, the constraints are **Pfaffian**

$$a_i^T(q)\dot{q} = 0 \quad i = 1,\dots,k \qquad \text{or} \qquad A^T(q)\dot{q} = 0$$

  i.e., they are linear in the velocities

- kinematic constraints may be **integrable**, that is, there may exist k functions h_i such that

$$\frac{\partial h_i(q(t))}{\partial q} = a_i^T(q) \qquad i = 1,\dots,k$$

  in this case, the kinematic constraints are indeed geometric constraints

a set of Pfaffian constraints is called **holonomic** if it is integrable (a geometric limitation); otherwise, it is called **nonholonomic** (a kinematic limitation) holonomic/nonholonomic constraints affect mobility in a **completely different** way:

for illustration, consider a single Pfaffian constraint

$$a^T(q)\dot{q} = 0$$

- if the constraint is **holonomic**, then it can be integrated as

  $$h(q) = c$$

  with $\frac{\partial h}{\partial q} = a^T(q)$ and c an integration constant

  $$\Downarrow$$

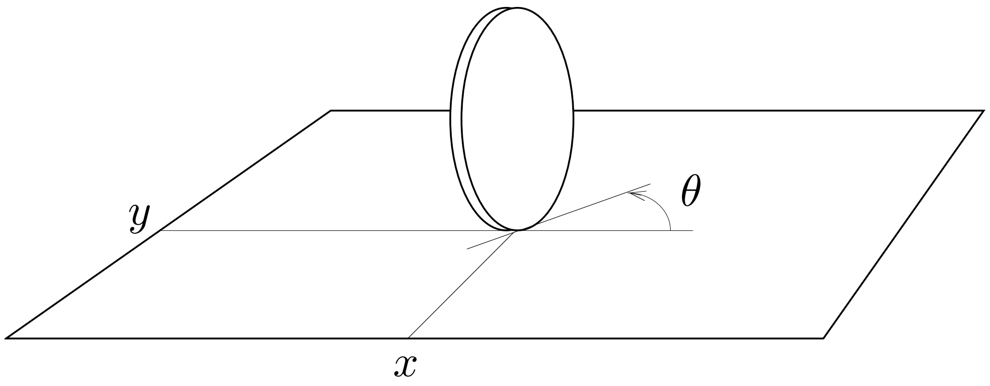
  the motion of the system is confined to lie on a particular level surface (**leaf**) of h, depending on the initial condition through $c = h(q_0)$

- if the constraint is **nonholonomic**, then it cannot be integrated

  $$\Downarrow$$

  although at each configuration the instantaneous motion (velocity) of the system is restricted to an $(n-1)$-dimensional space (the null space of the constraint matrix $a^T(q)$), **it is still possible to reach any configuration in $\mathcal{Q}$**

a canonical example of nonholonomy: the rolling disk

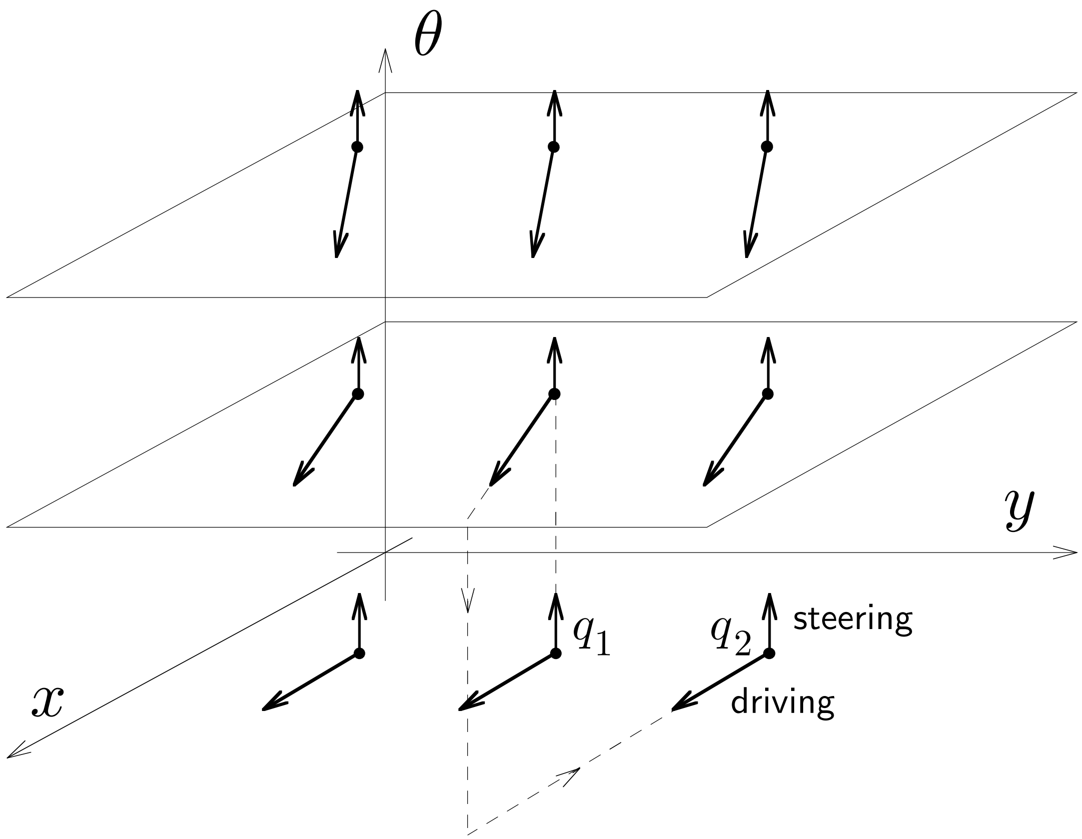


- generalized coordinates $q = (x, y, \theta)$
- **pure rolling** nonholonomic constraint $\quad \dot{x}\sin\theta - \dot{y}\cos\theta = 0 \quad \left(\dfrac{\dot{y}}{\dot{x}} = \tan\theta\right)$
- feasible velocities are contained in the null space of the constraint matrix

$$a^T(q) = (\sin\theta \quad -\cos\theta \quad 0) \qquad \Longrightarrow \qquad \mathcal{N}(a^T(q)) = \text{span}\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- **any** configuration $q_f = (x_f, y_f, \theta_f)$ can be reached:
  1. rotate the disk until it aims at (x_f, y_f)
  2. roll the disk until until it reaches (x_f, y_f)
  3. rotate the disk until until its orientation is θ_f

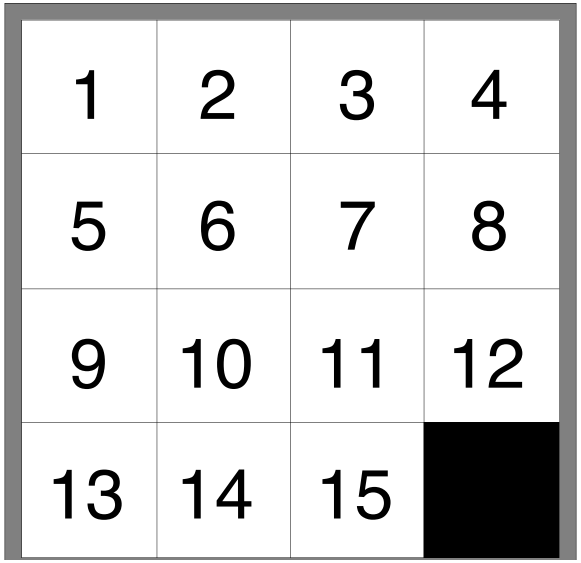
nonholonomy **in the configuration space** of the rolling disk



- at each q, only two instantaneous directions of motion are possible

- to move from q_1 to q_2 (**parallel parking**) an appropriate **maneuver** (sequence of moves) is needed; one possibility is to follow the dashed line

a less canonical example of nonholonomy: the fifteen puzzle



- generalized coordinates $q = (q_1, \ldots, q_{15})$
- each q_i may assume 16 different values corresponding to the cells in the grid; **legal** configurations are obtained when $q_i \neq q_j$ for $i \neq j$
- depending on the current configuration, a limited number (2 to 4) moves are possible
- **any** configuration with an **even** number of inversions can be reached by an appropriate sequence of moves

A Control Viewpoint

- holonomy/nonholonomy of constraints may be conveniently studied through a dual approach: look at the

directions in which motion is **allowed**

rather than

directions in which motion is **prohibited**

- there is a strict relationship between

capability of accessing every configuration

and

nonholonomy of the velocity constraints

- the interesting question is:

given two arbitrary points q_i and q_f,
when does a connecting trajectory $q(t)$ exist
which satisfies the kinematic constraints?

$\Downarrow$

. . . this is indeed a **controllability** problem!

