4. Elementary Theory of Nonlinear Feedback for Single-Input Single-Output Systems

4.1 Local Coordinates Transformations

Beginning with this Chapter, we will study – in order of increasing complexity – a series of problems concerned with the synthesis of *feedback control laws* for nonlinear systems of the form (1.2). We will discuss first the case of single-input single-output systems, whose simple structure lends itself to a rather elementary analysis, and then – in the next Chapter – a special class of multivariable systems, in which a straightforward extension of most of the theory developed for single-input single-output systems is possible. Finally – in the last four Chapters – we will present a set of more powerful tools for the analysis and the design of more general classes of nonlinear control systems.

The purpose of this introductory section is to show how single-input single-output nonlinear systems can be locally given, by means of a suitable change of coordinates in the state space, a "normal form" of special interest, on which several important properties can be elucidated.

The point of departure of the whole analysis is the notion of relative degree of the system, which is formally described in the following way. The single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{4.1}$$

is said to have *relative degree* r at a point x° if

(i) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r-1$

(ii) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Note that there may be points where a relative degree cannot be defined. This occurs, in fact, when the first function of the sequence

$$L_g h(x), L_g L_f h(x), \ldots, L_g L_f^k h(x), \ldots$$

which is not identically zero (in a neighborhood of x°) has a zero exactly at the point $x = x^\circ$. However, the set of points where a relative degree can be defined is clearly an open and dense subset of the set U where the system (4.1) is defined.

Example 4.1.1. Consider the equations describing a controlled Van der Pol oscillator in state space form

$$\dot{x} = f(x) + g(x)u = \begin{pmatrix} x_2 \\ 2\omega\zeta(1 - \mu x_1^2)x_2 - \omega^2 x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u .$$

Suppose the output function is chosen as

$$y = h(x) = x_1 .$$

In this case we have

$$L_g h(x) = \frac{\partial h}{\partial x} g(x) = (1 \quad 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

and

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) = (1 \quad 0) \begin{pmatrix} x_2 \\ 2\omega\zeta(1 - \mu x_1^2)x_2 - \omega^2 x_1 \end{pmatrix} = x_2 .$$

Moreover

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x) = (0 \quad 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

and thus we see that the system in question has relative degree 2 at any point x°.

However, if the output function is, for instance

$$y = h(x) = \sin x_2$$

then $L_g h(x) = \cos x_2$. The system has relative degree 1 at any point x°, provided that $(x^\circ)_2 \neq (2k+1)\pi/2$. If the point x° is such that this condition is violated, no relative degree can be defined. ◁

Remark 4.1.2. In order to compare the notion thus introduced with a familiar concept, let us calculate the relative degree of a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx . \end{aligned}$$

In this case, since $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$, it easily seen that

$$L_f^k h(x) = CA^k x$$

and therefore

$$L_g L_f^k h(x) = CA^k B .$$

Thus, the integer r is characterized by the conditions

$$\begin{aligned} CA^k B &= 0 \qquad \text{for all } k < r-1 \\ CA^{r-1} B &\neq 0 . \end{aligned}$$

It is well-known that the integer satisfying these conditions is exactly equal to the *difference* between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function

$$H(s) = C(sI - A)^{-1}B$$

of the system. ◁

We illustrate now a simple interpretation of the notion of relative degree, which is not restricted to the assumption of linearity considered in the previous Remark. Assume the system at some time t° is in the state $x(t^\circ) = x^\circ$ and suppose we wish to calculate the value of the output $y(t)$ and of its derivatives with respect to time $y^{(k)}(t)$, for $k = 1, 2, \ldots$, at $t = t^\circ$. We obtain

$$\begin{aligned} y(t^\circ) &= h(x(t^\circ)) = h(x^\circ) \\ y^{(1)}(t) &= \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f h(x(t)) + L_g h(x(t))u(t) \, . \end{aligned}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, i.e. for all t near t°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t)) \, .$$

This yields

$$\begin{aligned} y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{dx}{dt} &= \frac{\partial L_f h}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &= L_f^2 h(x(t)) + L_g L_f h(x(t))u(t) \, . \end{aligned}$$

Again, if the relative degree is larger than 2, for all t near t° we have $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t)) \, .$$

Continuing in this way, we get

$$\begin{aligned} y^{(k)}(t) &= L_f^k h(x(t)) \qquad \text{for all } k < r \text{ and all } t \text{ near } t^\circ \\ y^{(r)}(t^\circ) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(t^\circ) \, . \end{aligned}$$

Thus, the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^\circ$ in order to have the value $u(t^\circ)$ of the input explicitly appearing.

Note also that if

$$L_g L_f^k h(x) = 0 \qquad \text{for all } x \text{ in a neighborhood of } x^\circ \text{ and all } k \geq 0$$

(in which case no relative degree can be defined at any point around x°) then the output of the system is not affected by the input, for all t near t°. As a matter of fact, if this is the case, the previous calculations show that the Taylor series expansion of $y(t)$ at the point $t = t^\circ$ has the form

$$y(t) = \sum_{k=0}^{\infty} L_f^k h(x^\circ) \frac{(t - t^\circ)^k}{k!}$$

i.e. that $y(t)$ is a function depending only on the initial state and not on the input.

These calculations suggest that the functions $h(x)$, $L_f h(x), \ldots, L_f^{r-1} h(x)$ must have a special importance. As a matter of fact, it is possible to show that they can be used in order to define, at least partially, a local coordinates transformation around x°(recall that x° is a point where $L_g L_f^{r-1} h(x^\circ) \neq 0$). This fact is based on the following property.

Lemma 4.1.1. *The row vectors*

$$dh(x^\circ), dL_f h(x^\circ), \ldots, dL_f^{r-1} h(x^\circ)$$

are linearly independent.

In order to prove this Lemma, we illustrate first another property, which will also be used several other times in the sequel.

Lemma 4.1.2. *Let ϕ be a real-valued function and f, g vector fields, all defined in an open set U of $\mathbb{R}^n$. Then, for any choice of $s, k, r \geq 0$,*

$$\langle dL_f^s \phi(x), ad_f^{k+r} g(x) \rangle = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} \langle dL_f^{s+i} \phi(x), ad_f^k g(x) \rangle . \quad (4.2)$$

As a consequence, the two sets of conditions

(i) $L_g \phi(x) = L_g L_f \phi(x) = \ldots = L_g L_f^k \phi(x) = 0$ *for all* $x \in U$ (4.3)

(ii) $L_g \phi(x) = L_{ad_f g} \phi(x) = \cdots = L_{ad_f^k g} \phi(x) = 0$ *for all* $x \in U$ (4.4)

are equivalent.

Proof. The proof of (4.2) is easily obtained by induction on r, in view of the fact that

$$\begin{aligned} \langle dL_f^s \phi(x), ad_f^{k+r+1} g(x) \rangle &= \langle dL_f^s \phi(x), [f, ad_f^{k+r} g(x)] \rangle \\ &= L_f \langle dL_f^s \phi(x), ad_f^{k+r} g(x) \rangle - \langle dL_f^{s+1} \phi(x), ad_f^{k+r} g(x) \rangle . \end{aligned}$$

The equivalence of (4.3) and (4.4) is a straightforward consequence of (4.2). ◁

We can proceed now with the proof of Lemma 4.1.1.

Proof. Observe that by definition of relative degree, using (4.2) we obtain for all i, j such that $i + j \leq r - 2$

$$\langle dL_f^j h(x), ad_f^i g(x) \rangle = 0 \qquad \text{for all } x \text{ around } x^\circ$$

and

$$\langle dL_f^j h(x^\circ), ad_f^i g(x^\circ) \rangle = (-1)^{r-1-j} L_g L_f^{r-1} h(x^\circ) \neq 0$$

for all i, j such that $i + j = r - 1$.

The above conditions, all together, show that the matrix

$$\begin{pmatrix} dh(x^\circ) \\ dL_f h(x^\circ) \\ \cdots \\ dL_f^{r-1} h(x^\circ) \end{pmatrix} \begin{pmatrix} g(x^\circ) & ad_f g(x^\circ) & \ldots & ad_f^{r-1} g(x^\circ) \end{pmatrix} = \\ = \begin{pmatrix} 0 & \ldots & \langle dh(x^\circ), ad_f^{r-1} g(x^\circ) \rangle \\ 0 & \ldots & \star \\ \cdot & \ldots & \star \\ \langle dL_f^{r-1} h(x^\circ), g(x^\circ) \rangle & \star & \star \end{pmatrix} \tag{4.5}$$

has rank r and, thus, that the row vectors $dh(x^\circ), dL_f h(x^\circ), \ldots, dL_f^{r-1} h(x^\circ)$ are linearly independent. ◁

Lemma 4.1.1 shows that necessarily $r \leq n$ and that the r functions $h(x), L_f h(x), \ldots, L_f^{r-1} h(x)$ qualify as a partial set of new coordinate functions around the point x° (recall Proposition 1.2.3). As we shall see in a moment, the choice of these new coordinates entails a particularly simple structure for the equations describing the system. However, before doing this, it is convenient to summarize the results discussed so far in a formal statement, that also illustrates a way in which the set of new coordinates can be completed in case the relative degree r is strictly less than n.

Proposition 4.1.3. *Suppose the system has relative degree r at x°. Then $r \leq n$. Set*

$$\begin{aligned} \phi_1(x) &= h(x) \\ \phi_2(x) &= L_f h(x) \\ &\cdots \\ \phi_r(x) &= L_f^{r-1} h(x) . \end{aligned}$$

If r is strictly less than n, it is always possible to find $n - r$ more functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ such that the mapping

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \cdots \\ \phi_n(x) \end{pmatrix}$$

has a jacobian matrix which is nonsingular at x° and therefore qualifies as a local coordinates transformation in a neighborhood of x°. The value at x°

of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\phi_{r+1}(x), \ldots, \phi_n(x)$ *in such a way that*

$$L_g\phi_i(x) = 0 \qquad \text{for all } r+1 \leq i \leq n \text{ and all } x \text{ around } x^\circ .$$

Proof. By definition of relative degree, the vector $g(x^\circ)$ is nonzero, and, thus, the distribution $G = \text{span}\{g\}$ is nonsingular around x°. Being 1-dimensional, this distribution is also involutive. Therefore, by Frobenius' Theorem, we deduce the existence of $n-1$ real-valued functions, $\lambda_1(x), \ldots, \lambda_{n-1}(x)$, defined in a neighborhood of x°, such that

$$\text{span}\{d\lambda_1, \ldots, d\lambda_{n-1}\} = G^\perp . \tag{4.6}$$

It is easy to show that

$$\dim(G^\perp + \text{span}\{dh, dL_fh, \ldots, dL_f^{r-1}h\}) = n \tag{4.7}$$

at x°. For, suppose this is false. Then

$$G(x^\circ) \cap (\text{span}\{dh, dL_fh, \ldots, dL_f^{r-1}h\})^\perp(x^\circ) \neq \{0\}$$

i. e. the vector $g(x^\circ)$ annihilates all the covectors in

$$\text{span}\{dh, dL_fh, \ldots, dL_f^{r-1}h\}(x^\circ) \ .$$

But this is a contradiction, because by definition $\langle dL_f^{r-1}h(x^\circ), g(x^\circ)\rangle$ is nonzero.

From (4.6), (4.7) and from the fact that $\text{span}\{dh, dL_fh, \ldots, dL_f^{r-1}h\}$ has dimension r, by Lemma 4.1.1, we conclude that in the set $\{\lambda_1, \ldots, \lambda_{n-1}\}$ it is possible to find $n-r$ functions, without loss of generality $\lambda_1, \ldots, \lambda_{n-r}$, with the property that the n differentials $dh, dL_fh, \ldots, dL_f^{r-1}h, d\lambda_1, \ldots, d\lambda_{n-r}$, are linearly independent at x°. Since by construction the functions $\lambda_1, \ldots, \lambda_{n-r}$ are such that

$$\langle d\lambda_i(x), g(x)\rangle = L_g\lambda_i(x) = 0 \qquad \text{for all } x \text{ near } x^\circ \text{ and all } 1 \leq i \leq n-r$$

this establishes the required result. Note that any other set of functions of the form $\lambda_i'(x) = \lambda_i(x) + c_i$, where c_i is a constant, satisfies the same conditions, thus showing that the value of these functions at the point x° can be chosen arbitrarily. ◁

The description of the system in the new coordinates $z_i = \phi_i(x)$, $1 \leq i \leq n$, is found very easily. Looking at the calculations already carried out at the beginning, we obtain for $z_1, \ldots, z_r$

$$\begin{aligned}
\frac{dz_1}{dt} &= \frac{\partial\phi_1}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} = L_fh(x(t)) = \phi_2(x(t)) = z_2(t) \\
&\cdots \\
\frac{dz_{r-1}}{dt} &= \frac{\partial\phi_{r-1}}{\partial x}\frac{dx}{dt} = \frac{\partial(L_f^{r-2}h)}{\partial x}\frac{dx}{dt} = L_f^{r-1}h(x(t)) = \phi_r(x(t)) = z_r(t) \ .
\end{aligned}$$

For z_r we obtain

$$\frac{dz_r}{dt} = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t))u(t).$$

On the right-hand side of this equation we must now replace $x(t)$ with its expression as a function of $z(t)$, i.e. $x(t) = \Phi^{-1}(z(t))$. Thus, setting

$$\begin{aligned} a(z) &= L_g L_f^{r-1} h(\Phi^{-1}(z)) \\ b(z) &= L_f^r h(\Phi^{-1}(z)) \end{aligned}$$

the equation in question can be rewritten as

$$\frac{dz_r}{dt} = b(z(t)) + a(z(t))u(t)\ .$$

Note that at the point $z^\circ = \Phi(x^\circ)$, $a(z^\circ) \neq 0$ by definition. Thus, the coefficient $a(z)$ is nonzero for all z in a neighborhood of z°.

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However, if $\phi_{r+1}(x), \ldots, \phi_n(x)$ have been chosen in such a way that $L_g\phi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \phi_i}{\partial x}(f(x(t))+g(x(t))u(t)) = L_f\phi_i(x(t))+L_g\phi_i(x(t))u(t) = L_f\phi_i(x(t)).$$

Setting

$$q_i(z) = L_f\phi_i(\Phi^{-1}(z)) \qquad \text{for all } r+1 \le i \le n$$

the latter can be rewritten as

$$\frac{dz_i}{dt} = q_i(z(t))\ .$$

Thus, in summary, the state-space description of the system in the new coordinates will be as follows

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(z) + a(z)u \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\cdots \\ \dot{z}_n &= q_n(z)\ . \end{aligned} \tag{4.8}$$

In addition to these equations one has to specify how the output of the system is related to the new state variables. But, being $y = h(x)$, it is immediately seen that

$$y = z_1 \; . \tag{4.9}$$

The structure of these equations is best illustrated in the block diagram depicted in Fig. 4.1. The equations thus defined are said to be in *normal form.* We will find them useful in understanding how certain control problems can be solved.

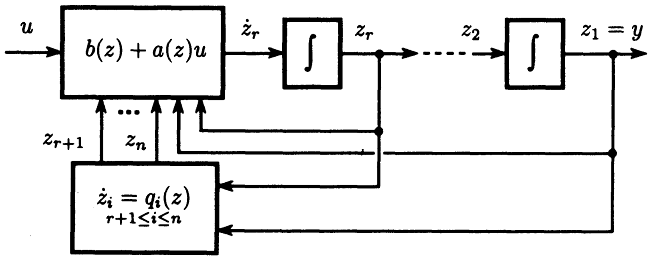

Fig. 4.1.

Remark 4.1.3. Note that sometimes it is not easy to construct $n-r$ functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ such that $L_g\phi_i(x) = 0$, because this, as shown in the proof of Proposition 4.1.3, amounts to solve a system of $n-r$ partial differential equations. Usually, it is much simpler to find functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ with the only property that the jacobian matrix of $\Phi(x)$ is nonsingular at x°, and this is sufficient to define a coordinates transformation. Using a transformation constructed in this way, one gets the same structure for the first r equations, i. e.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(z) + a(z)u \end{aligned}$$

but it is not possible to obtain anything special for the last $n-r$ ones, that therefore will appear in a form like

$$\begin{aligned} \dot{z}_{r+1} &= q_{r+1}(z) + p_{r+1}(z)u \\ &\cdots \\ \dot{z}_n &= q_n(z) + p_n(z)u \end{aligned}$$

with the input u explicitly present. ◁

Example 4.1.4. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} \exp(x_2) \\ 1 \\ 0 \end{pmatrix} u \\ y &= h(x) = x_3 \,. \end{aligned}$$

For this system we have

$$\begin{aligned} \frac{\partial h}{\partial x} &= (0 \quad 0 \quad 1), \qquad L_g h(x) = 0, \qquad L_f h(x) = x_2 \\ \frac{\partial (L_f h)}{\partial x} &= (0 \quad 1 \quad 0), \qquad L_g L_f h(x) = 1 \,. \end{aligned}$$

In order to find the normal form, we set

$$\begin{aligned} z_1 &= \phi_1(x) = h(x) = x_3 \\ z_2 &= \phi_2(x) = L_f h(x) = x_2 \end{aligned}$$

and we seek for a function $\phi_3(x)$ such that

$$\frac{\partial \phi_3}{\partial x} g(x) = \frac{\partial \phi_3}{\partial x_1} \exp(x_2) + \frac{\partial \phi_3}{\partial x_2} = 0 \,.$$

It is easily seen that the function

$$\phi_3(x) = 1 + x_1 - \exp(x_2)$$

satisfies this condition. This and the previous two functions define a transformation $z = \Phi(x)$ whose jacobian matrix

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -\exp(x_2) & 0 \end{pmatrix}$$

is nonsingular for all x. The inverse transformation is given by

$$\begin{aligned} x_1 &= -1 + z_3 + \exp(z_2) \\ x_2 &= z_2 \\ x_3 &= z_1 \,. \end{aligned}$$

Note also that $\Phi(0) = 0$. In the new coordinates the system is described by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= (-1 + z_3 + \exp(z_2)) z_2 + u \\ \dot{z}_3 &= (1 - z_3 - \exp(z_2))(1 + z_2 \exp(z_2)) \,. \end{aligned}$$

These equations are globally valid because the transformation we considered was a global coordinates transformation. ◁

Example 4.1.5. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} x_1x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2+2x_3 \\ 1 \\ 0 \end{pmatrix} u \\ y &= h(x) = x_4 \, . \end{aligned}$$

For this system we have

$$\begin{aligned} \frac{\partial h}{\partial x} &= (0 \quad 0 \quad 0 \quad 1), \qquad L_g h(x) = 0, \qquad L_f h(x) = x_1^2 + x_2 \\ \frac{\partial (L_f h)}{\partial x} &= (2x_1 \quad 1 \quad 0 \quad 0), \qquad L_g L_f h(x) = 2(1+x_3) \, . \end{aligned}$$

Note that $L_g L_f h(x) \neq 0$ if $x_3 \neq -1$. This means that we shall be able to find a normal form only locally, away from any point such that $x_3 = -1$.