- associate to the set of kinematic constraints a basis for their null space, i.e. a set of vectors g_j such that

$$a_i^T(q)g_j(q) = 0 \quad i = 1,\dots,k \quad j = 1,\dots,n-k$$

  or in matrix form

$$A^T(q)G(q) = 0$$

- feasible trajectories of the mechanical system are the solutions $q(t)$ of

$$\boxed{\dot{q} = \sum_{j=1}^{m} g_j(q)u_j = G(q)u} \qquad (*)$$

  for some input $u(t) \in I\!R^m$, $m = n-k$ (u: also called **pseudovelocities**)

- $(*)$ is a **driftless** (i.e., $u=0 \Rightarrow \dot{q}=0$) nonlinear system known as the **kinematic model** of the constrained mechanical system

- **controllability** of its whole configuration space is equivalent to **nonholonomy** of the original kinematic constraints

More General Nonholonomic Constraints

- one may also find Pfaffian constraints of the form

$$a_i^T(q)\dot{q} = c_i, \;\; i = 1, \ldots, k \quad \text{or} \quad A^T(q)\dot{q} = c$$

  with constant c_i

- these constraints are **differential** but **not** of a kinematic nature; for example, this form arises from conservation of an initial **non-zero** angular momentum in space robots

- the constrained mechanism is transformed into an equivalent control system by describing feasible trajectories $q(t)$ as solutions of

$$\dot{q} = f(q) + \sum_{i=1}^{m} g_i(q) u_i$$

  i.e., a nonlinear control system **with drift**, where $g_1(q), \ldots, g_m(q)$ are a basis of the null space of $A^T(q)$ and the drift vector f is computed through pseudoinversion

$$f(q) = A^{\#}(q)c = A(q)\left(A^T(q)A(q)\right)^{-1} c$$

MODELING EXAMPLES

source of nonholonomic constraints on motion:

- bodies in **rolling contact without slipping**
  - wheeled mobile robots (WMRs) or automobiles (wheels rolling on the ground with no skid or slippage)
  - dextrous manipulation with multifingered robot hands (fingertips on grasped objects)

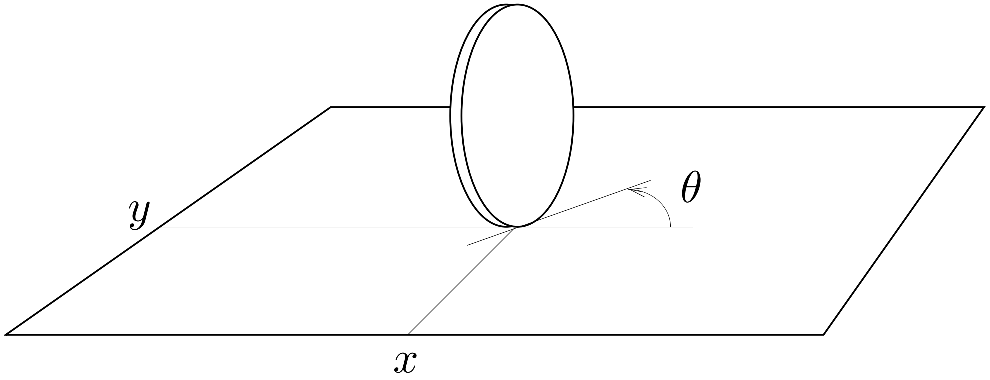
- **angular momentum conservation** in multibody systems
  - robotic manipulators floating in space (with no external actuation)
  - dynamically balancing hopping robots, divers or astronauts (in flying or mid-air phases)
  - satellites with reaction (or momentum) wheels for attitude stabilization

- special **control operation**

$$\dot{q} = G(q)u \qquad q \in \mathbb{R}^n \;\; u \in \mathbb{R}^m \;\; (m < n)$$

  - non-cyclic inversion schemes for redundant robots (m task commands for n joints)
  - floating underwater robotic systems ($m = 4$ velocity inputs for $n = 6$ generalized coords)

unicycle



- generalized coordinates $q = (x, y, \theta)$
- nonholonomic constraint $\dot{x} \sin\theta - \dot{y} \cos\theta = 0$
- a matrix whose columns span the null space of the constraint matrix is

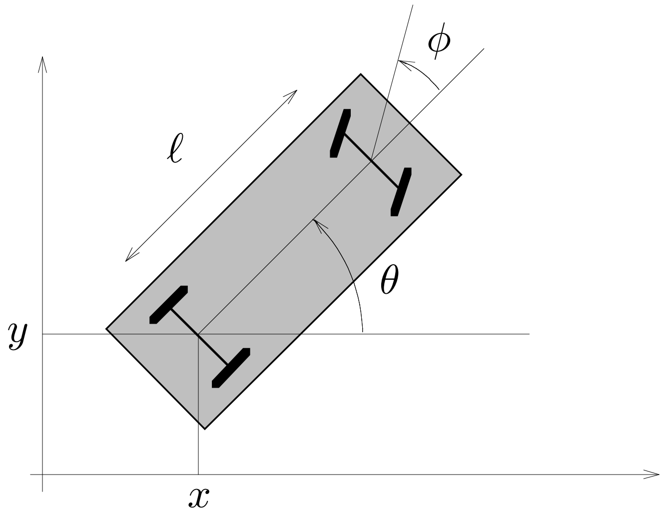
$$G(q) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \end{pmatrix}$$

- hence the kinematic model

$$\dot{q} = G(q)u = g_1(q)u_1 + g_2(q)u_2$$

with $u_1 =$ **driving**, $u_2 =$ **steering** velocity inputs

car-like robot



- 'bicycle' model: front and rear wheels collapse into two wheels at the axle midpoints
- generalized coordinates $q = (x, y, \theta, \phi)$ ϕ: steering angle
- nonholonomic constraints

$$\begin{array}{rcll} \dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) & = & 0 & \textbf{(front wheel)} \\ \dot{x}\sin\theta - \dot{y}\cos\theta & = & 0 & \textbf{(rear wheel)} \end{array}$$

- being the front wheel position

$$x_f = x + \ell\cos\theta \qquad y_f = y + \ell\sin\theta$$

the first constraint becomes

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta}\,\ell\cos\phi = 0$$

the constraint matrix is

$$A^T(q) = \begin{pmatrix} \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 \\ \sin\theta & -\cos\theta & 0 & 0 \end{pmatrix}$$

there are two physical alternatives for the controls:

(RD) choosing

$$G(q) = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{\ell}\tan\phi & 0 \\ 0 & 1 \end{pmatrix} \implies \dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where u_1 = **rear driving**, u_2 = **steering** inputs

- ◇ a ‘control singularity’ at $\phi = \pm\pi/2$, where vector field g_1 diverges

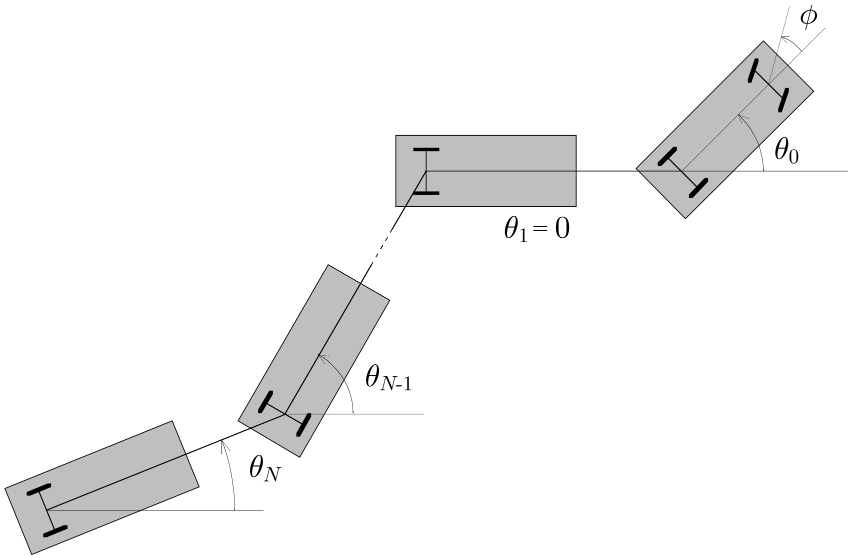
(FD) choosing

$$G(q) = \begin{pmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ \frac{1}{\ell}\sin\phi & 0 \\ 0 & 1 \end{pmatrix} \implies \dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where u_1 = **front driving**, u_2 = **steering** inputs

- ◇ no singularities in this case!