In order to find this normal form, we have to set first of all

$$\begin{aligned} z_1 &= \phi_1(x) = h(x) = x_4 \\ z_2 &= \phi_2(x) = L_f h(x) = x_2 + x_1^2 \end{aligned}$$

and then find $\phi_3(x)$, $\phi_4(x)$ which complete the transformation and are such that $L_g\phi_3(x) = L_g\phi_4(x) = 0$.

Suppose we do not want to search for these particular functions and we just take any choice of $\phi_3(x)$, $\phi_4(x)$ which completes the transformation. This can be done, e.g. by taking

$$\begin{aligned} z_3 &= \phi_3(x) = x_3 \\ z_4 &= \phi_4(x) = x_1 \, . \end{aligned}$$

The jacobian matrix of the transformation thus defined

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2x_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is nonsingular for all x, and the inverse transformation is given by

$$\begin{aligned} x_1 &= z_4 \\ x_2 &= z_2 - z_4^2 \\ x_3 &= z_3 \\ x_4 &= z_1 \, . \end{aligned}$$

Note also that $\Phi(0) = 0$. In these new coordinates the system is described by

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_4 + 2z_4(z_4(z_2 - z_4^2) - z_4^3) + (2 + 2z_3)u \\
\dot{z}_3 &= -z_3 + u \\
\dot{z}_4 &= -2z_4^3 + z_2 z_4 \, .
\end{aligned}$$

These equations are valid globally (because the transformation we considered was a global coordinates transformation), but they are not in normal form because of the presence of the input u in the equation for z_3.

If one wants to get rid of u in this equation, it is necessary to use a different function $\phi_3(x)$, making sure that

$$\frac{\partial \phi_3}{\partial x} g(x) = \frac{\partial \phi_3}{\partial x_2}(2 + 2x_3) + \frac{\partial \phi_3}{\partial x_3} = 0 \, .$$

An easy calculation shows that the function

$$\phi_3(x) = x_2 - 2x_3 - x_3^2$$

satisfies this equation. Using this new function and still taking $\phi_4(x) = x_1$ one finds a transformation (whose domain of definition does not include the points at which $x_3 = -1$) yielding the required normal form. ◁

4.2 Exact Linearization Via Feedback

As we anticipated at the beginning of the previous section, one of the main purposes of these notes is the analysis and the design of feedback control laws for nonlinear systems. In almost all situations, we assume the state x of the system being available for measurements, and we let the input of the system to depend on this state and, possibly, on external reference signals. If the value of the control at time t depends only on the values, at the same instant of time, of the state x and of the external reference input, the control is said to be a *Static* (or *Memoryless*) *State Feedback Control.* Otherwise, if the control depends also on a set of additional state variables, i.e. if this control is itself the output of an appropriate dynamical system having its own internal state, driven by x and by the external reference input, we say that a *Dynamic State Feedback Control* is implemented.

In a single-input single-output system, the most convenient structure for a Static State Feedback Control is the one in which the input variable u is set equal to

$$u = \alpha(x) + \beta(x)v \tag{4.10}$$

where v is the external reference input (see Fig. 4.2). In fact, the composition of this control with a system of the form

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}$$

yields a closed loop characterized by the similar structure

$$\begin{aligned}\dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x)v \, . \\ y &= h(x)\end{aligned}$$

The functions $\alpha(x)$ and $\beta(x)$ that characterize the control (4.10) are defined on a suitable open set of $\mathbb{R}^n$. For obvious reasons, $\beta(x)$ is assumed to be nonzero for all x in this set.

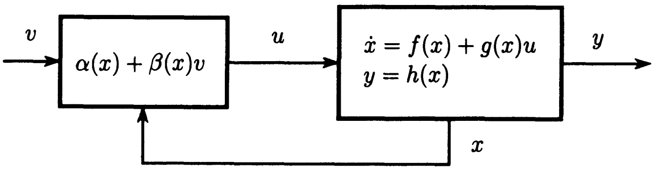

Fig. 4.2.

The first application that will be discussed is the use of state feedback (and change of coordinates in the state-space) to the purpose of transforming a given nonlinear system into a linear and controllable one. The point of departure of this study will be the normal form developed and illustrated in the previous section.

Consider a nonlinear system having relative degree $r = n$, i.e. exactly equal to the dimension of the state space, at some point $x = x^\circ$. In this case the change of coordinates required to construct the normal form is given exactly by

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \cdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{pmatrix}$$

i.e. by the function $h(x)$ and its first $n-1$ derivatives along $f(x)$. No extra functions are needed in order to complete the transformation. In the new coordinates

$$z_i = \phi_i(x) = L_f^{i-1} h(x) \qquad 1 \le i \le n$$

the system will appear described by equations of the form

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= b(z) + a(z)u\end{aligned}$$

where $z = (z_1, \ldots, z_n)$. Recall also that at the point $z^\circ = \Phi(x^\circ)$, and thus at all z in a neighborhood of z°, the function $a(z)$ is nonzero.

Suppose now the following state feedback control law is chosen

$$u = \frac{1}{a(z)}(-b(z) + v) \tag{4.11}$$

which indeed exists and is well-defined in a neighborhood of z°. The resulting closed loop system is governed by the equations (Fig. 4.3)

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned}$$

i.e. is *linear* and *controllable.* Thus we conclude that any nonlinear system with relative degree n at some point x° can be transformed into a system which, in a neighborhood of the point $z^\circ = \Phi(x^\circ)$, is linear and controllable. It is important to stress that the transformation in question consists of two basic ingredients

(i) a change of coordinates, defined locally around the point x°

(ii) a state feedback, also defined locally around the point x°.

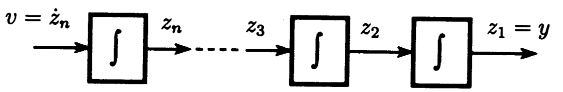

Fig. 4.3.

Remark 4.2.1. It is easily checked that the two transformations used in order to obtain the linear form can be interchanged. One can first use a feedback and then change the coordinates in the state space, and the result is the same. The feedback needed to this purpose is exactly the same feedback just used, but now expressed in the x coordinates, i.e. as

$$u = \frac{1}{a(\Phi(x))}(-b(\Phi(x)) + v) .$$

Comparing this with the expressions for $a(z)$ and $b(z)$ given in the previous section, one realizes that this feedback – expressed in terms of the functions $f(x)$, $g(x)$, $h(x)$ which characterize the original system – has the form

$$u = \frac{1}{L_g L_f^{n-1} h(x)} (-L_f^n h(x) + v) . \tag{4.12}$$

An immediate calculation shows that this feedback, together with the same change of coordinates used so far, exactly yields the same linear and controllable system already obtained. ◁

Remark 4.2.2. Note that if x° is an equilibrium point for the original nonlinear system, i.e. if $f(x^\circ) = 0$, and if also $h(x^\circ) = 0$, then $z^\circ = \Phi(x^\circ) = 0$. As a matter of fact

$$\begin{aligned} \phi_1(x^\circ) &= h(x^\circ) = 0 \\ \phi_i(x^\circ) &= \frac{\partial (L_f^{i-2} h)}{\partial x} f(x^\circ) = 0 \qquad \text{for all } 2 \le i \le n . \end{aligned}$$

Note also that a condition like $h(x^\circ) = 0$ can always be satisfied, by means of a suitable translation of the origin of the output space.

Thus, we conclude that if x° is an equilibrium point for the original system, and this system has relative degree n at x°, there is a feedback control law (defined in a neighborhood of x°) and a coordinates transformation (also defined in a neighborhood of x°) changing the system into a linear and controllable one, defined in a neighborhood of 0. ◁

Remark 4.2.3. On the linear system thus obtained one can impose new feedback controls, like for instance

$$v = Kz$$

with

$$K = (c_0 \dots c_{n-1})$$

chosen e.g. in order to assign a specific set of eigenvalues, or to satisfy an optimality criterion. Recalling the expression of the z_i's as functions of x, the feedback in question can be rewritten as

$$v = c_0 h(x) + c_1 L_f h(x) + \cdots + c_{n-1} L_f^{n-1} h(x) \tag{4.13}$$

i.e. in the form of a nonlinear feedback from the state x of the original description of the system. Note that the composition of (4.12) and (4.13) is again a state feedback, having the form

$$u = \frac{-L_f^n h(x) + \sum_{i=0}^{n-1} c_i L_f^i h(x)}{L_g L_f^{n-1} h(x)} . ◁$$

Example 4.2.4. Consider the system

$$\dot{x} = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} \exp(x_2) \\ \exp(x_2) \\ 0 \end{pmatrix} u$$

$$y = x_3 .$$

For this system we have

$$\begin{aligned} L_g h(x) &= 0, \qquad L_f h(x) = x_1 - x_2, \\ L_g L_f h(x) &= 0, \qquad L_f^2 h(x) = -x_1 - x_2^2, \\ L_g L_f^2 h(x) &= -(1 + 2x_2)\exp(x_2), \\ L_f^3 h(x) &= -2x_2(x_1 + x_2^2) . \end{aligned}$$

Thus, we see that the system has relative degree 3 (i.e. equal to n) at each point such that $1 + 2x_2 \neq 0$. Around any of such points, for instance around $x = 0$, the system can be transformed into a linear and controllable system by means of the feedback control

$$u = \frac{-2x_2(x_1 + x_2^2)}{(1 + 2x_2)\exp(x_2)} - \frac{1}{(1 + 2x_2)\exp(x_2)} v$$

and the change of coordinates

$$\begin{aligned} z_1 &= h(x) = x_3 \\ z_2 &= L_f h(x) = x_1 - x_2 \\ z_3 &= L_f^2 h(x) = -x_1 - x_2^2 . \end{aligned}$$

Note that both the feedback and the change of coordinates are defined only locally around $x = 0$. In particular, the feedback u is not defined at points x such that $1 + 2x_2 = 0$ and the jacobian matrix of the coordinates transformation is singular at these points.

In the new coordinates, the system appears as

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

which is linear and controllable. ◁

Of course, the basic feature of the system that made it possible to change it into a linear and controllable one was the existence of an "output" function $h(x)$ for which the system had relative degree exactly n (at x°). We shall see now that the existence of such a function is not only a sufficient – as the previous discussion shows – but also a necessary condition for the existence of a state feedback and a change of coordinates transforming a given system into a linear and controllable one.

More precisely, consider a system (without output)

$$\dot{x} = f(x) + g(x)u$$

and suppose the following problem is set: given a point x° find (if possible), a neighborhood U of x°, a feedback

$$u = \alpha(x) + \beta(x)v$$

defined on U, and a coordinates transformation $z = \Phi(x)$ also defined on U, such that the corresponding closed loop system

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v$$

in the coordinates $z = \Phi(x)$, is linear and controllable, i.e. such that

$$\left[\frac{\partial \Phi}{\partial x}(f(x) + g(x)\alpha(x))\right]_{x=\Phi^{-1}(z)} = Az \tag{4.14}$$

$$\left[\frac{\partial \Phi}{\partial x}(g(x)\beta(x))\right]_{x=\Phi^{-1}(z)} = B \tag{4.15}$$

for some suitable matrix $A \in \mathbb{R}^{n\times n}$ and vector $B \in \mathbb{R}^n$ satisfying the condition

$$\text{rank}(B\ AB\ \dots\ A^{n-1}B) = n\,.$$

This problem is the "single-input" version of the so-called *State Space Exact Linearization Problem*. The previous analysis has already established a sufficient condition for the existence of a solution; we show now that this condition is also necessary.

Lemma 4.2.1. *The State Space Exact Linearization Problem is solvable if and only if there exist a neighborhood U of x° and a real-valued function $\lambda(x)$, defined on U, such that the system*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= \lambda(x) \end{aligned}$$

has relative degree n at x°.

Proof. Clearly, we only have to show that the condition is necessary. We begin by showing an interesting feature of the notion of relative degree, namely that the latter is invariant under coordinates transformations and feedback. For, let $z = \Phi(x)$ be a coordinates transformation, and set

$$\bar{f}(z) = \left[\frac{\partial \Phi}{\partial x} f(x)\right]_{x=\Phi^{-1}(z)} \quad \bar{g}(z) = \left[\frac{\partial \Phi}{\partial x} g(x)\right]_{x=\Phi^{-1}(z)} \quad \bar{h}(z) = h(\Phi^{-1}(z)).$$

Then

$$\begin{aligned} L_{\bar{f}}\bar{h}(z) &= \frac{\partial \bar{h}}{\partial z}\bar{f}(z) = \left[\frac{\partial h}{\partial x}\right]_{x=\Phi^{-1}(z)} \left[\frac{\partial \Phi^{-1}}{\partial z}\right] \left[\frac{\partial \Phi}{\partial x}f(x)\right]_{x=\Phi^{-1}(z)} \\ &= \left[\frac{\partial h}{\partial x}f(x)\right]_{x=\Phi^{-1}(z)} = [L_f h(x)]_{x=\Phi^{-1}(z)}. \end{aligned}$$

Iterated calculations of this kind show that

$$L_{\bar{g}} L_{\bar{f}}^k \bar{h}(z) = [L_g L_f^k h(x)]_{x=\Phi^{-1}(z)}$$

from which it is easily concluded that the relative degree is invariant under coordinates transformation. As far as the feedback is concerned, note that

$$L_{f+g\alpha}^k h(x) = L_f^k h(x) \qquad \text{for all } 0 \le k \le r-1\,. \tag{4.16}$$

As a matter of fact, this equality is trivially true for $k = 0$. By induction, suppose is true for some $0 < k < r-1$. Then

$$L_{f+g\alpha}^{k+1} h(x) = L_{f+g\alpha} L_f^k h(x) = L_f^{k+1} h(x) + L_g L_f^k h(x)\alpha(x) = L_f^{k+1} h(x)$$

thus showing that the equality in question holds for $k+1$. From (4.16), one deduces that

$$L_{g\beta} L_{f+g\alpha}^k h(x) = 0 \qquad \text{for all } 0 \le k < r-1$$

and that, if $\beta(x^\circ) \neq 0$

$$L_{g\beta} L_{f+g\alpha}^{r-1} h(x^\circ) \neq 0\,.$$

This shows that r is invariant under feedback.

Now, let (A, B) be a reachable pair. Then, it is well-known from the theory of linear systems that there exists a nonsingular $n \times n$ matrix T and a $1 \times n$ vector k such that

$$T(A+Bk)T^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad TB = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}. \tag{4.17}$$

Suppose (4.14) and (4.15) hold and set

$$\begin{aligned} \bar{z} &= \bar{\Phi}(x) = T\Phi(x) \\ \bar{\alpha}(x) &= \alpha(x) + \beta(x)k\Phi(x). \end{aligned}$$

Then, it is easily seen that

$$\left[\frac{\partial\bar{\Phi}}{\partial x}(f(x)+g(x)\bar{\alpha}(x))\right]_{x=\bar{\Phi}^{-1}(\bar{z})} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \bar{z}$$

$$\left[\frac{\partial\bar{\Phi}}{\partial x}(g(x)\beta(x))\right]_{x=\bar{\Phi}^{-1}(\bar{z})} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix} .$$

From this, it is deduced that there is no loss of generality in assuming that the pair (A, B) which renders the (4.14)-(4.15) satisfied has the form indicated in the right-hand-sides of (4.17).

Define now the "output" function

$$y = (1\ 0\ \cdots\ 0)\bar{z} .$$

A straightforward calculation shows that the linear system with A and B in the form of the right-hand-sides of (4.17) and with this output function has exactly relative degree n. Thus, since the relative degree is invariant under feedback and coordinates transformation, the proof is complete. ◁

The problem of finding a function $\lambda(x)$ such that the relative degree of the system at x° is exactly n, namely a function such that

$$L_g\lambda(x) = L_gL_f\lambda(x) = \ldots = L_gL_f^{n-2}\lambda(x) = 0 \quad \text{for all } x \tag{4.18}$$

$$L_gL_f^{n-1}\lambda(x^\circ) \neq 0 \tag{4.19}$$

is apparently a problem involving the solution of a system of partial differential equations (namely the equations (4.18)), in which the unknown function $\lambda(x)$ is differentiated up to $n-1$ times, together with a condition (namely the condition (4.19)) which singles-out trivial solutions like e.g. $\lambda(x) = 0$. However, thanks to Lemma 4.1.2, this system is in fact equivalent to a system of first order partial differential equations, of a rather simple form. As a matter of fact, this Lemma exactly shows that the equations (4.18) are equivalent to

$$L_g\lambda(x) = L_{ad_fg}\lambda(x) = \ldots = L_{ad_f^{n-2}g}\lambda(x) = 0 \tag{4.20}$$

and that the nontriviality condition (4.19) is equivalent to

$$L_{ad_f^{n-1}g}\lambda(x^\circ) \neq 0 . \tag{4.21}$$

The existence of a function satisfying these relations is an easy consequence of Frobenius' Theorem, as it can be seen in the proof of the following result.