N-trailer system



- an FD car-like robot with N trailers, each hinged to the axle midpoint of the previous
- generalized coordinates $q = (x, y, \phi, \theta_0, \theta_1, \ldots, \theta_N) \in I\!R^{N+4}$

$$
\begin{aligned}
x, y &= \text{position of the car rear axle midpoint} \\
\phi &= \text{steering angle of the car (w.r.t. car body)} \\
\theta_0 &= \text{orientation angle of the car (w.r.t. } x\text{-axis)} \\
\theta_i &= \text{orientation angle of } i\text{-th trailer (w.r.t. } x\text{)}
\end{aligned}
$$

- the car is considered as the 0-th trailer

$$
\begin{aligned}
d_0 &= \ell = \text{car length} \\
d_i &= i\text{-th trailer length (hinge to hinge)}
\end{aligned}
$$

nonholonomic constraints:

steering wheel

$$\dot{x}_f \sin(\theta_0 + \phi) - \dot{y}_f \cos(\theta_0 + \phi) = 0$$

with

$$x_f = x + \ell \cos\theta_0 \qquad y_f = y + \ell \sin\theta_0$$

all other wheels

$$\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0 \qquad i = 0, 1, \ldots, N$$

being

$$x_i = x - \sum_{j=1}^{i} d_j \cos\theta_j \qquad y_i = y - \sum_{j=1}^{i} d_j \sin\theta_j$$

the constraints become

$$\begin{aligned} \dot{x}\sin(\theta_0 + \phi) - \dot{y}\cos(\theta_0 + \phi) - \dot{\theta}_0\,\ell\cos\phi &= 0 \\ \dot{x}\sin\theta_i - \dot{y}\cos\theta_i + \sum_{j=1}^{i}\dot{\theta}_j\,d_j\cos(\theta_i - \theta_j) &= 0 \qquad i = 0, 1, \ldots, N \end{aligned}$$

- the null space of the $N+2$ constraints is spanned by the two columns g_1, g_2 of

$$G(q) = \begin{pmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \\ \frac{1}{\ell}\tan\phi & 0 \\ -\frac{1}{d_1}\sin(\theta_1-\theta_0) & 0 \\ -\frac{1}{d_2}\cos(\theta_1-\theta_0)\sin(\theta_2-\theta_1) & 0 \\ \vdots & \vdots \\ -\frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_i-\theta_{i-1}) & 0 \\ \vdots & \vdots \\ -\frac{1}{d_N}\left(\prod_{j=1}^{N-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_N-\theta_{N-1}) & 0 \end{pmatrix}$$

- the kinematic model is $\dot{q} = g_1(q)u_1 + g_2(q)u_2$

  with $u_1 =$ **(rear) driving**, $u_2 =$ **steering** inputs for the front car

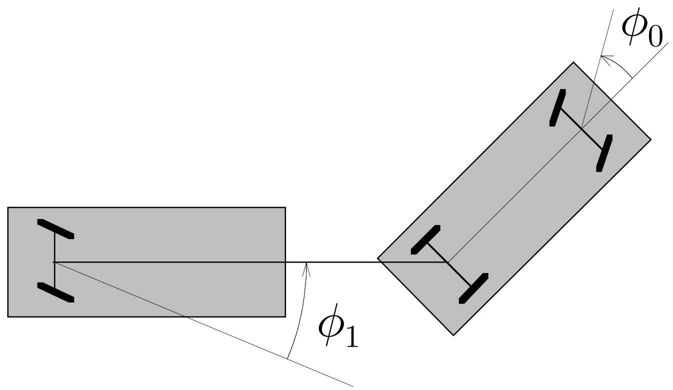
- an alternative way to derive kinematic equations

$$\begin{aligned} \dot{\theta}_i &= -\frac{1}{d_i}\sin(\theta_i-\theta_{i-1})\nu_{i-1} \\ & \qquad\qquad\qquad\qquad i = 1,\ldots,N \\ \nu_i &= \nu_{i-1}\cos(\theta_i-\theta_{i-1}) \end{aligned}$$

  with $\nu_i =$ linear (forward) velocity of the i-th trailer ($\nu_0 = u_1$)

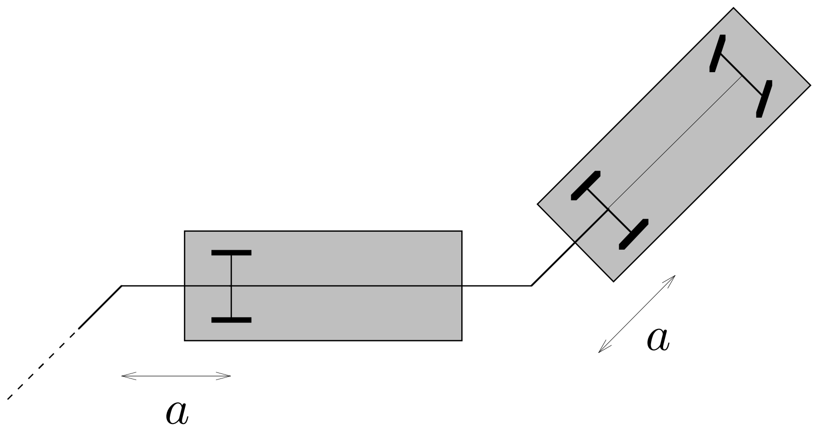
other wheeled mobile robots

- **firetruck**



6 configuration variables, 3 differential constraints, 3 control inputs (car driving and steering, trailer steering)

- N-trailer system with **nonzero hooking**



when $a \neq 0$ and $N \geq 2$, this system **cannot** be converted in chained form (later)

TOOLS FROM DIFFERENTIAL GEOMETRY

- a smooth **vector field** $f : \mathbb{R}^n \mapsto T_q\mathbb{R}^n$ is a smooth mapping from each point of $\mathbb{R}^n$ to the tangent space $T_q\mathbb{R}^n$

- if f defines the rhs of a differential equation

$$\dot{q} = f(q)$$

  the **flow** $\phi_t^f(q)$ of the vector field f is the mapping which associates to each q the solution evolving from q, i.e., it satisfies

$$\frac{d}{dt}\phi_t^f(q) = f(\phi_t^f(q))$$

  with the **group** property $\phi_t^f \circ \phi_s^f = \phi_{t+s}^f$

  in linear systems, $f(q) = Aq$, the flow is $\phi_t^f = e^{At}$

- considering two vector fields g_1 and g_2 as in

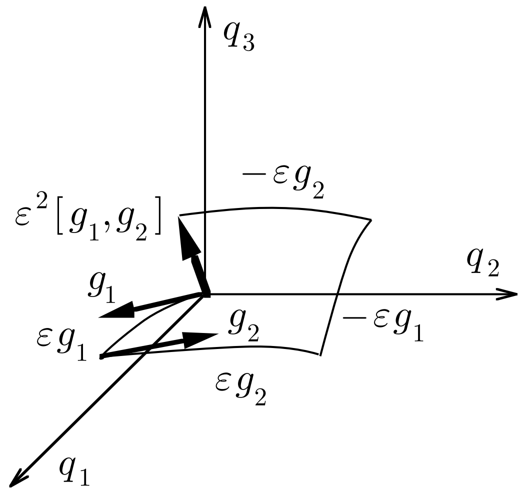
$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

  the composition of their flows (obtained by taking $u_1 = \{0,1\}$ and $u_2 = \{1,0\}$ or vice versa) is **non-commutative**

$$\phi_t^{g_1} \circ \phi_s^{g_2} \neq \phi_s^{g_2} \circ \phi_t^{g_1}$$

- starting at q_0, an infinitesimal flow of time ϵ along g_1, then g_2, then $-g_1$, and finally $-g_2$, yields (R. Brockett: *'a computation everybody should do once in his life'*)

$$q(4\epsilon) = \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(q_0) = q_0 + \epsilon^2 \left(\frac{\partial g_2}{\partial q} g_1(q_0) - \frac{\partial g_1}{\partial q} g_2(q_0) \right) + O(\epsilon^3)$$



- **Lie bracket** of two vector fields f, g

$$[g_1, g_2](q) = \frac{\partial g_2}{\partial q} g_1(q) - \frac{\partial g_1}{\partial q} g_2(q)$$

- g_1 and g_2 **commute** if $[g_1, g_2] = 0$; moreover,

$$[g_1, g_2] = 0 \quad \Rightarrow \quad q(4\epsilon) = q_0 \text{ (zero net flow)}$$

- **properties** of Lie brackets

$$[f,g] = -[g,f] \qquad \textbf{skew-symmetry}$$
$$[f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0 \qquad \textbf{Jacobi identity}$$

in linear single input systems, $f(q) = Aq$, $g(q) = b$,

$$\begin{aligned} [f,g] &= -Ab & \qquad [f,[f,g]] = A^2 b \\ [f,[f,[f,g]]] &= -A^3 b & \qquad \ldots \end{aligned}$$

- a smooth **distribution** Δ associated with a set of smooth vector fields $\{g_1, \ldots, g_m\}$ assigns to each point q a subspace of its tangent space defined as

$$\begin{aligned} \Delta &= \text{span}\,\{g_1, \ldots, g_m\} \\ &\Updownarrow \\ \Delta_q &= \text{span}\,\{g_1(q), \ldots, g_m(q)\} \subset T_q I\!\!R^n \end{aligned}$$

- a distribution is **regular** if dim $\Delta_q = \text{const}$, $\forall q$

- a distribution is **involutive** if it is closed under the Lie bracket operation

$$\Delta \text{ involutive} \quad \Longleftrightarrow \quad \forall g_i, g_j \in \Delta \quad [g_i, g_j] \in \Delta$$

- the **involutive closure** $\bar{\Delta}$ of a distribution Δ is its closure under the Lie bracket operation

Controllability of Nonholonomic Systems

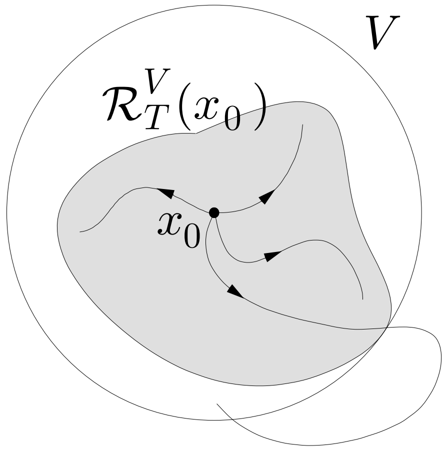
consider a nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j \qquad \text{(NCS)}$$

with state $x \in \mathcal{M} \simeq I\!R^n$, and input in the class $\mathcal{U}$ of piecewise-continuous time functions