Lemma 4.2.2. *There exists a real-valued function $\lambda(x)$ defined in a neighborhood U of x° solving the partial differential equations (4.20), and satisfying the nontriviality condition (4.21), if and only if*

(i) *the matrix* $\left(g(x^\circ)\ ad_f g(x^\circ)\ \dots\ ad_f^{n-2} g(x^\circ)\ ad_f^{n-1} g(x^\circ)\right)$ *has rank n,*

(ii) *the distribution* $D = \text{span}\{g, ad_f g, \dots, ad_f^{n-2} g\}$ *is involutive in a neighborhood of x°.*

Proof. Suppose a function $\lambda(x)$ satisfying (4.20) and (4.21) exists. Then, from the proof of Lemma 4.1.1, in particular from the nonsingularity of the matrix (4.5), we deduce that the n vectors

$$g(x^\circ), ad_f g(x^\circ), \dots, ad_f^{n-2} g(x^\circ), ad_f^{n-1} g(x^\circ)$$

are linearly independent. This proves the necessity of (i). If (i) holds then the distribution D is nonsingular and $(n-1)$-dimensional around x°. The equations (4.20), that can be rewritten as

$$d\lambda(x)\left(g(x)\ ad_f g(x)\ \dots\ ad_f^{n-2} g(x)\right) = 0\,, \tag{4.22}$$

tell us that the differential $d\lambda(x)$ is a basis of the 1-dimensional codistribution $D^\perp$ around x°. So, by Frobenius' Theorem, the distribution D is involutive, and this proves the necessity of (ii). Conversely, suppose (i) holds. Then the distribution D is nonsingular and $(n-1)$-dimensional around x°. If also (ii) holds, by Frobenius' Theorem we know there exists a real-valued function $\lambda(x)$, defined in a neighborhood U of x° whose differential $d\lambda(x)$ spans $D^\perp$, i.e. solves the partial differential equation (4.20). Moreover, $d\lambda(x)$ also satisfies (4.21), because otherwise $d\lambda(x)$ would be annihilated by a set of n linearly independent vectors, i.e. a contradiction. ◁

We can at this point summarize the results established so far in the following formal statement

Theorem 4.2.3. *Suppose a system*

$$\dot{x} = f(x) + g(x)u$$

is given. The State Space Exact Linearization Problem is solvable near a point x° (i.e. there exists an "output" function $\lambda(x)$ for which the system has relative degree n at x°) if and only if the following conditions are satisfied

(i) *the matrix* $\left([g(x^\circ)\ ad_f g(x^\circ)\ \dots\ ad_f^{n-2} g(x^\circ)\ ad_f^{n-1} g(x^\circ)\right)$ *has rank n,*

(ii) *the distribution* $D = \text{span}\{g, ad_f g, \dots, ad_f^{n-2} g\}$ *is involutive near x°.*

On the basis of the previous discussion, it is now clear that the procedure leading to the construction of a feedback $u = \alpha(x) + \beta(x)v$ and of a coordinates transformation $z = \Phi(x)$ solving the State Space Exact Linearization problem consists of the following steps

- from $f(x)$ and $g(x)$, construct the vector fields

$$g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x), ad_f^{n-1} g(x)$$

and check the conditions (i) and (ii),
- if both are satisfied, solve for $\lambda(x)$ the partial differential equation (4.20),
- set

$$\alpha(x) = \frac{-L_f^n \lambda(x)}{L_g L_f^{n-1} \lambda(x)} \qquad \beta(x) = \frac{1}{L_g L_f^{n-1} \lambda(x)} \tag{4.23}$$

- set

$$\Phi(x) = \mathrm{col}(\lambda(x), L_f \lambda(x), \ldots, L_f^{n-1} \lambda(x)) \ . \tag{4.24}$$

The feedback defined by the functions (4.23) is called the *linearizing feedback* and the new coordinates defined by (4.24) are called the *linearizing coordinates*. We illustrate now the whole Exact Linearization procedure in a simple example.

Example 4.2.5. Consider the system

$$\dot{x} = \begin{pmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix} u \ .$$

In order to check whether or not this system can be transformed into a linear and controllable system via state feedback and coordinates transformation, we have to compute the functions $ad_f g(x)$ and $ad_f^2 g(x)$ and test the conditions of Theorem 4.2.3.

Appropriate calculations show that $ad_f g(x) =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{pmatrix} - \begin{pmatrix} 0 & x_3 & 1+x_2 \\ 1 & 0 & 0 \\ x_2 & 1+x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ x_1 \\ -(1+x_1)(1+2x_2) \end{pmatrix}$$

and that

$$ad_f^2 g(x) = \begin{pmatrix} (1+x_2)(1+2x_2)(1+x_1) - x_3 x_1 \\ x_3(1+x_2) \\ -x_3(1+x_2)(1+2x_2) - 3x_1(1+x_1) \end{pmatrix} .$$

At $x = 0$, the matrix

$$\begin{pmatrix} g(x) & ad_f g(x) & ad_f^2 g(x) \end{pmatrix}_{x=0} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

has rank 3 and therefore the condition (i) is satisfied. It is also easily checked that the product $[g, ad_f g](x)$ has a form

$$[g, ad_f g](x) = \begin{pmatrix} 0 \\ \star \\ \star \end{pmatrix}$$

and therefore also the condition (ii) is satisfied, because the matrix

$$(\, g(x) \quad ad_f g(x) \quad [g, ad_f g](x)\,)$$

has rank 2 for all x near $x = 0$.

In the present case, it is easily seen that a function $\lambda(x)$ that solves the equation

$$\frac{\partial \lambda}{\partial x}(\, g(x) \quad ad_f g(x)\,) = 0$$

is given by

$$\lambda(x) = x_1 \ .$$

From our previous discussion, we know that considering this as "output" will yield a system having relative degree 3 (i.e. equal to n) at the point $x = 0$. We double-check and observe that

$$L_g\lambda(x) = 0, \quad L_gL_f\lambda(x) = 0, \quad L_gL_f^2\lambda(x) = (1+x_1)(1+x_2)(1+2x_2) - x_1x_3$$

and $L_gL_f^2\lambda(0) = 1$. Locally around $x = 0$, the system will be transformed into a linear and controllable one by means of the state feedback

$$u = \frac{-L_f^3\lambda(x) + v}{L_gL_f^2\lambda(x)} =$$

$$= \frac{-x_3^2(1+x_2) - x_2x_3(1+x_2)^2 - x_1(1+x_1)(1+2x_2) - x_1x_2(1+x_1) + v}{(1+x_1)(1+x_2)(1+2x_2) - x_1x_3}$$

and the coordinates transformation

$$\begin{aligned} z_1 &= \lambda(x) = x_1 \\ z_2 &= L_f\lambda(x) = x_3(1+x_2) \\ z_3 &= L_f^2\lambda(x) = x_3x_1 + (1+x_1)(1+x_2)x_2 \ .\triangleleft \end{aligned}$$

Remark 4.2.6. Using the above result, it is easily seen that any nonlinear system whose state space has dimension $n = 2$ can be transformed into a linear system, via state feedback and change of coordinates, around a point x°, if and only if the matrix

$$(\, g(x^\circ) \quad ad_f g(x^\circ)\,)$$

has rank 2. As a matter of fact, this is exactly the condition (i) of the previous Theorem, and condition (ii) is always satisfied, because $D = \text{span}\{g\}$ is 1-dimensional. In this case it is always possible to find a function $\lambda(x) = \lambda(x_1, x_2)$, defined locally around x°, such that

$$\frac{\partial \lambda}{\partial x} g(x) = \frac{\partial \lambda}{\partial x_1} g_1(x_1, x_2) + \frac{\partial \lambda}{\partial x_2} g_2(x_1, x_2) = 0 . \triangleleft$$

Remark 4.2.7. The condition (i) of Theorem 4.2.3 has the following interesting interpretation. Suppose the vector field $f(x)$ has an equilibrium at $x^\circ = 0$, i.e. $f(0) = 0$, and consider for $f(x)$ an expansion of the form

$$f(x) = Ax + f_2(x)$$

with

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \quad \text{and} \quad \left[\frac{\partial f_2}{\partial x}\right]_{x=0} = 0$$

which separates the linear approximation Ax from the higher-order term $f_2(x)$. Consider also for $g(x)$ an expansion of the form

$$g(x) = B + g_1(x)$$

with $B = g(0)$. These expansions characterize the *linear approximation* of the system at $x = 0$, which is defined as

$$\dot{x} = Ax + Bu .$$

An easy calculation shows that the vector fields $ad_f^k g(x)$ can be expanded in the following way

$$ad_f^k g(x) = (-1)^k A^k B + p_k(x)$$

where $p_k(x)$ is a function such that $p_k(0) = 0$. As a matter of fact, the expansion in question is trivially true for $k = 0$. By induction, suppose is true for some k. Then, by definition

$$\begin{aligned} ad_f^{k+1} g(x) &= \frac{\partial (ad_f^k g)}{\partial x} f(x) - \frac{\partial f}{\partial x} ad_f^k g(x) \\ &= \frac{\partial p_k}{\partial x}(Ax + f_2(x)) - (A + \frac{\partial f_2}{\partial x})((-1)^k A^k B + p_k(x)) \\ &= (-1)^{k+1} A^{k+1} B + p_{k+1}(x) \end{aligned}$$

where $p_{k+1}(x)$, by construction, is zero at $x = 0$.

From this, we see that the condition (i) of Theorem 4.2.3 (written at $x^\circ = 0$) is equivalent to the condition

$$\text{rank}(B \ AB \ \ldots \ A^{n-1}B) = n$$

i. e. to the condition that *the linear approximation of the system at x=0 is controllable.*

In other words, we conclude that the controllability of the linear approximation of the system at $x = x^\circ$ is a necessary condition for the solvability of the State Space Exact Linearization Problem. $\triangleleft$

Remark 4.2.8. It is interesting to observe that the conditions (i) and (ii) of Theorem 4.2.3 imply the involutivity of the distribution

$$D_k = \text{span}\{g, ad_f g, \dots, ad_f^k g\}$$

for any $1 \le k \le n-3$. As a matter of fact, since (i) and (ii) imply the existence of a $\lambda(x)$ such that (4.18) and (4.19) hold, from Lemma 4.1.2 it follows that

$$\begin{aligned} d\lambda(x)\,(\,g(x) \quad ad_f g(x) \quad \cdots \quad ad_f^k g(x)\,) &= 0 \\ dL_f\lambda(x)\,(\,g(x) \quad ad_f g(x) \quad \cdots \quad ad_f^k g(x)\,) &= 0 \\ &\cdots \\ dL_f^{n-k-2}\lambda(x)\,(\,g(x) \quad ad_f g(x) \quad \cdots \quad ad_f^k g(x)\,) &= 0\,. \end{aligned}$$

These equalities show that

$$\text{span}\{d\lambda, dL_f\lambda, \dots, dL_f^{n-k-2}\lambda\} \subset D_k^{\perp}\,.$$

Moreover, since (Lemma 4.1.1) the differentials $d\lambda, dL_f\lambda, \dots, dL_f^{n-k-2}\lambda$ are linearly independent around x° and $D_k^{\perp}$ has dimension $n-k-1$ around x° (as a consequence of assumption (i)), it is concluded that $D_k^{\perp}$ is spanned by exact differentials. Then, by Frobenius' Theorem, D_k is involutive.

We see from this property that the involutivity of all the distributions D_k, $1 \le k \le n-2$, is a *necessary* condition for the solvability of the Exact State Space Linearization Problem. $\triangleleft$

Remark 4.2.9. Note that if the State Space Exact Linearization Problem is solved by means of a feedback and a coordinates transformation $z = \Phi(x)$ defined in a neighborhood U of x°, the corresponding linear system is defined on the open set $\Phi(U)$. For obvious reasons, it is interesting to have $\Phi(U)$ containing the origin of $\mathbb{R}^n$, and in particular to have $\Phi(x^\circ) = 0$. In this case, in fact, one could for instance use linear systems theory concepts in order to asymptotically stabilize at $z = 0$ the transformed system and then use the stabilizer thus found in a composite loop to the purpose of stabilizing the nonlinear system at $x = x^\circ$ (see Remark 4.2.3).

This is indeed the case when x° is an equilibrium of the vector field $f(x)$. In this case, in fact, choosing the solution $\lambda(x)$ of the differential equation with the additional constraint $\lambda(x^\circ) = 0$, as is always possible, one gets $\Phi(x^\circ) = 0$, as already shown at the beginning of the section (see Remark 4.2.2).

If x° is not an equilibrium of the vector field $f(x)$, one can manage to have this occurring by means of feedback. As a matter of fact, the condition $\Phi(x^\circ) = 0$, replaced into (4.14), necessarily yields

$$\left[\frac{\partial\Phi}{\partial x}(f(x) + g(x)\alpha(x))\right]_{x=x^\circ} = 0$$

i.e.

$$f(x^\circ) + g(x^\circ)\alpha(x^\circ) = 0 .$$

This clearly expresses the fact that the point x° is an equilibrium of the vector field $f(x) + g(x)\alpha(x)$, and can be obtained if and only if the vectors $f(x^\circ)$ and $g(x^\circ)$ are such that

$$f(x^\circ) = cg(x^\circ)$$

where c is a real number. If this is the case, an easy calculation shows that the linearizing coordinates are still zero at x° (if $\lambda(x)$ is such), because, for all $2 \le i \le n$

$$L_f^{i-1}\lambda(x^\circ) = cL_gL_f^{i-2}\lambda(x^\circ) = 0 .$$

Moreover, the linearizing feedback $\alpha(x)$ is such that

$$\alpha(x^\circ) = -\frac{L_f^n\lambda(x^\circ)}{L_gL_f^{n-1}\lambda(x^\circ)} = -c$$

as expected. ◁

Remark 4.2.10. Note that a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

having relative degree strictly less than n could as well meet the requirements (i) and (ii) of Theorem 4.2.3. If this is the case, there will be a ***different*** "output" function, say $\lambda(x)$, with respect to which the system will have relative degree exactly n. Starting from this new function it will be possible to construct a feedback $u = \alpha(x) + \beta(x)v$ and a change of coordinates $z = \Phi(x)$, that will transform the state space equation

$$\dot{x} = f(x) + g(x)u$$

into a linear and controllable one. However, the real output of the system, in the new coordinates

$$y = h(\Phi^{-1}(z))$$

will in general continue to be a ***nonlinear*** function of the state z. ◁

If the system has relative degree $r < n$, for some given output $h(x)$, and either the conditions of Lemma 4.2.2 – for the existence of another output for which the relative degree is equal to n – are not satisfied, or more simply one doesn't like to embark oneself in the solution of the partial differential equation (4.20) yielding such an output, it is still possible to obtain – by means of state feedback – a system which is ***partially*** linear. As a matter of fact, setting again

$$u = \frac{1}{a(z)}(-b(z) + v) \tag{4.25}$$

on the normal form of the equations, one obtains, if $r < n$, a system like

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\cdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= v \\
\dot{z}_{r+1} &= q_{r+1}(z) \\
&\cdots \\
\dot{z}_n &= q_n(z) \\
y &= z_1 \,.
\end{aligned} \tag{4.26}$$

This system clearly appears decomposed into a *linear subsystem*, of dimension r, which is the only one responsible for the input-output behavior, and a possibly nonlinear subsystem, of dimension $n-r$, whose behavior however does not affect the output (Fig.4.4).

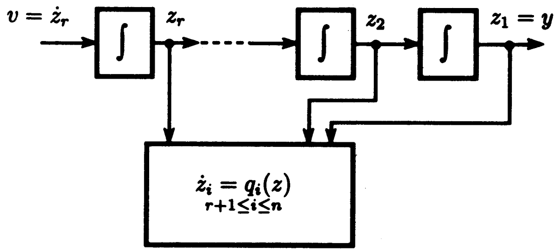

Fig. 4.4.

We summarize this result for convenience in a formal statement where, for more generality, the linearizing feedback is specified in terms of the functions $f(x)$, $g(x)$ and $h(x)$ characterizing the original description of the system.

Proposition 4.2.4. *Consider a nonlinear system having relative degree r at a point x°. The state feedback*

$$u = \frac{1}{L_g L_f^{r-1} h(x)}(-L_f^r h(x) + v) \tag{4.27}$$

transforms this system into a system whose input-output behavior is identical to that of a linear system having a transfer function

$$H(s) = \frac{1}{s^r} \,.$$

4.3 The Zero Dynamics

In this section we introduce and discuss an important concept, that in many instances plays a role exactly similar to that of the "zeros" of the transfer function in a linear system. We have already seen that the relative degree r of a linear system can be interpreted as the difference between the number of poles and the number of zeros in the transfer function. In particular, any linear system in which r is strictly less than n has zeros in its transfer function. On the contrary, if $r = n$ the transfer function has no zeros; thus, the systems considered at the beginning of the previous section are in some sense analogue to linear systems without zeros. We shall see in this section that this kind of analogy can be pushed much further.

Consider a nonlinear system with r strictly less than n and look at its normal form. In order to write the equations in a slightly more compact manner, we introduce a suitable vector notation. In particular, since there is no specific need to keep track individually of each one of the last $n - r$ components of the state vector, we shall represent all of them together as

$$\eta = \begin{pmatrix} z_{r+1} \\ \cdots \\ z_n \end{pmatrix} .$$

Sometimes, whenever convenient and not otherwise required, we shall represent also the first r components together, as

$$\xi = \begin{pmatrix} z_1 \\ \cdots \\ z_r \end{pmatrix} .$$

With the help of these notations, the normal form of a single-input single-output nonlinear system having $r < n$ (at some point of interest x°, e.g. an equilibrium point) can be rewritten as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(\xi, \eta) + a(\xi, \eta)u \\ \dot{\eta} &= q(\xi, \eta) . \end{aligned}$$

Recall that, if x° is such that $f(x^\circ) = 0$ and $h(x^\circ) = 0$, then necessarily the first r new coordinates z_i are 0 at x°. Note also that it is always possible to choose arbitrarily the value at x° of the last $n - r$ new coordinates, thus in particular being 0 at x°. Therefore, without loss of generality, one can assume that $\xi = 0$ and $\eta = 0$ at x°. Thus, if x° was an equilibrium for the system in the original coordinates, its corresponding point $(\xi, \eta) = (0, 0)$ is an equilibrium for the system in the new coordinates and from this we deduce that

$$\begin{aligned} b(\xi,\eta) &= 0 \qquad \text{at } (\xi,\eta)=(0,0) \\ q(\xi,\eta) &= 0 \qquad \text{at } (\xi,\eta)=(0,0)\ . \end{aligned}$$

Suppose now we want to analyze the following problem, called the *Problem of Zeroing the Output.* Find, if any, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t=0$, such that the corresponding output $y(t)$ of the system is identically zero for all t in a neighborhood of $t=0$. Of course, we are interested in finding *all* such pairs (x°, u°) and not simply in the trivial pair $x^\circ = 0, u^\circ = 0$ (corresponding to the situation in which the system is initially at rest and no input is applied). We perform this analysis on the normal form of the system.

Recalling that in the normal form

$$y(t) = z_1(t)\ ,$$

we see that the constraint $y(t)=0$ for all t implies

$$\dot z_1(t) = \dot z_2(t) = \ldots = \dot z_r(t) = 0\ ,$$

that is $\xi(t)=0$ for all t.