- denote its unique solution at time $t \geq 0$ by $x(t, 0, x_0, u)$, with input $u(\cdot)$, and $x(0) = x_0$
- (NCS) is **controllable** if $\forall x_1, x_2 \in \mathcal{M}$, $\exists T < \infty$, $\exists u$: $[0, T] \rightarrow \mathcal{U}$: $x(T, 0, x_1, u) = x_2$
- the set of states **reachable** from x_0 **within** time $T > 0$, with trajectories contained in a neighborhood V of x_0, is denoted by

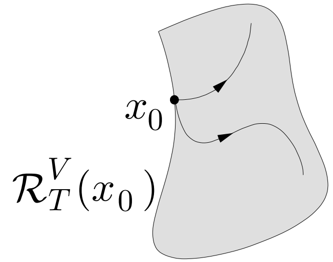
$$\mathcal{R}_T^V(x_0) = \bigcup_{\tau \leq T} \mathcal{R}^V(x_o, \tau)$$



where $\mathcal{R}^V(x_0, \tau) = \{x \in \mathcal{M} \mid x(\tau, 0, x_0, u) = x, \forall t \in [0, \tau],\ x(t, 0, x_0, u) \in V\}$

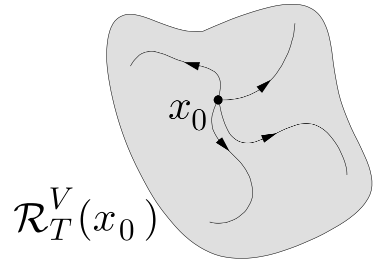
- (NCS) is **locally accessible** (LA) from x_0 if $\forall V$, a neighborhood of x_0, and $\forall T > 0$

$$\mathcal{R}_T^V(x_o) \supset \Omega, \quad \text{with } \Omega \text{ some non-empty open set}$$



- (NCS) is **small-time locally controllable** (STLC) from x_0 if $\forall V$, a neighborhood of x_0, and $\forall T > 0$

$$\mathcal{R}_T^V(x_o) \supset \Psi, \quad \text{with } \Psi \text{ some neighborhood of } x_0$$



- STLC $\Rightarrow$ controllability $\Rightarrow$ LA (not vice versa)

- LA is checked through an algebraic test
  - let $\bar{\mathcal{C}}$ be the involutive closure of the distribution associated with $\{f, g_1, g_2, \ldots, g_m\}$
  - **Chow Theorem** (1939): (NCS) is LA from x_0 if and only if

$$\dim \bar{\mathcal{C}}(x_0) = n \qquad \textbf{accessibility rank condition}$$

  - an algorithmic test:

$$\bar{\mathcal{C}} = \text{span}\left\{v \in \bigcup_{k\geq 0} \mathcal{C}^k\right\} \quad \text{with} \quad \begin{cases} \mathcal{C}^0 = \text{span}\{f, g_1, \ldots, g_m\} \\ \mathcal{C}^k = \mathcal{C}^{k-1} + \text{span}\{[f, v], [g_j, v], j = 1, .., m : v \in \mathcal{C}^{k-1}\} \end{cases}$$

- only **sufficient** conditions exists for STLC

- however, for driftless control systems:

$$\text{LA} \iff \text{controllability} \iff \text{STLC}$$

- this equivalence holds also whenever

$$f(x) \in \text{span}\,\{g_1(x), \ldots, g_m(x)\} \qquad \forall x \in \mathcal{M}$$

  ('trivial' drift)

- if the driftless control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

  is controllable, then its **dynamic extension**

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{m} g_i(x) v_i \\ \dot{v}_i &= u_i \qquad i = 1, \ldots, m \end{aligned}$$

  is also controllable (and vice versa)

- in the linear case $\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j = Ax + Bu$, all controllability definitions are equivalent and the associated tests reduce to the well-known Kalman rank condition:

$$\text{rank}\,(\,B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B\,) = n$$

- a controllability test is a nonholonomy test!

a set of k Pfaffian constraints $A(q)\dot{q} = 0$ is nonholonomic if and only if the associated kinematic model

$$\dot{q} = G(q)u = \sum_{i=1}^{m} g_i(q)u_i \qquad m = n - k$$

is controllable, that is

$$\dim \bar{\mathcal{C}} = n$$

being $\bar{\mathcal{C}}$ the involutive closure of the distribution associated with $g_1, \dots, g_m$

$$\Downarrow$$

for a nonholonomic system, it is always possible to design **open-loop** commands that drive the system from any state to any other state (**nonholonomic path planning**)

Stabilizability of Nonholonomic Systems

for a nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j = f(x) + g(x)u$$

one would like to build a **feedback control** law of the form

$$u = \alpha(x) + \beta(x)v$$

in such a way that either

a) a desired closed-loop equilibrium point x_e is made asymptotically stable, or

b) a desired feasible closed-loop trajectory $x_d(t)$ is made asymptotically stable

- feedback laws are essential in motion control to counteract the presence of disturbances as well as modeling inaccuracies
- in linear systems, controllability directly implies asymptotic (actually, exponential) stabilizability at x_e by **smooth** (actually, linear) state feedback

$$\alpha(x) = K(x - x_e)$$

- if the linear approximation of the system at x_e

$$\dot{\delta x} = A\delta x + B\delta u \qquad \delta x = x - x_e,\ \delta u = K\delta x$$

  is controllable, then the original system can be locally smoothly stabilized at x_e (a **sufficient** condition)

- in the presence of **uncontrollable eigenvalues at zero**, nothing can be concluded (except that smooth exponential stability is not achievable)

- for kinematic models of nonholonomic systems $\dot{q} = G(q)u$, the linear approximation around x_e has **always** uncontrollable eigenvalues at zero since

$$A \equiv 0 \quad \text{and} \quad \text{rank}\, B = \text{rank}\, G(q_e) = m < n$$

- however, there are **necessary** conditions for the existence of a C^0-stabilizing state feedback law (next slide)

- whenever these conditions fail, two alternatives are left:

  a) **discontinuous feedback** $\quad u = \alpha(x),\ \alpha \in \bar{C}^0$

  b) **time-varying feedback** $\quad u = \alpha(x, t),\ \alpha \in C^1$

Brockett stabilization theorem (1983)

if the system

$$\dot{x} = f(x, u)$$

is locally asymptotically C^1-stabilizable at x_e, then the image of the map

$$f : \mathcal{M} \times \mathcal{U} \rightarrow I\!R^n$$

contains some **neighborhood** *of x_e* (a **necessary** condition)

a special case: the **driftless** system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

with linearly independent vectors $g_i(x_e)$, i.e.,

$$\text{rank} \begin{pmatrix} g_1(x_e) & g_2(x_e) & \dots & g_m(x_e) \end{pmatrix} = m$$

is locally asymptotically C^1-stabilizable at x_e **if and only if** $m \geq n$

$$\Downarrow$$

nonholonomic mechanical systems
(either in kinematic or dynamic form)
cannot be stabilized at a point by smooth feedback

Examples

- **unicycle** $(n = 3)$

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad g_3 = [g_1, g_2] = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$\dim \bar{\mathcal{C}} = 3$ for all q

- **car-like robot (RD)** $(n = 4)$

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ -1/\ell\cos^2\phi \\ 0 \end{pmatrix}$$

$$g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell\cos^2\phi \\ \cos\theta/\ell\cos^2\phi \\ 0 \\ 0 \end{pmatrix}$$

$\dim \bar{\mathcal{C}} = 4$ away from the singularity at $\phi = \pm\pi/2$ of g_1

- **car-like robot (FD)** $(n = 4)$

$$g_1 = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_3 = [g_1, g_2] = \begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ -\cos\phi/\ell \\ 0 \end{pmatrix}$$

$$g_4 = [g_1, g_3] = \begin{pmatrix} -\sin\theta/\ell \\ \cos\theta/\ell \\ 0 \\ 0 \end{pmatrix}$$

  $\dim \bar{\mathcal{C}} = 4$ for all q

- **N-trailer system** $(n = N + 4)$ $\qquad \dim \bar{\mathcal{C}} = n$ for all q

- **all** the previous WMRs are controllable (STLC); **none** of these is smoothly stabilizable

NONHOLONOMIC MOTION PLANNING

- the objective is to build a sequence of **open-loop** input commands that steer the system from q_i to q_f satisfying the nonholonomic constraints

- there exist **canonical** model structures for which the steering problem can be solved efficiently
  - chained form
  - power form
  - Caplygin form

- interest in the **transformation** of the original model equation into one of these forms

- such model structures allow also a simpler design of feedback stabilizers (necessarily, non-smooth or time-varying)

- we limit the analysis to the case of systems **with two inputs**, where the three above forms are equivalent (via a coordinate transformation)

Chained Forms

- a $(2, n)$ **chained form** is a two-input driftless control system

$$\dot{z} = g_1(z)v_1 + g_2(z)v_2$$

in the following form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1 \end{aligned}$$