Thus, we see that when the output of the system is identically zero its state is constrained to evolve in such a way that also $\xi(t)$ is identically zero. In addition, the input $u(t)$ must necessarily be the unique solution of the equation

$$0 = b(0,\eta(t)) + a(0,\eta(t))u(t)$$

(recall that $a(0,\eta(t)) \neq 0$ if $\eta(t)$ is close to 0). As far as the variable $\eta(t)$ is concerned, it is clear that, being $\xi(t)$ identically zero, its behavior is governed by the differential equation

$$\dot\eta(t) = q(0,\eta(t))\ . \tag{4.28}$$

From this analysis we deduce the following facts. If the output $y(t)$ has to be zero, then necessarily the initial state of the system must be set to a value such that $\xi(0)=0$, whereas $\eta(0)=\eta^\circ$ can be chosen arbitrarily. According to the value of η°, the input must be set as

$$u(t) = -\frac{b(0,\eta(t))}{a(0,\eta(t))}$$

where $\eta(t)$ denotes the solution of the differential equation

$$\dot\eta(t) = q(0,\eta(t)) \qquad \text{with initial condition } \eta(0)=\eta^\circ.$$

Note also that for each set of initial data $\xi=0$ and $\eta=\eta^\circ$ the input thus defined is the *unique* input capable to keep $y(t)$ identically zero for all times.

The dynamics of (4.28) correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics, which are rather important in many of our developments, are called the *zero dynamics* of the system.

Remark 4.3.1. In order to understand why we used the terminology "zero" dynamics in dealing with the dynamical system (4.28), it is convenient to examine how these dynamics are related to the zeros of the transfer function in a linear system. Let

$$H(s) = K\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}$$

denote the transfer function of a linear system (where r characterizes, as expected, the relative degree). Suppose the numerator and denominator polynomials are relatively prime and consider a minimal realization of $H(s)$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ K \end{pmatrix}$$

$$C = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let us now calculate its normal form. For the first r new coordinates we know we have to take

$$\begin{aligned} z_1 &= Cx = b_0x_1 + b_1x_2 + \cdots + b_{n-r-1}x_{n-r} + x_{n-r+1} \\ z_2 &= CAx = b_0x_2 + b_1x_3 + \cdots + b_{n-r-1}x_{n-r+1} + x_{n-r+2} \\ &\cdots \\ z_r &= CA^{r-1}x = b_0x_r + b_1x_{r+1} + \cdots + b_{n-r-1}x_{n-1} + x_n. \end{aligned}$$

For the other $n-r$ new coordinates we have some freedom of choice (provided that the conditions stated in Proposition 4.1.3 are satisfied), but the simplest one is

$$\begin{aligned} z_{r+1} &= x_1 \\ z_{r+2} &= x_2 \\ &\cdots \\ z_n &= x_{n-r}\,. \end{aligned}$$

This is indeed an admissible choice because the corresponding coordinates transformation $z = \Phi(x)$ has a jacobian matrix with the following structure

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} (\cdots) & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \star & \star & \cdots & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix} \end{pmatrix}$$

and therefore nonsingular.

In the new coordinates we obtain equations in normal form, which, because of the linearity of the system, have the following structure

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= R\xi + S\eta + Ku \\ \dot{\eta} &= P\xi + Q\eta \end{aligned}$$

where R and S are row vectors and P and Q matrices, of suitable dimensions. The zero dynamics of this system, according to our previous definition, are those of

$$\dot{\eta} = Q\eta \, .$$

The particular choice of the last $n - r$ new coordinates (i.e. of the elements of η) entails a particularly simple structure for the matrix Q. As a matter of fact, is easily checked that

$$\begin{aligned} \frac{dz_{r+1}}{dt} &= \frac{dx_1}{dt} = x_2(t) = z_{r+2}(t) \\ &\cdots \\ \frac{dz_{n-1}}{dt} &= \frac{dx_{n-r-1}}{dt} = x_{n-r}(t) = z_n(t) \\ \frac{dz_n}{dt} &= \frac{dx_{n-r}}{dt} = x_{n-r+1}(t) = -b_0 x_1(t) - \cdots - b_{n-r-1}x_{n-r}(t) + z_1(t) \\ &= -b_0 z_{r+1}(t) - \cdots - b_{n-r-1}z_n(t) + z_1(t) \end{aligned}$$

from which we deduce that

$$Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{pmatrix}.$$

From the particular form of this matrix, it is clear that the eigenvalues of Q coincide with the zeros of the numerator polynomial of $H(s)$, i.e. with the zeros of the transfer function. Thus it is concluded that in a linear system the zero dynamics are linear dynamics with eigenvalues coinciding with the zeros of the transfer function of the system. ◁

Remark 4.3.2. The calculations carried out in the previous Remark are also useful in showing that the linear approximation, at $\eta = 0$, of the zero dynamics of a system coincide with the zero dynamics of the linear approximation of the system at $x = 0$, i. e. that the operations of taking the linear approximation and calculating the zero dynamics commute.

In order to check this, all we have to show is that the linear approximation of equations in normal form coincides with the normal form of the linear approximation of the original description of the system and this amounts only to show that the relative degree of the system and that of its linear approximation are the same. To this end, suppose that the system has relative degree r at $x = 0$. Consider the expansions already introduced in Remark 4.2.7

$$\begin{aligned} f(x) &= Ax + f_2(x) \\ g(x) &= B + g_1(x) \end{aligned}$$

and, in addition, expand $h(x)$ (which is 0 at $x = 0$) as

$$h(x) = Cx + h_2(x)$$

where

$$C = \left[\frac{\partial h}{\partial x}\right]_{x=0} \qquad \text{and} \qquad \left[\frac{\partial h_2}{\partial x}\right]_{x=0} = 0\,.$$

An easy calculation shows, by induction, that

$$L_f^k h(x) = CA^k x + d_k(x)$$

where $d_k(x)$ is a function such that

$$\left[\frac{\partial d_k}{\partial x}\right]_{x=0} = 0\,.$$

From this one deduces that

$$\begin{aligned} CA^k B &= L_g L_f^k h(0) = 0 \qquad \text{for all } k < r-1 \\ CA^{r-1} B &= L_g L_f^{r-1} h(0) \neq 0 \end{aligned}$$

i.e. that the relative degree of the linear approximation of the system at $x = 0$ is exactly r.

From this fact, it is concluded that taking the linear approximation of equations in normal form, based on expansions of the form

$$\begin{aligned} b(\xi,\eta) &= R\xi + S\eta + b_2(\xi,\eta) \\ a(\xi,\eta) &= K + a_1(\xi,\eta) \\ q(\xi,\eta) &= P\xi + Q\eta + q_2(\xi,\eta) \end{aligned}$$

yields a linear system in normal form. Thus, the jacobian matrix

$$Q = \left[\frac{\partial q}{\partial \eta}\right]_{(\xi,\eta)=0}$$

which describes the linear approximation at $\eta = 0$ of the zero dynamics of the original nonlinear system has eigenvalues which coincide with the zeros of the transfer function of the linear approximation of the system at $x = 0$. ◁

Example 4.3.3. Suppose we want to calculate the zero dynamics of the system already analyzed in the Example 4.1.4. The only thing we have to do is to set $z_1 = z_2 = 0$ in the last equation of the normal form of the equations and get

$$\dot{z}_3 = -z_3 \ .$$

These are the zero dynamics of the system. ◁

Example 4.3.4. Suppose we want to analyze the zero dynamics of the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} x_3 - x_2^3 \\ -x_2 \\ x_1^2 - x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} u \\ y &= x_1 \ . \end{aligned}$$

For this system we have

$$L_g h(x) = 0 \qquad L_f h(x) = x_3 - x_2^3 \qquad L_g L_f h(x) = 1 + 3x_2^2 \ .$$

We can calculate a normal form by taking

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_3 - x_2^3 \\ z_3 &= x_2 + x_3 \end{aligned}$$

which is a globally defined coordinates transformation. Using these new coordinates we obtain equations of the following form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= b(z_1, z_2, z_3) + a(z_1, z_2, z_3)u \\ \dot{z}_3 &= z_1^2 - z_3 \ . \end{aligned}$$

The constraint $y(t) = 0$ for all t imposes $z_1(t) = z_2(t) = 0$ for all t, and this shows that when the output is identically zero the state must necessarily evolve on the curve (see Fig 4.5)

$$M = \{x \in \mathbb{R}^3 : x_1 = 0 \text{ and } x_3 = x_2^3\}$$

and be governed by its zero dynamics

$$\dot{z}_3 = -z_3 \ . ◁$$

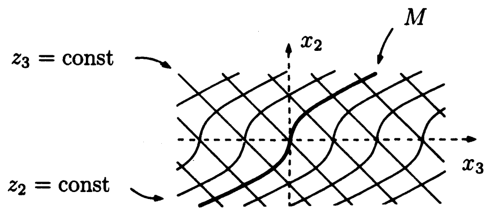

Fig. 4.5.