- denoting repeated Lie brackets as $\text{ad}_{g_1}^k g_2$

$$\text{ad}_{g_1} g_2 = [g_1, g_2] \qquad \text{ad}_{g_1}^k g_2 = [g_1, \text{ad}_{g_1}^{k-1} g_2]$$

one has

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \text{ad}_{g_1}^k g_2 = \begin{pmatrix} 0 \\ \vdots \\ (-1)^k \\ \vdots \\ 0 \end{pmatrix}$$

in which $(-1)^k$ is the $(k+2)$-th entry

- a one-chain system is **completely nonholonomic (controllable)** since the n vectors

$$\{g_1, g_2, \ldots, \mathrm{ad}^i_{g_1} g_2, \ldots\} \quad i = 1, \ldots, n-2$$

  are independent

- v_1 is called the **generating** input, z_1 and z_2 are called **base variables**

- if v_1 is (piecewise) constant, the system in chained form behaves like a (piecewise) linear system

- chained systems are a generalization of first- and second-order controllable systems for which sinusoidal steering from z_i to z_f minimizes the integral norm of the input

- different input commands can be used, e.g.
  - sinusoidal inputs
  - piecewise constant inputs
  - polynomial inputs

steering with sinusoidal inputs

- it is a two-phase method:

  I. steer the base variables z_1 and z_2 to their desired values z_{f1} and z_{f2} (in finite time)

  II. for each z_{k+2}, $k \geq 1$, steer z_{k+2} to its final value $z_{f,k+2}$ using

  $$v_1 = \alpha \sin \omega t \qquad v_2 = \beta \cos k\omega t$$

  over one period $T = 2\pi/\omega$, where α, β are such that

  $$\frac{\alpha^k \beta}{k!(2\omega)^k} = z_{f,k+2}(T) - z_{k+2}(0)$$

  this guarantees $z_i(T) = z_i(0) = z_{fi}$ for $i < k$

  in phase II, this step-by-step procedure adjusts one variable at a time by exploiting the closed-form integrability of the system equations under sinusoidal inputs

- phase II can be executed also all at once, choosing

  $$\begin{aligned} v_1 &= a_0 + a_1 \sin \omega t \\ v_2 &= b_0 + b_1 \cos \omega t + \ldots + b_{n-2} \cos(n-2)\omega t \end{aligned}$$

  and solving numerically for the $n+1$ unknowns in terms of the desired variation of the $n-2$ states

steering with piecewise constant inputs

- an idea coming from multirate digital control, with the travel time T divided in subintervals of length δ over which constant inputs are applied

$$\begin{aligned} v_1(\tau) &= v_{1,k} \\ v_2(\tau) &= v_{2,k} \end{aligned} \qquad \tau \in [(k-1)\delta, k\delta)$$

- it is convenient to keep v_1 always constant and take $n-1$ subintervals so that

$$T = (n-1)\delta \qquad\qquad v_1 = \frac{z_{f1} - z_{01}}{T}$$

  and the $n-1$ constant values of input v_2

$$v_{2,1},\ v_{2,2},\ \ldots,\ v_{2,n-1}$$

  are obtained solving a triangular linear system coming from the closed-form integration of the model equations

- if $z_{f1} = z_{01}$, an intermediate point must be added

- for small δ, a fast motion but with large inputs

steering with polynomial inputs

- idea similar to piecewise constant input, but with improved **smoothness** properties w.r.t. time (remember that kinematic models are controlled at the (pseudo)velocity level)

- the controls are chosen as

$$\begin{aligned} v_1 &= \text{sign}(z_{f1} - z_{01}) \\ v_2 &= c_0 + c_1 t + \ldots + c_{n-2} t^{n-2} \end{aligned}$$

  with $T = z_{f1} - z_{01}$ and $c_0, \ldots, c_n$ obtained solving the linear system coming from the closed-form integration of the model equations

$$M(T) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix} + m(z_i, T) = \begin{pmatrix} z_{f2} \\ z_{f3} \\ \vdots \\ z_{fn} \end{pmatrix}$$

  with $M(T)$ nonsingular for $T \neq 0$

- if $z_{f1} = z_{01}$, an intermediate point must be added

- for small T, a fast motion but with large inputs

- **unicycle**

  the change of coordinates

$$\begin{aligned} z_1 &= x \\ z_2 &= \tan\theta \\ z_3 &= y \end{aligned}$$

  and input transformation

$$\begin{aligned} u_1 &= v_1/\cos\theta \\ u_2 &= v_2\cos^2\theta \end{aligned}$$

  yield

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \end{aligned}$$

  other, globally defined transformations are possible

- **unicycle with N trailers**

  an 'ad hoc' transformation can be found (it starts using as (x, y) the position of the **last trailer** instead of the position of the trailing car)

- **car-like robot (RD)**

scaling first u_1 by $\cos\theta$

$$
\begin{aligned}
\dot{x} &= u_1 \\
\dot{y} &= u_1 \tan\theta \\
\dot{\theta} &= \frac{1}{\ell} u_1 \sec\theta \tan\phi \\
\dot{\phi} &= u_2
\end{aligned}
$$

then setting

$$
\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{\ell} \sec^3\theta \tan\phi \\
z_3 &= \tan\theta \\
z_4 &= y
\end{aligned}
$$

and

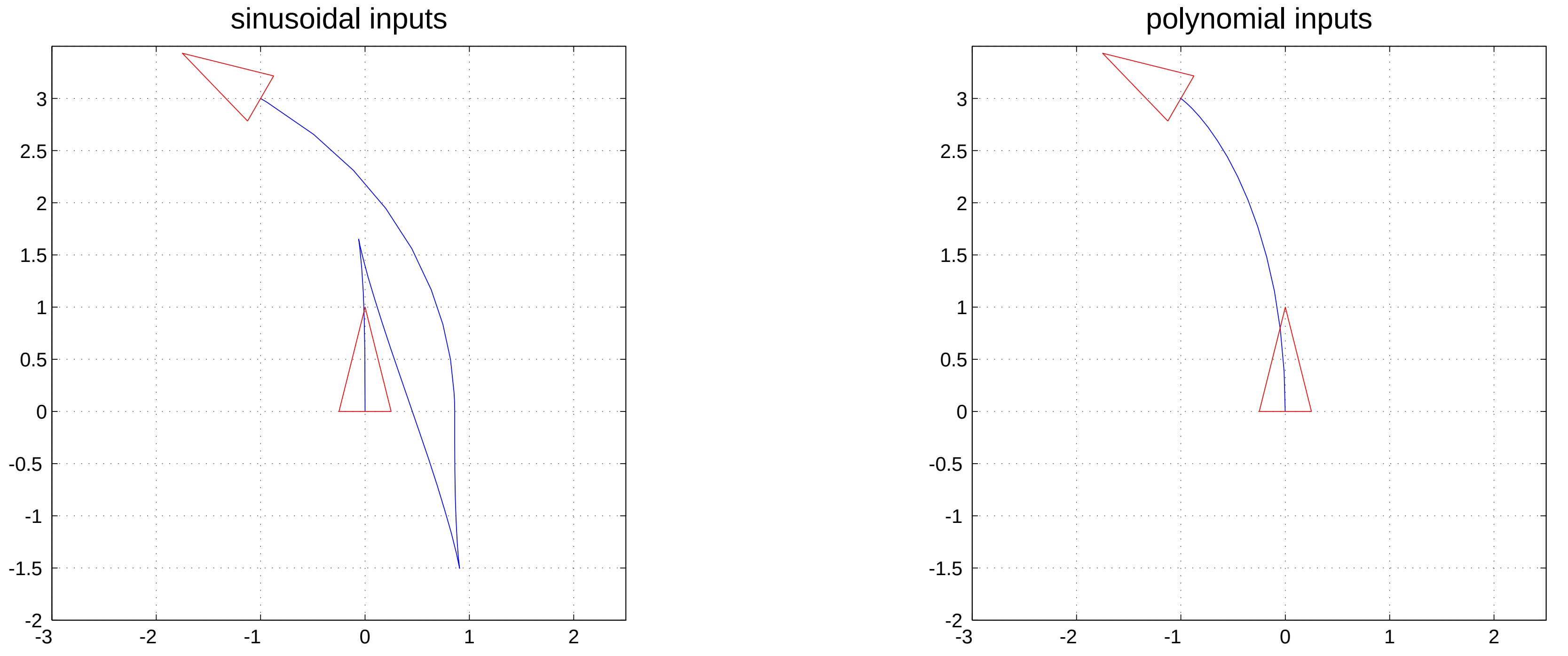
$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= -\frac{3}{\ell} v_1 \sec\theta \sin^2\phi + \frac{1}{\ell} v_2 \cos^3\theta \cos^2\phi
\end{aligned}
$$

yields

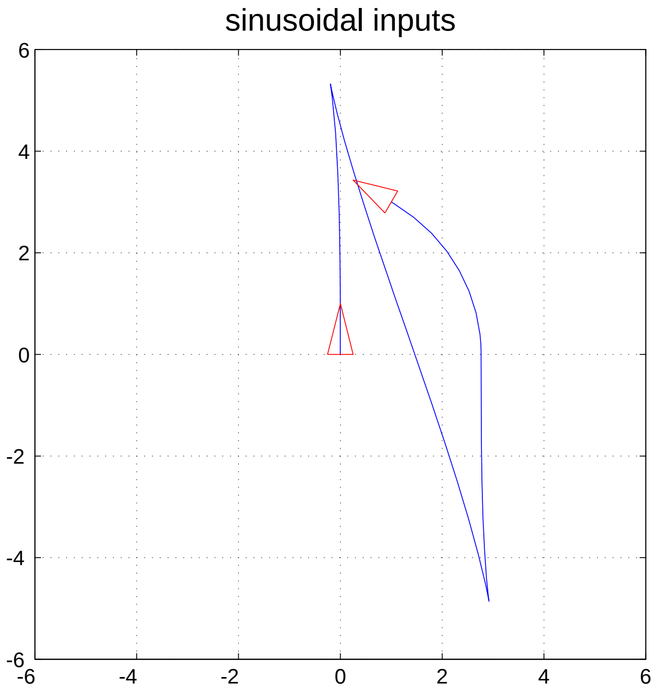
$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1 \\
\dot{z}_4 &= z_3 v_1
\end{aligned}
$$

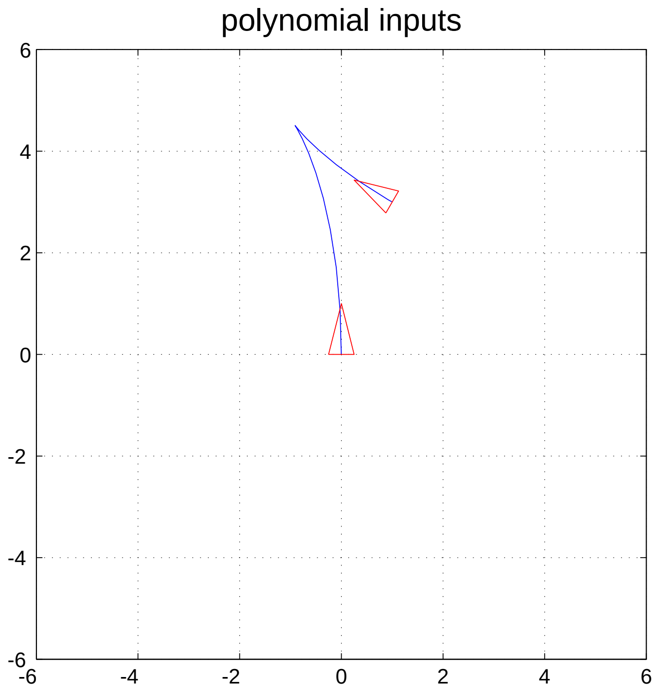
Path Planning for the Unicycle

simulation 1: $q_i = (-1, 3, 150°)$, $q_f = (0, 0, 90°)$



simulation 2: $q_i = (1, 3, 150°)$, $q_f = (0, 0, 90°)$





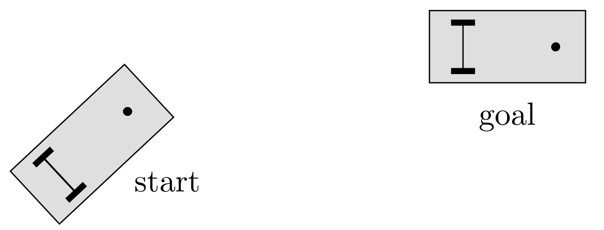
FEEDBACK CONTROL OF NONHOLONOMIC SYSTEMS

Basic Problems

- target system: **unicycle**
  - the kinematic models of most single-body WMRs can be reduced to a unicycle
  - most of the presented design techniques can be systematically extended to chained-form transformable systems

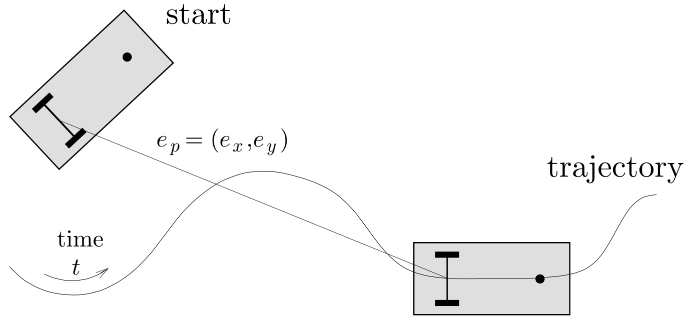
- basic motion tasks

(a) point-to-point motion (PTPM)



(a)

(b) trajectory following (TF)



- PTPM via feedback: **posture stabilization**
  - w.l.o.g., the origin $(0,0,0)$ is assumed to be the desired posture
  - a **nonsquare** ($q \in I\!R^3$, $u \in I\!R^2$) state regulation problem
  - need to use discontinuous/time-varying feedback in view of Brockett Theorem
  - poor, erratic transient performance is often obtained (inefficient, unsafe in the presence of obstacles)

- TF via feedback: **asymptotic tracking**
  - the desired trajectory $q_d(t)$ must be feasible, i.e., must comply with the nonholonomic constraints
  - a **square** ($e_p \in I\!R^2$, $u \in I\!R^2$) error zeroing problem
  - smooth feedback can be used here because the linear approximation along a nonvanishing trajectory is controllable (see later)

$$\Downarrow$$

asymptotic tracking is easier (and more useful) **than posture stabilization for nonholonomic systems**

Asymptotic Tracking

- a reference output trajectory $(x_d(t), y_d(t))$ is given
- control action: **feedforward** + **error feedback**

  error may be defined w.r.t. the reference output (**output error**) or the associated reference state (**state error**)

- given an initial posture and a desired trajectory $(x_d(t), y_d(t))$ there is a **unique** associated state trajectory $q_d(t) = (x_d(t), y_d(t), \theta_d(t))$ which can be computed as

$$\theta_d(t) = \text{ATAN2}\left(\dot{y}_d(t), \dot{x}_d(t)\right) + k\pi \qquad k = 0, 1$$

- **feedforward command generation**: we have

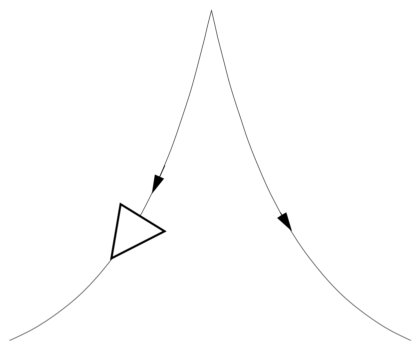
$$\begin{aligned} u_{d1}(t) &= \pm\sqrt{\dot{x}_d^2(t) + \dot{y}_d^2(t)} \\ u_{d2}(t) &= \frac{\ddot{y}_d(t)\dot{x}_d(t) - \ddot{x}_d(t)\dot{y}_d(t)}{\dot{x}_d^2(t) + \dot{y}_d^2(t)} \end{aligned}$$

– the choice of sign for $u_{d1}(t)$ produces forward or backward motion

– to be exactly reproducible, $(x_d(t), y_d(t))$ should be twice differentiable

– $\theta_d(t)$ may be computed off-line and used in order to define a state error

– if $u_{d1}(\bar{t}) = 0$ for some $\bar{t}$ (e.g., at a cusp)



neither $u_{d2}(\bar{t})$ nor $\theta_d(\bar{t})$ are defined

$\Rightarrow$ a continuous motion is guaranteed by keeping the same orientation attained at $\bar{t}^-$

asymptotic tracking: controllability

linear approximation along $q_d(t) = (x_d(t), y_d(t), \theta_d(t))$

- define:
  u_{d1}, u_{d2} the inputs associated to $q_d(t)$
  $\tilde{q} = q - q_d$ the state tracking error
  $\tilde{u}_1 = u_1 - u_{d1}$ and $\tilde{u}_2 = u_2 - u_{d2}$ the input variations

- the linear approximation along $q_d(t)$ is

$$\dot{\tilde{q}} = \begin{pmatrix} 0 & 0 & -u_{d1} \sin\theta_d \\ 0 & 0 & u_{d1} \cos\theta_d \\ 0 & 0 & 0 \end{pmatrix} \tilde{q} + \begin{pmatrix} \cos\theta_d & 0 \\ \sin\theta_d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$$

  a time-varying system

  $\Rightarrow$ the N&S controllability condition is that the controllability Gramian is nonsingular

- a simpler analysis can be performed by 'rotating' the state tracking error

$$\tilde{q}_R = \begin{pmatrix} \cos\theta_d & \sin\theta_d & 0 \\ -\sin\theta_d & \cos\theta_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{q}$$

  according to the reference orientation θ_d

- we get

$$\dot{\tilde{q}}_R = \begin{pmatrix} 0 & u_{d2} & 0 \\ -u_{d2} & 0 & u_{d1} \\ 0 & 0 & 0 \end{pmatrix} \tilde{q}_R + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$$

- when u_{d1} and u_{d2} are constant, the linearization becomes time-invariant and controllable, since

$$\left(B \;\; AB \;\; A^2B \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & -u_{d2}^2 & u_{d1}u_{d2} \\ 0 & 0 & -u_{d2} & u_{d1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 3 provided that either u_{d1} or u_{d2} is nonzero

$\Rightarrow$ the kinematic model of the unicycle can be locally asymptotically stabilized by linear feedback along trajectories consisting **of linear or circular paths** executed at a constant velocity

(actually: the same can be proven for **any** nonvanishing trajectory)

linear control design

- designed using a (slightly different) linear approximation along the reference trajectory

- define the state tracking error e as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{pmatrix}$$