Although all the properties illustrated so far were discovered and discussed using the normal form, it is not difficult to arrive at similar conclusions starting from equations in different forms. If, for instance, one has not been able to find exactly the normal form because of the difficulty in constructing functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ with the property that $L_g\phi_i(x) = 0$ (see Remark 4.1.3), one can still identify the zero dynamics of the system working on equations of the form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\cdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= b(\xi, \eta) + a(\xi, \eta)u \\
\dot{\eta} &= q(\xi, \eta) + p(\xi, \eta)u \, .
\end{aligned}$$

As a matter of fact, having seen that the zero dynamics of the system describe its behavior when the output is forced to be zero, we impose this condition on the equations above. We obtain, as before, $\xi(t) = 0$ and

$$0 = b(0, \eta(t)) + a(0, \eta(t))u(t) \, .$$

Replacing $u(t)$ from this equation into the last one, yields a differential equation for $\eta(t)$

$$\dot{\eta} = q(0, \eta) - p(0, \eta)\frac{b(0, \eta)}{a(0, \eta)}$$

which describes the zero dynamics in the new coordinates chosen.

Example 4.3.5. Suppose we want to calculate the zero dynamics of the system already analyzed in the Example 4.1.5. In this case we don't have the normal form, but the calculation of the zero dynamics is still very easy. Setting $z_1 = z_2 = 0$ in the second equation yields

$$u = -\frac{z_4 - 4z_4^4}{2 + 2z_3} \; .$$

Replacing this, and $z_1 = z_2 = 0$, in the third and fourth equation yields

$$\begin{aligned} \dot{z}_3 &= -z_3 - \frac{z_4 - 4z_4^4}{2 + 2z_3} \\ \dot{z}_4 &= -2z_4^3 \end{aligned}$$

which describes the zero dynamics of the system. ◁

The problem of zeroing the output could also have been analyzed directly on the original form of the equations. Keeping in mind the calculations already done at the beginning of section 4.1, it is easy to deduce that $y^{(i-1)}(t) = 0$ implies $L_f^{i-1}h(x(t)) = 0$, for all $1 \leq i \leq r$. Thus, as expected, the system has to evolve on the subset

$$Z^\star = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \cdots = L_f^{r-1}h(x) = 0\}$$

which, locally around x°, is exactly the set of points whose new coordinates $z_1, \ldots, z_r$ are 0 (see Fig. 4.6). If one writes the additional constraint

$$0 = y^{(r)}(t) = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t)$$

this turns out to be exactly the same constraint previously obtained for $u(t)$, but now expressed in terms of the functions which characterize the original equations.

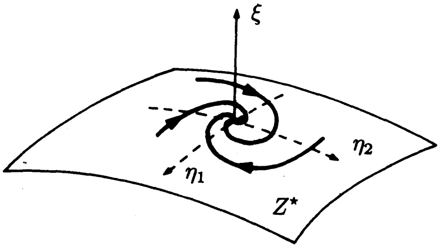

Fig. 4.6.

Note that, since the differentials $dL_f^i h(x)$, $0 \leq i \leq r-1$, are linearly independent at x° (Lemma 4.1.1), the set $Z^\star$ is a *smooth manifold* of dimension $n - r$, near x°. The state feedback

$$u^\star(x) = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)}$$

by construction is such that

$$\begin{pmatrix} dh(x) \\ dL_f h(x) \\ \cdots \\ dL_f^{r-1} h(x) \end{pmatrix} (f(x) + g(x)u^\star(x))$$

$$= \begin{pmatrix} L_f h(x) + L_g h(x) u^\star(x) \\ L_f^2 h(x) + L_g L_f h(x) u^\star(x) \\ \cdots \\ L_f^r h(x) + L_g L_f^{r-1} h(x) u^\star(x) \end{pmatrix} = \begin{pmatrix} L_f h(x) \\ L_f^2 h(x) \\ \cdots \\ L_f^{r-1} h(x) \\ 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} dh(x) \\ dL_f h(x) \\ \cdots \\ dL_f^{r-1} h(x) \end{pmatrix} (f(x) + g(x)u^\star(x)) = 0$$

for all $x \in Z^\star$ (because $h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$ if $x \in Z^\star$) and therefore the vector field

$$f^\star(x) = f(x) + g(x)u^\star(x)$$

is *tangent* to $Z^\star$. As a consequence, any trajectory of the closed loop system

$$\dot{x} = f^\star(x)$$

starting at a point of $Z^\star$ remains in $Z^\star$ (for small values of t). The restriction $f^\star(x)|_{Z^\star}$ of $f^\star(x)$ to $Z^\star$ is a well-defined vector field of $Z^\star$, which exactly describes – in a coordinate-free setting – the zero dynamics of the system.

We will illustrate in the sequel a series of relevant issues in which the notion of zero dynamics, and in particular its asymptotic properties, plays an important role. For the time being we can show, for instance, how the zero dynamics are naturally imposed as internal dynamics of a closed loop system whose input-output behavior has been rendered linear by means of state feedback. For, consider again a system in normal form and suppose the feedback control law (4.25) is imposed, under which the input-output behavior becomes identical with that of a linear system consisting of a string of r integrators between input and output (see Fig. 4.4). The closed loop system thus obtained is described by the equations (4.26), that can be rewritten in the form

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ \dot{\eta} &= q(\xi, \eta) \\ y &= C\xi \end{aligned}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$
$$C = (1 \ \ 0 \ \cdots \ 0).$$

If the linear subsystem is initially at rest and no input is applied, then $y(t) = 0$ for all values of t, and the corresponding internal dynamics of the whole (closed loop) system are exactly those of (4.28), namely the zero dynamics of the open loop system.

We conclude the section by showing that the interpretation of

$$\dot{\eta}(t) = q(0, \eta(t)) ,$$

as of the dynamics describing the internal behavior of the system when the output is forced to track exactly the output $y(t) = 0$, can easily be extended to the case in which the output to be tracked is any arbitrary function. Consider the following problem, which is called the *Problem of Reproducing the Reference Output* $y_R(t)$. Find, if any, pairs consisting of an initial state x° and of an input function $u^\circ(\cdot)$, defined for all t in a neighborhood of $t = 0$, such that the corresponding output $y(t)$ of the system coincides exactly with $y_R(t)$ for all t in a neighborhood of $t = 0$. Again, we are interested in finding *all* such pairs (x°, u°). Proceeding as before, we deduce that $y(t) = y_R(t)$ necessarily implies

$$z_i(t) = y_R^{(i-1)}(t) \qquad \text{for all } t \text{ and all } 1 \le i \le r .$$

Setting

$$\xi_R(t) = \text{col}(y_R(t), y_R^{(1)}(t), \ldots, y_R^{(r-1)}(t)) \tag{4.29}$$

we then see that the input $u(t)$ must necessarily satisfy

$$y_R^{(r)}(t) = b(\xi_R(t), \eta(t)) + a(\xi_R(t), \eta(t))u(t)$$

where $\eta(t)$ is a solution of the differential equation

$$\dot{\eta}(t) = q(\xi_R(t), \eta(t)) . \tag{4.30}$$

Thus, if the output $y(t)$ has to track exactly $y_R(t)$, then necessarily the initial state of the system must be set to a value such that $\xi(0) = \xi_R(0)$, whereas $\eta(0) = \eta^\circ$ can be chosen arbitrarily. According to the value of η°, the input must be set as

$$u(t) = \frac{y_R^{(r)}(t) - b(\xi_R(t), \eta(t))}{a(\xi_R(t), \eta(t))} \tag{4.31}$$

where $\eta(t)$ denotes the solution of the differential equation (4.30) with initial condition $\eta(0) = \eta^\circ$. Note also that for each set of initial data $\xi(0) = \xi_R(0)$ and $\eta(0) = \eta^\circ$ the input thus defined is the *unique* input capable of keeping $y(t) = y_R(t)$ for all times.

The (forced) dynamics (4.30) clearly correspond to the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to track exactly $y_R(t)$. Note that the relations (4.30) and (4.31) describe a system with input $\xi_R(t)$, output $u(t)$ and state $\eta(t)$ that can be interpreted as a *realization* of the *inverse* of the original system.

4.4 Local Asymptotic Stabilization

In this section we illustrate how the notion of zero dynamics can be helpful in dealing with the problem of asymptotically stabilizing a nonlinear system at a given equilibrium point. Suppose, as usual, a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u$$

is given, with $f(x)$ having an equilibrium point at x° that, without loss of generality, we assume to be $x^\circ = 0$. The problem we want to discuss is the one of finding a smooth state feedback

$$u = \alpha(x)$$

defined locally around the point $x^\circ = 0$ and preserving the equilibrium, i.e. such that $\alpha(0) = 0$, with the property that the corresponding closed loop system

$$\dot{x} = f(x) + g(x)\alpha(x)$$

has a locally asymptotically stable equilibrium at $x = 0$. We shall refer to it as to the *Local Asymptotic Stabilization Problem.*

First of all, we review a rather well-known property, by discussing to what extent the possibility of solving the problem in question depends on the properties of the linear approximation of the system near $x^\circ = 0$. To this end, recall that the linear approximation of a system having an equilibrium at $x^\circ = 0$ is defined by expanding $f(x)$ and $g(x)$ as (see Remark 4.2.7)

$$\begin{aligned} f(x) &= Ax + f_2(x) \\ g(x) &= B + g_1(x) \end{aligned}$$

with

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \qquad \text{and} \qquad B = g(0)\ .$$

From the point of view of the stability properties of the closed loop system, the importance of the linear approximation is essentially related to the following result.