- use a nonlinear transformation of velocity inputs

$$\begin{aligned} u_1 &= u_{d1} \cos e_3 - v_1 \\ u_2 &= u_{d2} - v_2 \end{aligned}$$

- the error dynamics becomes

$$\dot{e} = \begin{pmatrix} 0 & u_{d2} & 0 \\ -u_{d2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 \\ \sin e_3 \\ 0 \end{pmatrix} u_{d1} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

- linearizing around the reference trajectory, one obtains the same linear time-varying equations as before, now with state e and input (v_1, v_2)

- define the ‘linear’ feedback law

$$\begin{aligned} v_1 &= -k_1\, e_1 \\ v_2 &= -k_2\, \text{sign}(u_{d1}(t))\, e_2 - k_3\, e_3 \end{aligned}$$

with gains

$$k_1 = k_3 = 2\zeta a \qquad k_2 = \frac{a^2 - u_{d2}(t)^2}{|u_{d1}(t)|}$$

- the closed-loop characteristic polynomial is $(\lambda+2\zeta a)(\lambda^2+2\zeta a\lambda+a^2)$, $\quad \zeta \in (0,1) \quad a > 0$

- a convenient **gain scheduling** is achieved letting

$$a = a(t) = \sqrt{u_{d2}^2(t) + b u_{d1}^2(t)} \quad \Longrightarrow \quad k_1 = k_3 = 2\zeta\sqrt{u_{d2}^2(t) + b u_{d1}^2(t)}, \quad k_2 = b\,|u_{d1}(t)|$$

  these gains go to zero when the state trajectory stops (and local controllability is lost)

- the actual controls are **nonlinear** and **time-varying**

- even if the eigenvalues are constant, local asymptotic stability is not guaranteed as the system is still time-varying

  $\Rightarrow$ a Lyapunov-based analysis is needed

nonlinear control design

for the previous error dynamics, define

$$\begin{aligned} v_1 &= -k_1(u_{d1}(t), u_{d2}(t))\, e_1 \\ v_2 &= -\bar{k}_2\, u_{d1}(t) \frac{\sin e_3}{e_3}\, e_2 - k_3(u_{d1}(t), u_{d2}(t))\, e_3 \end{aligned}$$

with constant $\bar{k}_2 > 0$ and positive, continuous gain functions $k_1(\cdot,\cdot)$ and $k_3(\cdot,\cdot)$

theorem *if u_{d1}, u_{d2}, $\dot{u}_{d1}$ $\dot{u}_{d2}$ are bounded, and if $u_{d1}(t) \not\to 0$ or $u_{d2}(t) \not\to 0$ as $t \to \infty$, the above control globally asymptotically stabilizes the origin $e = 0$*

proof based on the Lyapunov function

$$V = \frac{\bar{k}_2}{2}\left(e_1^2 + e_2^2\right) + \frac{e_3^2}{2}$$

nonincreasing along the closed-loop solutions

$$\dot{V} = -k_1 \bar{k}_2 e_1^2 - k_3 e_3^2 \leq 0$$

$\Rightarrow$ $\|e(t)\|$ is bounded, $\dot{V}(t)$ is uniformly continuous, and $V(t)$ tends to some limit value

$\Rightarrow$ using Barbalat lemma, $\dot{V}(t)$ tends to zero

$\Rightarrow$ analyzing the system equations, one can show that $(u_{d1}^2 + u_{d2}^2)e_i^2$ $(i = 1, 2, 3)$ tends to zero so that, from the persistency of the trajectory, the thesis follows ∎

dynamic feedback linearization

- define the output as $\eta = (x, y)$; differentiation w.r.t. time yields

$$\dot{\eta} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Rightarrow$ cannot recover u_2 from first-order differential information

- add an integrator on the linear velocity input

$$u_1 = \xi, \qquad \dot{\xi} = a \qquad \Rightarrow \qquad \dot{\eta} = \xi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

new input a is the unicycle **linear acceleration**

- differentiating further

$$\ddot{\eta} = \begin{pmatrix} \cos\theta & -\xi\sin\theta \\ \sin\theta & \xi\cos\theta \end{pmatrix} \begin{pmatrix} a \\ u_2 \end{pmatrix}$$

- **assuming** $\xi \neq 0$, we can let

$$\begin{pmatrix} a \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\xi\sin\theta \\ \sin\theta & \xi\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

obtaining

$$\ddot{\eta} = \begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

- the resulting dynamic compensator is

$$\begin{aligned} \dot{\xi} &= v_1 \cos\theta + v_2 \sin\theta \\ u_1 &= \xi \\ u_2 &= \frac{v_2 \cos\theta - v_1 \sin\theta}{\xi} \end{aligned}$$

- as the dynamic compensator is 1-dim, we have $n + 1 = 4$, equal to the total number of output differentiations

  $\Rightarrow$ in the new coordinates

$$\begin{aligned} z_1 &= x \\ z_2 &= y \\ z_3 &= \dot{x} = \xi \cos\theta \\ z_4 &= \dot{y} = \xi \sin\theta \end{aligned}$$

  the system is fully linearized and described by two chains of second-order input-output integrators

$$\begin{aligned} \ddot{z}_1 &= v_1 \\ \ddot{z}_2 &= v_2 \end{aligned}$$

- the dynamic feedback linearizing controller has a potential singularity at $\xi = u_1 = 0$, i.e., when the unicycle is not rolling

  a singularity in the dynamic extension process is **structural** for nonholonomic systems

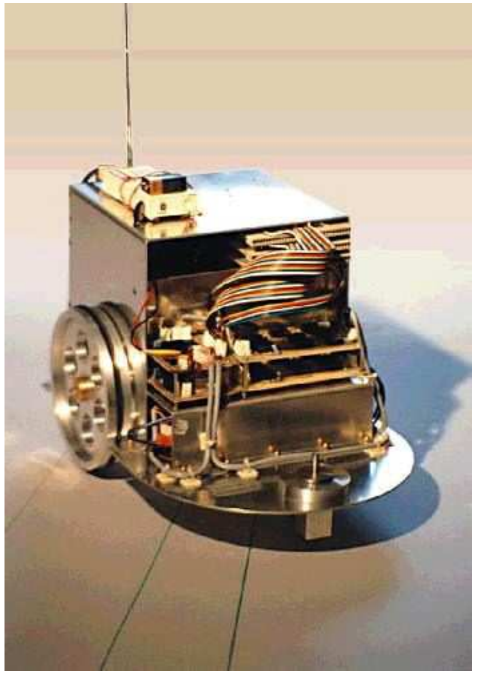
- for the (exactly) linearized system, a globally exponentially stabilizing feedback is

$$\begin{aligned} v_1 &= \ddot{x}_d(t) + k_{p1}(x_d(t) - x) + k_{d1}(\dot{x}_d(t) - \dot{x}) \\ v_2 &= \ddot{y}_d(t) + k_{p2}(y_d(t) - y) + k_{d2}(\dot{y}_d(t) - \dot{y}) \end{aligned}$$

  with PD gains $k_{pi} > 0$, $k_{di} > 0$, for $i = 1, 2$

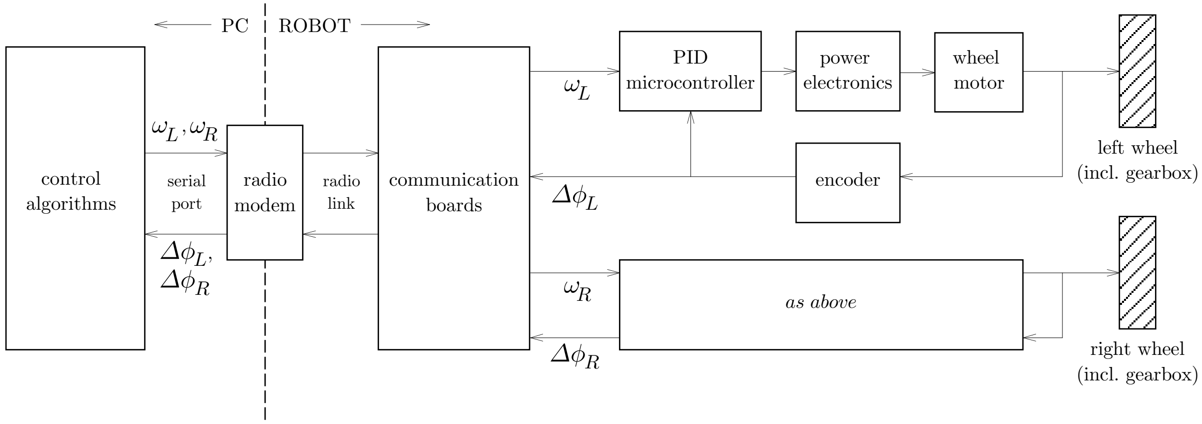
  - the desired trajectory $(x_d(t), y_d(t))$ must be smooth and **persistent**, i.e., $u_{d1}^2 = \dot{x}_d^2 + \dot{y}_d^2$ must never go to zero
  - cartesian transients are linear
  - $\dot{x}$ and $\dot{y}$ can be computed as a function ξ and θ; alternatively, one can use estimates of $\dot{x}$ and $\dot{y}$ obtained from odometric measurements
  - for **exact tracking**, one needs $q(0) = q_d(0)$ and $\xi(0) = u_{d1}(0)$ ($\Rightarrow$ pure feedforward)

experiments with SuperMARIO



- a two-wheel differentially-driven vehicle (with caster)
- the aluminum chassis measures $46 \times 32 \times 30.5$ cm (l/w/h) and contains two motors, transmission elements, electronics, and four 12 V batteries; total weight about 20 kg
- each wheel independently driven by a DC motor (peak torque ≈ 0.56 Nm); each motor equipped with an encoder (200 pulse/turn) and a gearbox (reduction ratio 20)
- typical nonidealities of electromechanical systems: friction, gear backlash, wheel slippage, actuator deadzone and saturation
- due to robot and motor dynamics, discontinuous velocity commands cannot be realized

two-level control architecture



- control algorithms (with reference generation) are written in C^{++} and run with a sampling time of $T_s = 50$ ms on a remote server

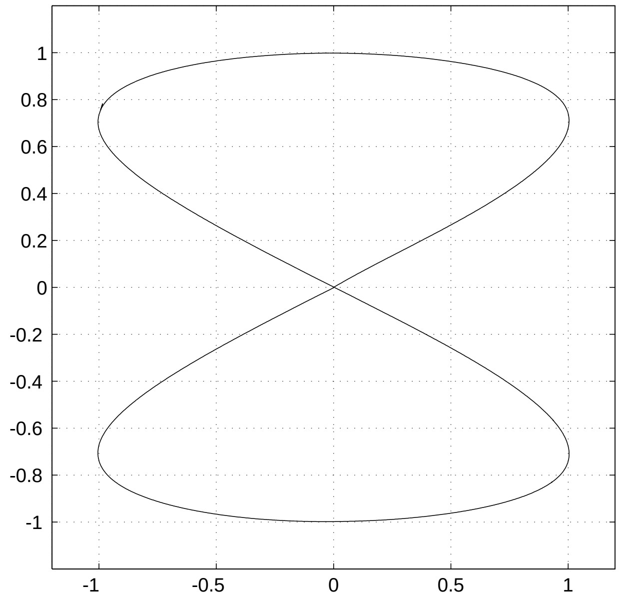
- the PC communicates through a radio modem with the serial communication boards on the robot

- actual commands are the angular velocities ω_R and ω_L of right and left wheel (instead of driving and steering velocities u_1 and u_2):

$$u_1 = \frac{r\,(\omega_R + \omega_L)}{2} \qquad u_2 = \frac{r\,(\omega_R - \omega_L)}{d}$$

  with $d =$ axle length, $r =$ wheel radius

- reconstruction of the current robot state based on encoder data (**dead reckoning**) experiments on an eight-shaped trajectory



- the reference trajectory

$$x_d(t) = \sin\frac{t}{10} \qquad y_d(t) = \sin\frac{t}{20} \qquad t \in [0, T]$$

  starts from the origin with $\theta_d(0) = \pi/6$ rad

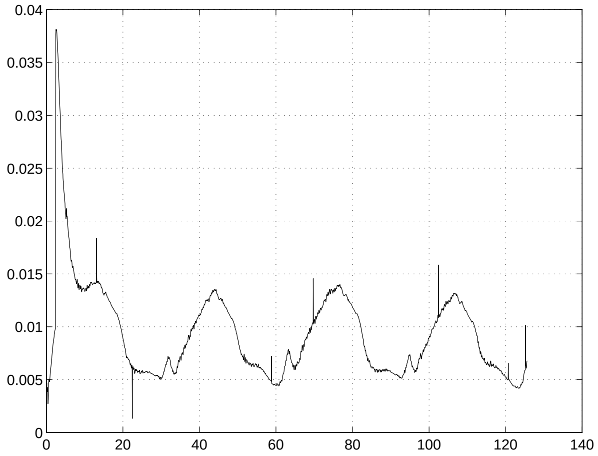
- a full cycle is completed in $T = 2\pi \cdot 20 \approx 125$ s

- the reference initial velocities are

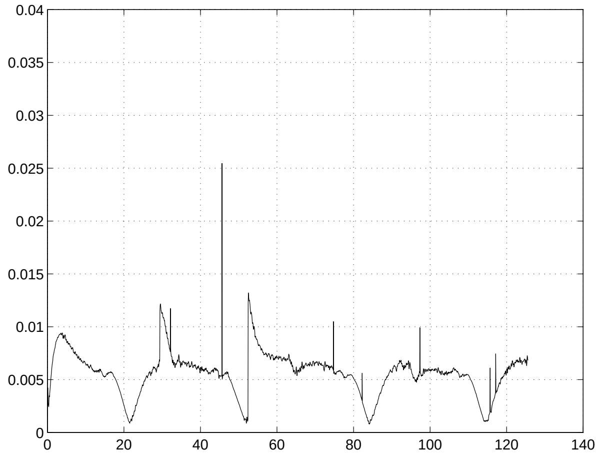
$$u_{d1}(0) \simeq 0.1118 \text{ m/s}, \qquad u_{d2}(0) = 0 \text{ rad/s}.$$

experiment 1: the robot initial state is **on** the reference trajectory

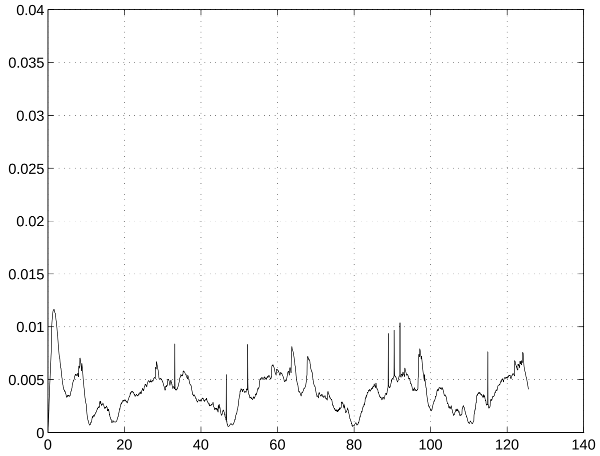
tracking error norm



linear design

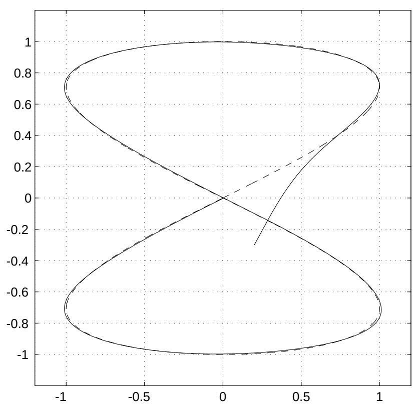


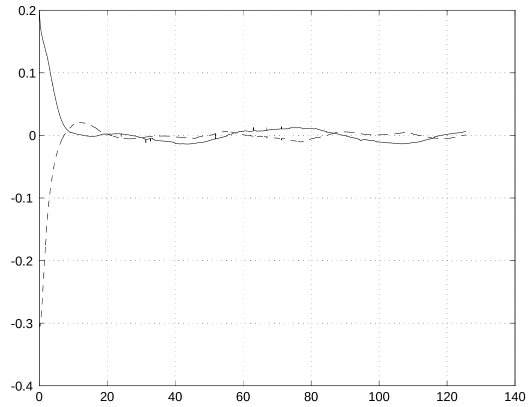
nonlinear design



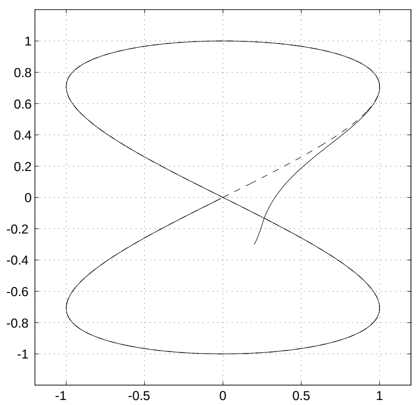
feedback linearization

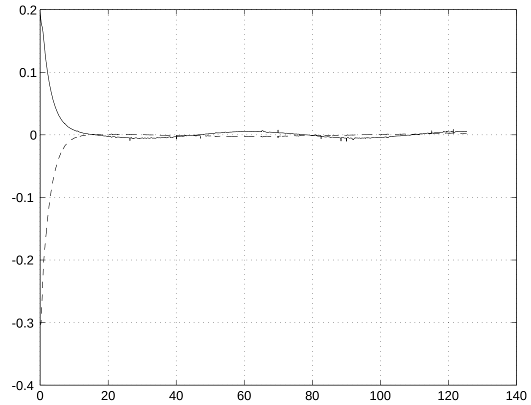
experiment 2: the robot initial state is **off** the reference trajectory





linear design





feedback linearization

Posture Stabilization: A Bird's Eye View

- the main obstruction is the non-smooth stabilizability of WMRs at a point

- two main approaches
  - **time-varying** stabilizers: an exogenous time-varying signal is injected in the controller
  - **discontinuous** stabilizers: the controller is time invariant but discontinuous at the origin

- drawbacks: slow convergence (time-varying), oscillatory transient (both)

- improvements
  - **mixed time-varying/discontinuous** stabilizers
  - **non-Lyapunov, discontinuous** stabilizers: through coordinate transformations that circumvent Brockett's obstruction or via dynamic feedback linearization

    $\hookrightarrow$ excellent transient performance!